Stabilizability theorem of discrete-time nonlinear systems with scalar parameters

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Abstract: This paper advances [1] by deducing a stabilizability theorem for discrete-time nonlinear systems with scalar parameters, which takes a step forward to the complete characterization of feedback limitations in discrete-time adaptive nonlinear control. It is well-known that exponent 4 is an important critical number to characterize the feedback capability for the basic discrete-time scalar-parameter systems, which are governed by power functions. As an application of our theorem, a new critical number 2 is derived for a typical class of discrete-time nonlinear stochastic systems with scalar parameters.

Key words: feedback limitations; adaptive control; least squares; stabilizability; nonlinear systems; discrete time


1 Introduction

Most works on nonlinear adaptive control in the literature are focused on continuous-time systems[2–4]. But adaptive control between continuous- and discrete-time systems are rather different. As a matter of fact, a large class of continuous-time nonlinear systems can be globally stabilized by applying nonlinear damping or back-stepping techniques, no matter how fast their growth rates are[5–6]. However, the situation in the discrete-time case is different.

A heuristic result derived by [7] is that feedback limitations exist for discrete-time adaptive nonlinear control. [7] studied a basic discrete-time nonlinear random system with a scalar parameter:

\[ y_{t+1} = \theta y_t^b + u_t + w_{t+1}, \]

and demonstrated that \( b = 4 \) is the critical exponent for the stabilizability. Soon afterwards [8] established an “impossibility theorem” for the multi-parameter system

\[ y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_n y_t^{b_n} + u_t + w_{t+1}. \tag{1} \]

A polynomial rule on \( b_1, \cdots, b_n \) was introduced in the theorem to describe the nonlinear growth rates that fail all feedback control laws in stabilizing system (1). Lately, [9] proved that the polynomial rule in fact serves as the necessary and sufficient condition of the stabilizability of system (1). Besides, some initial research
on nonlinear parameterized systems with scalar parameters indicates that $b = 4$ is also an important exponent for the stabilizability.\cite{10}. Meanwhile, a parallel theory in the deterministic framework on feedback limitations has been developed accordingly (see \cite{11–15}).

The aforementioned systems are all in polynomial forms. For the following relatively general system

$$y_{t+1} = \theta^T f(y_t) + u_t + w_{t+1}, \quad \theta \in \mathbb{R}^n,$$

one may wonder if $|f(x)| = O(|x|^4)$ is still the limit of nonlinear growth rates for discrete-time stabilizable systems? The first answer appeared in \cite{1}. This work showed that for $n = 1$, system (2) is possible to be stabilized by a discrete-time feedback controller, even if it grows exponentially fast. The density of a regular set is defined in \cite{1} to determine the stabilizability of system (2). As a matter of a fact, \cite{1} provided a quantitative criterion of the stabilizability of system (2) together with Theorem 2.6 in \cite{1}, takes a step further to the two densities for stabilizable systems and unstabilizable systems in discrete time. But there is still a gap between the two densities for stabilizable systems and unstabilizable systems. A theorem established here, which together with Theorem 2.6 in \cite{1}, takes a step further to illustrate that for the stabilizability of system (2).

$$\beta$$

is a new critical number here. Note that for any $b < 4$, $\lim_{x \to +\infty} \frac{|f(x)|}{x^b} = +\infty$, $\lim_{x \to +\infty} \frac{|f(x)|}{x^4} = 0$.

It confirms the fact that $x^4$ cannot serve as the critical growth rate for system (3). As a matter of fact, if $\beta \in (0, 2]$, systems (3)–(4) can be stabilized by the least-squares based self-tuning regulator (LS-STR), which is defined later by (6)–(7). Fig. 1 simulates the stability of the closed-loop systems (3) (4) (6) and (7) for $\beta = 2$, $\sigma_0 = 0$, $\theta_0 = 0$ and $P_0 = 1$. Of course, it just a simple simulation of trajectory and can not provide more information about the stability with probability 1, we need strictly analysis to confirm the criticality of $\beta = 2$.

**Example 1** Consider system (3) with

$$f(x) = \begin{cases} x^3, & x \in [-e^\beta, e^\beta], \\ |x|^{|\beta|-1(\log(\log|x|))}, & x > e^\beta \text{ or } x < -e^\beta, \end{cases}$$

where $\beta > 0$. The system is globally stabilizable whenever $\beta \in (0, 2]$ and unstabilizable if $\beta > 2$. Obviously, $\beta = 2$ is a new critical number here. Note that for any $b < 4$.

$\lim_{x \to +\infty} \frac{|f(x)|}{x^b} = +\infty$, $\lim_{x \to +\infty} \frac{|f(x)|}{x^4} = 0$.

The critical number $\beta = 2$ in Example 1 cannot be deduced directly from the existing works. It origi-
nates from two theorems stated below. For this, assume $h : [0, +\infty) \to [0, +\infty)$ is a nonnegative monotone increasing piecewise continuous function and satisfies $h(|x|) = O(x^a) + O(1)$. Let $g(x) = |x|^{-\frac{1}{4}}h^{-1}(|x|)$, where $h^{-1}$ denotes the inverse function of $h$.

**Theorem 1** Under Assumptions A1)–A2), system (3) is globally stabilizable if $|f(x)| = O(h(|x|))$, where $h$ is chosen so that for some $\mu > \frac{1}{16}$,

$$
\liminf_{t \to +\infty} \inf_{x \in [r^T, r^T_1]} x^\mu g(x) > 0, \forall r_2 > r_1 > e^2.
$$

(5)

**Remark 1** Example 1 with $\beta \in (0, 2]$ follows from the fact that if we let $h(x) = f(x)$, then system (3) is globally stabilizable, according to Theorem 1 with $g(x) = |x|^{-\frac{1}{4}h^{-1}(|x|)}$. In this case, (5) holds. The proof is contained in Appendix.

On the other hand, Example 1 with $\beta > 2$ is unstable due to

**Theorem 2** Under Assumptions A1)–A2), system (3) is unstabilizable if there is a $\delta > 0$ such that

$$
\sup_{x \in \mathbb{R}} \ell(S_h \cap [x - l, x + l]) = O\left(\frac{1}{\log \log f}\right),
$$

where $S_h = \{x : |f(x)| \leq h(|x|)\}$ with $h$ satisfying

$$
\sum_{t=1}^{+\infty} \sup_{x \in [r^T, +\infty)} x^{-1/16^2} g(x) < +\infty.
$$

**Remark 2** The unstabilizability part of Example 1 is a direct consequence of Theorem 2, by taking $g(x) = x^{1/16(\log \log |x|)^2}$ with $\beta > 2$ (see Appendix for details).

3 Proof

### 3.1 Technique Lemmas

The feedback control law in this paper is designed based on the least-squares (LS) algorithm, which can be recursively defined by

$$
\begin{aligned}
\theta_{t+1} &= \theta + a_t P_t \phi_t (y_{t+1} - u_t - \phi_t^T \theta_t), \\
P_{t+1} &= P_t - a_t P_t \phi_t \phi_t^T P_t, \quad P_0 > 0,
\end{aligned}
$$

(6)

where $\phi_t \triangleq (1 + \phi_t^T P_t \phi_t)^{-1}$ and $(\theta_0, P_0)$ denotes a deterministic initial value. Let

$$
u_t = -\theta_t f(y_t), \quad t \geq 0.
$$

(7)

By the closed-loop system (3)(6)–(7),

$$
\begin{aligned}
\hat{\theta}_t &= \frac{\theta_0 - \sum_{i=0}^{t-1} \phi_i w_{i+1}}{r_{t-1}}, \\
y_{t+1} &= \hat{\theta}_t f(y_t) + w_{t+1},
\end{aligned}
$$

where $\hat{\theta}_t \triangleq \theta - \theta_t, r_{t-1} \triangleq P_0^{-1}, r_t \triangleq P_{t+1}^{-1} = P_0^{-1} + \sum_{i=0}^{t} \phi_i^2, \quad t \geq 0$. Notice that the LS algorithm (6) is exact-

ly the standard Kalman filter for $\theta \sim N(\theta_0, P_0)$, then

$$
\theta_t = E[\theta|F_t^\prime], \quad P_t = E[(\hat{\theta}_t)^2|F_t^\prime].
$$

So, $y_{t+1}$ is conditionally Gaussian distributed given $F_t^\prime$. For each $t \geq 0$, the conditional mean and variance satisfy

$$
m_t \triangleq E[y_{t+1}|F_t^\prime] = u_t + \theta_t \phi_t, \quad \sigma_t^2 \triangleq \text{Var}(y_{t+1}|F_t^\prime) = 1 + \phi_t P_t \phi_t = \frac{\phi_t^2}{r_{t-1}} + 1 = \frac{r_t}{r_{t-1}}, \quad \text{a.s.}
$$

We first present several technique lemmas under Assumptions A1)–A2).

**Lemma 1**[1] Let $\{c_t\}_{t \geq 1}$ be a sequence satisfying

$$
\liminf_{t \to +\infty} \frac{c_t}{\log t} > 0,
$$

then

$$
\sum_{t=1}^{+\infty} \int_{|x| > c_t} e^{-x^2} \, dx < +\infty.
$$

**Lemma 2**[1] If $\ell \{\{f(x) > 0\}\} > 0$, then

$$
\liminf_{t \to +\infty} \frac{r_t}{t} > 0, \quad \text{a.s.}
$$

**Lemma 3**[1] Let $f(x) = O(|x|^a) + O(1)$ for some $a \geq 1$ and let $x_{\min} \leq x_{\max}$ denote the two solutions of equation $x^2 - (a - 2)x + 1 = 0$. If $\ell \{\{f(x) > 0\}\} > 0$, then the following two statements hold:

i) $D_1 = D_2$ with $D_1 \triangleq \{\liminf_{t \to +\infty} \log r_t \geq 1 + x_{\min}\}$

ii) $P(D_3) = 0$ with $D_3 \triangleq \{\limsup_{t \to +\infty} \log r_t > 1 + x_{\max}\}$

**Lemma 4** If (5) holds, then

$$
\liminf_{x \to +\infty} \frac{g(x)}{\log x} > 0.
$$

**Proof** Suppose $\liminf_{x \to +\infty} \frac{g(x)}{\log x} \leq 0$. Then, there exists an infinite sequence $\{x_n\}_{n \geq 1}$ satisfying $\lim_{n \to +\infty} x_n = +\infty$ and

$$
g(x_n) < \frac{1}{n} \log x_n.
$$

(8)

Observe that for any $r_2 > r_1 > e^2$,

$$
r_2^{i+1} > r_2^i, \quad t \geq 2
$$

and hence

$$
\bigcup_{t \geq 2} [r_1^t, r_2^t] = [r_4^1, +\infty).
$$

Therefore, for any sufficiently large $n$, there is a positive integer $k_n$ with $\lim_{n \to +\infty} k_n = +\infty$ such that

$$
x_n \in [r_1^{2^{k_n}}, r_2^{2^{k_n}}].
$$

(9)
This together with (8) and (9) yields
\[
\limsup_{n \to +\infty} \inf_{x \in [t_n^{a_n}, t_n^{a_n+1}]} x^{-\mu^k_n} \frac{g(x)}{\log k_n} \leq \limsup_{n \to +\infty} x^{-\mu^k_n} \frac{\log x_n}{n \log k_n} \leq \limsup_{n \to +\infty} x^{-\mu^k_n} \frac{\log x_n}{n \log k_n} = 0,
\]
and consequently,
\[
\liminf_{t \to +\infty} \inf_{x \in [t_n^{a_n}, t_n^{a_n+1}]} x^{-\mu^k_n} \frac{g(x)}{\log t} \leq 0.
\]
We thus draw a contradiction of (5). QED.

3.2 Proof of Theorem 1

As already claimed in the proof of [1, Theorem 2.2], it suffices to show the stabilization for \(t(\{x : |f(x)| > 0\}) > 0\). Under this condition, \(\liminf_{t \to +\infty} \frac{r_t}{r_t} > 0\) almost surely due to Lemma 2. Denote
\[
s_m \equiv \frac{\log f^2(y_m)}{\log r_m} - 2,
\]
\[
S \equiv \left\{ \lim_{t \to +\infty} \frac{\log r_t}{\log r_{t-1}} = 2 \right\} = \left\{ \lim_{t \to +\infty} s_t = 0 \right\},
\]
\[
U_m \equiv \left\{ s_{m-1} \leq 0, s_m \geq 0 \right\},
\]
\[
V_m^{C} \equiv \left\{ s_m \geq \frac{2s_{m-1}}{2 + s_{m-1}} - \frac{C}{m^2} \right\}, C \geq 0.
\]
Since the proof of [1, Theorem 2.2] indicates
\[
\left\{ \sup_{t} \sigma_t < +\infty \right\} \subset \left\{ \frac{1}{t} \sum_{k=0}^{t} y_k^2 = O(1) \right\},
\]
taking account of Lemma 3 with \(a = 4\), the remainder of the proof is sufficient to verify
\[
P(S) = 0.
\]
Without loss of generality, suppose \(|f(x)| \leq h(|x|)\) for all \(x \in \mathbb{R}\). Therefore, as long as \(m\) is sufficiently large,
\[
P(U_{m+1} | F_m) = \frac{1}{\sqrt{2\pi}} \int_{|x| > \sqrt{\log r_m}} e^{-x^2/2} dx < +\infty
\]
and
\[
P(V_m^C | F_m) = \frac{1}{\sqrt{2\pi}} \int_{|x| > \sqrt{\log r_m}} e^{-x^2/2} dx < +\infty
\]
Observe that \(\liminf_{x \to +\infty} \frac{g(x)}{\log x} > 0\) in view of Lemma 4.
Furthermore, according to \(\liminf_{m \to +\infty} \frac{r_m}{m} > 0\) and \(S = \left\{ \lim_{t \to +\infty} s_t = 0 \right\}\), we have
\[
\liminf_{m \to +\infty} \frac{g(r_m)}{\log m} > 0
\]
and
\[
\liminf_{m \to +\infty} \frac{1}{\log m} g(r_m) > 0\]
Let \(C = 0\). By virtue of (11)–(14) and Lemma 1, it deduces that
\[
\sum_{m=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{|x| > (1 + f_0) - |g(r_m)|} e^{-x^2/2} dx < +\infty
\]
and
\[
\sum_{m=1}^{\infty} P(V_m^C | F_m) < +\infty
\]
Using Borel-Cantelli-Levy theorem, one has
\[
\sum_{m=1}^{\infty} I_{V_m^C} < +\infty \quad \text{on} \quad S,
\]
\[
\sum_{m=1}^{\infty} I_{V_m^C} = +\infty
\]
Now, according to (15), on \(S\), \(\{ s_t \}_{t \in \mathbb{N}}\) either satisfies
\[
0 < s_t < \frac{2s_{t-1}}{2 + s_{t-1}}
\]
or
\[
s_t < \frac{2s_{t-1}}{2 + s_{t-1}} < 0,
\]
where \(t\) is sufficiently large. However, if (17) holds,
\[
|s_t| > \frac{2|s_{t-1}|}{2 + s_{t-1}} > |s_{t-1}|, \text{ and then } \lim_{t \to +\infty} s_t \neq 0,
\]
which contradicts to the definition of $S$. Thus, $\{s_t\}_{t \geq 1}$ satisfies (16) on $S$, which means

$$S \subseteq \bigcup_{n \geq 1} \bigcap \{0 < s_t < \frac{2s_{t-1}}{2 + s_{t-1}} \} \subseteq \bigcup_{n \geq 1} \bigcap \{0 < s_t < \frac{2}{2r + t - n} \}. \quad (18)$$

On the other hand, since $\lim inf_{t \to \infty} \frac{r_t}{t} > 0$ almost surely,

$$S \subseteq \bigcup_{n \geq 1} \{ r_n > e^2 \} = \bigcup_{n \geq 1} \bigcup_{s \in Q^n, s > e^2} \{ r_n \in (s, s + 1) \}. \quad (19)$$

Denote

$$W_n^s \triangleq \bigcap_{t \geq n} \{0 < s_t < \frac{2}{2r + t - n} \},$$

$$T_n^s \triangleq \{ r_n \in (s, s + 1) \},$$

(18) and (19) leads to

$$S \subseteq \bigcup_{n \geq 1} \bigcap \{ r_n \in (s, s + 1) \} \subseteq \bigcup_{n \geq 1} \bigcup_{s \in Q^n, s > e^2} \{ T_n^s \cap W_n^s \}.$$ 

To show (10), we only need to prove that for any $r, s \in Q^+$, $s > e^2$ and $n \geq 1$,

$$P(\{ T_n^s \cap W_n^s \}) = 0. \quad (20)$$

Now, fix $r, s \in Q^+$, $s > e^2$ and $n \geq 1$. Assume $W \triangleq \bigcup_{n \geq 1} W_n^s$ satisfies $P(W) > 0$, we next show that on $W$,

$$r_m \in [s^{2^{m-n}}, (C_1 \cdot (s + 1))^{(m-n+1)2^{m-n}}], \quad m \geq n, \quad (21)$$

where $C_1 = 1 + \frac{1}{s}$.

In fact, $r_m \geq s^{2^{m-n}}$ clearly holds. Moreover, for $m > n$,

$$C_1 r_m \leq C_1 r_{m-1} + C_1 r_{m-1}^{2/(2r + m - n)} \leq C_1 r_{m-1}^{2/(2r + m - n)} (1 + r_{m-1}^{-1} - 2/(2r + m - n)) < C_1 r_{m-1}^{2/(2r + m - n)} C_1^2 < (C_1 r_{m-1})^{(m-n+1)2^{m-n}},$$

then

$$r_m < C_1^{-1} (C_1 r_{m-1})^{(m-n+1)2^{m-n}} \leq C_1^{-1} (C_1 r_{m-1})^{(m-n+1)2^{m-n}} < (C_1 (1 + s))^{(m-n+1)2^{m-n}} < (C_1 (1 + s))^{(m-n+1)2^{m-n}},$$

and hence (21) follows.

Now, take some $\nu_1 \in (e^2, s)$, $\nu_2 > C_1 (s + 1)$ and $C \in \left[ \frac{1}{2}, 8\mu \right]$. On $W$, one has

$$r_m^{1+s_m/(2+s_m)-C/2m^2} \geq s^{2^{m-n}} (1-C/2m^2) > \nu_1^{s_m}. \quad (22)$$

and

$$r_m^{1+s_m/(2+s_m)-C/2m^2} \geq (C_1 \cdot (s + 1))^{(m-n+1)2^{m-n}} (1+s_m/(2+s_m)) < (C_1 \cdot (s + 1))^{(m-n+1)2^{m-n}} ((2r + m - n + 3)/(2r + m - n + 1)) < (C_1 \cdot (s + 1))^{(m-n-3)2^{m-n}} < \nu_2^{(m-n)2^{m-n}}, \quad (23)$$

where $m$ is sufficiently large. Denote

$$Y_t \triangleq \inf_{[\nu_1^2, \nu_2^2]} x^{-\mu(x^2)} g(x),$$

then by (21)–(23) and $\lim inf_{t \to +\infty} \frac{r_t}{t} > 0$, for any sufficiently large $m$,

$$R_m^C = \frac{r_m^{-C/8m^2} (r_m^{1+s_m/(2+s_m)-C/2m^2})}{(1 + r_m^{-1} - s_m/(2r + m - n))} \geq \frac{r_m^{-C/8m^2} (r_m^{1+s_m/(2+s_m)-C/2m^2})}{(1 + r_m^{-1} - s_m/(2r + m - n))} \geq \frac{(r_m^{1+s_m/(2+s_m)-C/2m^2})}{(1 + r_m^{-1} - s_m/(2r + m - n))} \geq \frac{1}{2},$$

$$g((r_m^{1+s_m/(2+s_m)-C/2m^2})) \geq \frac{1}{2},$$

$$\frac{1}{2} Y_{m-n} \text{ on } W,$$

where $R_m^C$ is defined in (12). Moreover, by virtue of Lemma 1 and (5) with $r_1 = \nu_1$ and $r_2 = \nu_2$,

$$\sum_{m=n+1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{|x| \geq Y_m^{-1/2}} e^{-x^2/2} \, dx < +\infty. \quad (24)$$

With (12), (24) and (25), it is straightforward that

$$\sum_{m=n+1}^{\infty} P(Y_{m+1}^C, Y_m^C) < +\infty \text{ on } W,$$

which shows $\sum_{m=n+1}^{\infty} I_{Y_{m+1}^C} < +\infty$ almost surely on set $W$, in view of Borel-Cantelli-Levy theorem. As a result, as long as $m$ is sufficiently large,

$$0 < s_m < \frac{2}{2 + s_m-1} - \frac{C}{m^2} \text{ a.s. on } W.$$

For $m > n + 1$, denote

$$\rho_m \triangleq 2 - (m-n)s_m,$$

$$d_m \triangleq \frac{2}{2 + s_m-1} - s_m,$$

since $s_m < \frac{2}{2r + m - n} < \frac{2}{m-n} \text{ on } W$, one has

$$\rho_m \in (0, 2), \quad d_m > \frac{C}{m^2} \text{ and }$$

$$s_m < \frac{2}{2r + m - n} < \frac{2}{m-n},$$

$$\rho_m \in (0, 2), \quad d_m > \frac{s_m-1}{2 + s_m-1} - s_m,$$

$$\rho_m \in (0, 2), \quad d_m > \frac{C}{m^2} \text{ and }$$

$$s_m < \frac{2}{2r + m - n} < \frac{2}{m-n} - \rho_m.$$
\[
\frac{\rho_{m-1}(2 - \rho_{m-1})}{2(m - n) - \rho_{m-1}} + \rho_m - \rho_{m-1} \leq \frac{1}{2(m - n - 1)} + \rho_m - \rho_{m-1}.
\]

Therefore,
\[
m(\rho_m - \rho_{m-1}) \geq \frac{C(m - n)}{m} - \frac{m}{2(m - n - 1)},
\]

which, by noting that \(C > \frac{1}{2}\) infers \(\liminf m(\rho_m - \rho_{m-1}) > 0\), and hence \(\lim_{m \to +\infty} \rho_m = +\infty\). This contradicts to the fact that \(\rho_m < 2\). We thus conclude \(P(W) = 0\). That is, (20) holds and hence Theorem 1 is proved.

4 Conclusions

The stabilizability theorem in this paper, combining with Theorem 2 derived by [1], tries to elaborate on the characterization of feedback limitations in discrete-time adaptive nonlinear control. Although the stabilizability and unstabilizability conditions presented here are very close, it still calls for further efforts on the critical stabilizability condition.

References:


Appendix: Proof of Remarks 1 and 2

We prove Remarks 1 and 2 in this appendix.

Proof of Remark 1. Given \(\beta \in (0, 2]\), let \(\mu = \frac{1}{12} > \frac{1}{16}\) and \(q_1(x) = x^{1/16} \log |x| \beta^3\), then for any \(r_2 > r_1 > e^2\) and \(x \in [r_1^2, r_2^2]\), one has
\[
x^{-\mu t^2} q_1(x) = x^{(1/16)\log |x| \beta^3} - 1/12t^2 \geq x^{1/16(\log |x| \beta^3) + 1/12t^2} = x^{1/16(\log x \beta^3) + 1/12t^2} \geq 1/16(\log x \beta^3) + 1/12t^2 = 1/16(t \log 2 + (0.75 - \log 2)^2) - 1/12t^2 = r_1^2/36t^2,
\]
where \(t\) is sufficiently large. Thus,
\[
\liminf_{t \to +\infty} \inf_{x \in [r_1^2, r_2^2]} x^{-\mu t^2} q_1(x) \log t \geq \liminf_{t \to +\infty} r_1^2/36t^2 \log^{-1} t = +\infty.
\]

Therefore, for any \(f(x)\) given by Example (4) with \(\beta \in (0, 2]\), if we can show
\[
|x|^{-\mu} f^{-1}(|x|) \geq q_1(x)
\]
holds for all sufficiently large \(|x|\), then by using (a1), \(f(x)\) satisfies the conditions of Theorem 1 and Remark 1 follows.

To this end, denote \(A_1 \triangleq \log \log |x|\) and \(A_2 \triangleq \frac{1}{4} + \frac{1}{16A_1}\). Since \(\log A_2 \in (-\log 2, 0)\) for \(|x| > e^2\), it yields
\[
1/4 + A_1^{1/3} \geq (A_1 + \log A_2)^{1/3}.
\]

Note that for any sufficiently large \(|x|\), (a3) is equivalent to
\[
1 \geq A_2(4 - \frac{1}{(\log(A_2 \log |x|))^{3/2}}),
\]
and hence
\[
|x| \geq |x| A_2(4 - \frac{1}{(\log(A_2 \log |x|))^{3/2}}) = f(|x|^{1/4} q_1(x)),
\]
which is exactly (a2) as desired.

QED.

Proof of Remark 2. For \(\beta > 2\), let \(g(x) = x^{1/12} \log(|x| \beta^3)\). Then, for any \(x \in [e^{\beta^3}, +\infty)\),
\[
x^{-1/16^2} g(x) = x^{1/12} \log(|x| \beta^3) - 1/16^2 \leq x^{1/12(\log 2^2) - 1/16^2} \leq e^{(1/12(\log 2^2) - 1/16^2)} t^2 = o(t^2),
\]

as $t \to +\infty$. Consequently,

\[
\sum_{t=1}^{+\infty} \sup_{x \in [x^3, +\infty)} x^{-1/16t^2} g(x) < \infty.
\]

This implies that if for all sufficiently large $|x|$,

\[
|x|^{-1/4} f^{-1}(|x|) \leq g(x), \tag{a4}
\]

then $f(x)$ satisfies the condition of Theorem 2.

As above, denote $A_3 \triangleq \frac{1}{4} + \frac{1}{12A_1^2} \in \left(\frac{1}{4}, 1\right)$ for $|x| > e^e$.

Then,

\[
\frac{1}{3} + A_3^2 \leq \frac{4}{3} (A_1 + \log A_3)^3,
\]

and hence for any sufficiently large $|x|$,

\[
|x| \leq |x|^A (4 - 1/(\log(A_3 \log |x|))^B).
\]

Since $f(x)$ is defined by (4), similar to the argument of Remark 1, (a4) follows and we complete the proof. QED.