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基于最优状态转移的非因果稳定逆

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摘要: 本文研究非最小相位系统的精确跟踪问题. 理想情况下, 非最小相位系统针对参考轨迹的精确跟踪可以通过非因果稳定逆方法实现, 但控制输入需从负无穷处开始作用. 而在实际情况下应用非因果稳定逆算法时, 控制输入通过延拓提前作用的时间是有限的, 只能得到近似的跟踪效果. 本文提出了一种基于最优状态转移的非因果稳定逆算法, 能够在实际情况下实现非最小相位系统对参考轨迹的精确跟踪, 放松了稳定逆方法对系统的初始状态和延拓时间的限制, 而且在相同跟踪效果的条件下, 比近似稳定逆方法的延拓时间更短. 对比仿真结果验证了所提方法的性能.

关键词: 稳定逆; 状态转移; 非因果; 非最小相位

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Non-causal stable inversion based on optimal state to state transition

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Abstract: Non-causal stable inversion method can be used to track the reference trajectory precisely under some strict conditions for non-minimum phase systems; these conditions require the control input to start action from negative infinity on the time axis. However, in the actual situation, the control input of the stable inversion method must be truncated and only acts in a finite extended time interval; this results in an approximate tracking instead of the precise tracking. In this paper, we propose for non-minimum phase systems a revised non-causal stable inversion method based on optimal state-to-state transition, it can achieve a precise tracking of the reference trajectory in the actual situation, regardless of the arbitrary initial system state or the arbitrary extended time. With the same tracking precision guaranteed, the proposed method has a shorter extended time in comparison with the approximate stable inversion method. The better performance of the proposed method is validated through simulation results.

Key words: stable inversion; state to state transition; non-causal; non-minimum phase

1 Introduction

Tracking problem is a hot topic. It is widely seen in the areas of flexible manipulator [1], aircraft [2], hard disk[3], etc., which are all non-minimum phase systems. Inversion technique and output regulation are regular methods to solve tracking problems. Non-stable zeroes are canceled by the output regulation method but it limits to asymptotic tracking[4]. While, precise tracking with unique input and state is realized by the inversion technique[5]. Classical inversion technique functions well in the minimum phase system but leads to divergent solutions in the non-minimum phase system. However, Bounded solutions can be obtained by the stable inversion method in the non-minimum phase system proposed by Devasia, etc.[6] and Chen, etc.[7]. Future information of the reference trajectory is needed for the stable inversion method, which ensures stability by giving up causal characteristic, thus it is non-causal.

Since the stable inversion method was proposed, its theory has been constantly improved and its application has been increasingly extended. As seen from its definition[6], zeroes were not allowed in the imaginary axis. Devasia dealt with the asymptotic tracking problems with zeroes in or near the imaginary axis[8] and the precise tracking problems of nonlinear time varying systems[9]. Sogo[10] analysed the stable inversion method in the frequency domain by two-sided Laplace transformation. Liu, etc.[11] proposed expanded Laguerre basis function based iterative learn-

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ing control to approximate stable inversion solution. Taylor,etc.[12] proposed the finite difference method considering stable inversion as a two point boundary value problem. Hunt,etc.[13] proposed a two-step procedure method to handle control problems of perturbed model like aircraft where the stable inversion method is applied in an error system. Since future information of the reference trajectory can be attained by iterative learning, stable inversion method were often combined with iterative learning. Liu, etc.[14] proposed the open-closed-loop optimal iterative learning control based stable inversion. Its concepts of prolongation and translation of the reference trajectory revealed the necessary procedure of the stable inversion method used in the actual situation.

Non-casual stable inversion was calculated offline owing to the need of the whole reference trajectory. Zou, etc.[15] proposed the preview-based stable inversion to calculate the stable inversion solution online using a finite window of trajectory rather than the whole trajectory. In a word, the dynamic tracking process of the time interval $[t_i, t_{\mathrm{f}}]$ can be seen from Devasia, etc.[6], while the dynamic tracking process of the time interval $[t_{\mathrm{f}}, \infty]$ can be seen form Zou[16].

Little attention was paid to the dynamic process in the time interval $[-\infty, t_i]$ of the reference trajectory about stable inversion method. Liu, etc.[14] used the way of prolongation and translation of the reference trajectory originally defined in $[t_i, t_{\mathrm{f}}]$. Devasia, etc.[6, 8] also expanded the reference trajectory to negative infinity as far as possible. Longer the expanded trajectory was, better the tracking effect would be. It was a way of approximate tracking in nature.

This paper analyses the dynamic process of the stable inversion method in the time interval $[-\infty, t_i]$ and puts forward the concept of stable initial state defined at $t = t_i$ and validate the equivalent precise tracking in $[t_i, t_{\mathrm{f}}]$ between the proposed method used in the actual situation and the classical method defined in the ideal situation. So precise tracking in the actual situation is achieved by introducing the optimal state to state transition technique in the proposed method.

The proposed method is based on the optimal state transition (OST) technique introduced from Lewis, etc.[17]. Devasia, etc.[18–21] handled the output transition problem of the continuous and discrete linear system using OST technique. Wang, etc.[22] found an optimal output transition trajectory by OST technique thus solving a class of aperiodic tracking transition problem with stable inversion method. This paper also uses OST technique to handle the precise tracking problem with stable inversion method in the actual situation, which has achieved three achievements comparing with the classical stable inversion method: 1) The expanded time interval of the reference trajectory is greatly decreased under uniform overall tracking error interval; 2) Precise tracking in the actual situation is achieved by the proposed method while the classical stable inversion is defined in the ideal situation and approximate tracking is obtained in the actual situation; 3) Precise tracking in $[t_i, t_{\mathrm{f}}]$ is achieved regardless of an arbitrary initial state or an arbitrary extended time .

This paper is organised as follows. Section 2 introduces the stable inversion method and the system state is decomposed; Section 3 proposes the concept of Stable Initial State which is of vital importance to achieve precise tracking; Section 4 introduces the pre-action process of the classical stable inversion; Section 5 introduces the OST technique; Section 6 gives the simulation results comparing the proposed method and the classical method.

2 Stable inversion

A SISO linear system is considered in this paper:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + \boldsymbol{B}u(t), \\ y(t) &= \boldsymbol{C}\boldsymbol{x}(t). \end{aligned} \tag{1}$$

Definition 1[23] Equation (1) is a non-minimum phase linear time invariant system with no zeroes in the imaginary axis. As to the reference trajectory y_{d} satisfying $y_{\mathrm{d}}^{(i)} \in L_1 \cap L_\infty$, $i = 0, 1, \cdots, r$, r is the smallest positive integer as $\boldsymbol{C}A^{r-1}\boldsymbol{B} \neq 0$ namely the relative degree. Bounded $\boldsymbol{x}_{\mathrm{d}}(t)$ and $u_{\mathrm{d}}(t)$ are existed satisfying

$$\begin{aligned} \dot{\boldsymbol{x}}_{\mathrm{d}}(t) &= A\boldsymbol{x}_{\mathrm{d}}(t) + \boldsymbol{B}u_{\mathrm{d}}(t), \\ y_{\mathrm{d}}(t) &= \boldsymbol{C}\boldsymbol{x}_{\mathrm{d}}(t), \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times 1}$, $\boldsymbol{C} \in \mathbb{R}^{1\times n}$ are known parameters, then $u_{\mathrm{d}}(t) \to 0$, $\boldsymbol{x}_{\mathrm{d}}(t) \to 0$ as $t \to \pm\infty$. $\boldsymbol{x}_{\mathrm{d}}(t)$ and $u_{\mathrm{d}}(t)$ are defined as the stable inversion of the reference trajectory $y_{\mathrm{d}}(t)$.

Definition 2 The relative degree is r, and then the outer state is defined as the output and its derivatives (till the $(r-1)$th order).

$$\boldsymbol{\xi}(t) = [y(t)\ \ \frac{\mathrm{d}y(t)}{\mathrm{d}t}\ \ \cdots\ \ \frac{\mathrm{d}^{r-1}y(t)}{\mathrm{d}t^{r-1}}]^{\mathrm{T}}. \tag{2}$$

Definition 3 The original reference trajectory is defined in the time interval $[t_i, t_{\mathrm{f}}]$ of $y_{\mathrm{d}}(t)$. $y_{\mathrm{d}}(t)$ is the extended reference trajectory defined in the time interval $t \in (-\infty, \infty)$. It's continuous differentiable from the first order till the r-th order with $y_{\mathrm{d}} \equiv 0$ for $\forall t \leqslant t_i$ and $\forall t \geqslant t_{\mathrm{f}}$.

The system state is decomposed as the outer state $\boldsymbol{\xi}(t)$ and the inner state $\boldsymbol{\eta}(t)$ namely the internal dynamic by introducing an invertible matrix T_1 as follows:

$$\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} = T_1\boldsymbol{x}(t) \tag{3}$$

and

$$\hat{A} = T_1 A T_1^{-1} = \begin{bmatrix}\hat{A}_1 & \hat{A}_2\\ \hat{A}_3 & \hat{A}_4\end{bmatrix},$$

$$\hat{\boldsymbol{B}} = T_1\boldsymbol{B} = \begin{bmatrix}\hat{\boldsymbol{B}}_1\\ \hat{\boldsymbol{B}}_2\end{bmatrix},\ \hat{\boldsymbol{C}} = \boldsymbol{C}T_1^{-1}.$$

Substituting Eq.(3) to Eq.(1), We get

$$\dot{\boldsymbol{\eta}}(t) = \hat{A}_3\boldsymbol{\xi}(t) + \hat{A}_4\boldsymbol{\eta}(t) + \hat{\boldsymbol{B}}_2 u(t). \tag{4}$$

Differentiating the output till the input is appeared

$$\frac{\mathrm{d}^r y(t)}{\mathrm{d}t^r} = CA^r\boldsymbol{x}(t) + CA^{r-1}\boldsymbol{B}u(t) = A_{\mathrm{x}}\boldsymbol{x}(t) + B_{\mathrm{y}}u(t), \tag{5}$$

where $B_{\mathrm{y}} = \boldsymbol{C}A^{r-1}\boldsymbol{B}$, it is invertible because of the well-defined relative degree assumption, the input can be rewritten as

$$u(t) = B_{\mathrm{y}}^{-1}\left(\frac{\mathrm{d}^r y(t)}{\mathrm{d}t^r} - A_{\mathrm{x}}\boldsymbol{x}(t)\right). \tag{6}$$

Substituting Eq.(3) into Eq.(6), we get

$$u(t) = B_{\mathrm{y}}^{-1}[y^{(r)}(t) - A_{\xi}\boldsymbol{\xi}(t) - A_{\eta}\boldsymbol{\eta}(t)], \tag{7}$$

where $\begin{bmatrix}A_{\xi} & A_{\eta}\end{bmatrix} = A_{\mathrm{x}}T_1^{-1}$.

If the reference trajectory $y_{\mathrm{d}}(t)$ is given and exact tracking is maintained, $y(t) = y_{\mathrm{d}}(t)$. Then the outer state is written as

$$\boldsymbol{\xi}_{\mathrm{d}}(t) = [y_{\mathrm{d}}(t)\ \ \frac{\mathrm{d}y_{\mathrm{d}}(t)}{\mathrm{d}t}\ \ \cdots\ \ \frac{\mathrm{d}^{r-1}y_{\mathrm{d}}(t)}{\mathrm{d}t^{r-1}}]^{\mathrm{T}}. \tag{8}$$

If a bounded solution $\eta_{\mathrm{d}}(t)$ can be found then the exact tracing input can be written as

$$u_{\mathrm{d}}(t) = B_{\mathrm{y}}^{-1}[y_{\mathrm{d}}^{(r)}(t) - A_{\xi}\boldsymbol{\xi}_{\mathrm{d}}(t) - A_{\eta}\boldsymbol{\eta}_{\mathrm{d}}(t)]. \tag{9}$$

Then Eq.(4) becomes

$$\dot{\boldsymbol{\eta}}_{\mathrm{d}}(t) = \tilde{A}_{\eta}\boldsymbol{\eta}_{\mathrm{d}}(t) + \tilde{B}_{\eta}\boldsymbol{Y}_{\mathrm{d}}(t), \tag{10}$$

where

$$\tilde{A}_{\eta} = \hat{A}_4 - \hat{\boldsymbol{B}}_2 B_{\mathrm{y}}^{-1}A_{\eta},$$

$$\tilde{B}_{\eta} = [\hat{A}_3 - \hat{\boldsymbol{B}}_2 B_{\mathrm{y}}^{-1}A_{\xi}\ \ \hat{\boldsymbol{B}}_2 B_{\mathrm{y}}^{-1}],$$

$$\boldsymbol{Y}_{\mathrm{d}}(t) = [\boldsymbol{\xi}_{\mathrm{d}}(t)\ \ y_{\mathrm{d}}^{(r)}(t)]^{\mathrm{T}}. \tag{11}$$

The system is restricted to be hyperbolic, which means none of the zeros of the system (1) lies on the imaginary axis of the complex plane. Then there exits a transformation T_2 that the inner state equation can be decomposed into the stable inner state equation and the unstable inner state equation.

$$\begin{bmatrix}\dot{\boldsymbol{\eta}}_{\mathrm{ds}}(t)\\ \dot{\boldsymbol{\eta}}_{\mathrm{du}}(t)\end{bmatrix} = \begin{bmatrix}\tilde{A}_{\eta\mathrm{s}} & 0\\ 0 & \tilde{A}_{\eta\mathrm{u}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}_{\mathrm{ds}}(t)\\ \boldsymbol{\eta}_{\mathrm{du}}(t)\end{bmatrix} + \begin{bmatrix}\widetilde{B}_{\eta\mathrm{s}}\\ \widetilde{B}_{\eta\mathrm{u}}\end{bmatrix}\boldsymbol{Y}_{\mathrm{d}}(t), \tag{12}$$

where

$$\begin{bmatrix}\boldsymbol{\eta}_{\mathrm{ds}}(t)\\ \boldsymbol{\eta}_{\mathrm{du}}(t)\end{bmatrix} = T_2\boldsymbol{\eta}_{\mathrm{d}}(t),\ T_2\widetilde{B}_{\eta} = \begin{bmatrix}\widetilde{B}_{\eta\mathrm{s}}\\ \widetilde{B}_{\eta\mathrm{u}}\end{bmatrix},$$

$$T_2\tilde{A}_{\eta}T_2^{-1} = \begin{bmatrix}\tilde{A}_{\eta\mathrm{s}} & 0\\ 0 & \tilde{A}_{\eta\mathrm{u}}\end{bmatrix}.$$

The stable inner state equation is obtained from Eq.(12) as

$$\dot{\boldsymbol{\eta}}_{\mathrm{ds}}(t) = \tilde{A}_{\eta\mathrm{s}}\boldsymbol{\eta}_{\mathrm{ds}}(t) + \tilde{B}_{\eta\mathrm{s}}\boldsymbol{Y}_{\mathrm{d}}(t). \tag{13}$$

Its result is

$$\boldsymbol{\eta}_{\mathrm{ds}}(t) = \int_{-\infty}^{t} \mathrm{e}^{\tilde{A}_{\eta\mathrm{s}}(t-\tau)}\tilde{B}_{\eta\mathrm{s}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau. \tag{14}$$

In the same way, the unstable inner state equation is obtained from Eq.(12) as

$$\dot{\boldsymbol{\eta}}_{\mathrm{du}}(t) = \tilde{A}_{\eta\mathrm{u}}\boldsymbol{\eta}_{\mathrm{du}}(t) + \tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(t). \tag{15}$$

Since $\tilde{A}_{\eta\mathrm{u}}$ has unstable eigenvalues, a divergent solution will be obtained by forward causal integration, while a convergent solution can be obtained by backward non-causal integration. Its solution is

$$\boldsymbol{\eta}_{\mathrm{du}}(t) = -\int_{t}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau. \tag{16}$$

Remark 1 As to the stable inner state, the result by forward integration is bounded while divergent by backward integration. As to the unstable inner state, the result by forward integration is divergent while bounded by backward integration.

3 Stable initial state

Definition 4 The actual physical system runs from time $t = t_0(t_0 < t_i)$, so t_0 is the initial point and t_i is defined in Definition 3. If the system state $\boldsymbol{x}(t)$ is equal to $\boldsymbol{x}_{\mathrm{d}}(t)$ for $t \geqslant t_i$ from Definition 1, the system state at t_i is defined as Stable Initial State.

Stable inversion is a two point boundary value problem[12], the system state is a zero vector when time tends to positive or negative infinity. In the ideal situation, the input always exists when $t \in (-\infty, \infty)$, while in the actual situation, the input functions from t_0, which violates the definition of the stable inversion. However, when the system state at t_i is Stable Initial State, the desired tracking performance from Definition 1 will appear in the following time interval $[t_i, \infty)$.

Theorem 1 According to Definition 4, the system state $\boldsymbol{x}(t)$ equals to the desired state $\boldsymbol{x}_{\mathrm{d}}(t)$ for $t \geqslant t_i$, if and only if Stable Initial State is in the following form:

$$\boldsymbol{x}(t_i) = T_1^{-1}\begin{bmatrix} \mathbf{0}_{r\times 1} \\ T_2^{-1}\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\eta}_{\mathrm{u}}(t_i)\end{bmatrix}\end{bmatrix}_{n\times 1}, \tag{17}$$

$$\boldsymbol{\eta}_{\mathrm{u}}(t_i) = -\int_{t_i}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau.$$

Proof

Necessity The system state at time t_i from Eq.(3) is

$$\boldsymbol{x}(t_i) = T_1^{-1}\begin{bmatrix} \boldsymbol{\xi}(t_i)_{r\times 1} \\ \boldsymbol{\eta}(t_i)\end{bmatrix}_{n\times 1}. \tag{18}$$

The inner state can be decomposed as

$$\boldsymbol{\eta}(t_i) = T_2^{-1}\begin{bmatrix} \boldsymbol{\eta}_{\mathrm{s}}(t_i) \\ \boldsymbol{\eta}_{\mathrm{u}}(t_i)\end{bmatrix}. \tag{19}$$

Then the system state at time t_i can be synthesized as

$$\boldsymbol{x}(t_i) = T_1^{-1}\begin{bmatrix} \boldsymbol{\xi}(t_i)_{r\times 1} \\ T_2^{-1}\begin{bmatrix} \boldsymbol{\eta}_{\mathrm{s}}(t_i) \\ \boldsymbol{\eta}_{\mathrm{u}}(t_i)\end{bmatrix}\end{bmatrix}_{n\times 1}.$$

The components of the ideal state $\boldsymbol{x}_{\mathrm{d}}(t)$ can be seen from Eqs.(8)(14)(16), the outer state at time x_i is

$$\boldsymbol{\xi}_{\mathrm{d}}(t_i) = [y_{\mathrm{d}}(t_i)\ \ \frac{\mathrm{d}y_{\mathrm{d}}(t_i)}{\mathrm{d}t}\ \ \cdots\ \ \frac{\mathrm{d}^{\mathrm{r}-1}y_{\mathrm{d}}(t_i)}{\mathrm{d}t^{r-1}}]^{\mathrm{T}}. \tag{20}$$

The stable inner state at time x_i is

$$\boldsymbol{\eta}_{\mathrm{ds}}(t_i) = \int_{-\infty}^{t_i} \mathrm{e}^{\tilde{A}_{\eta\mathrm{s}}(t_i-\tau)}\tilde{B}_{\eta\mathrm{s}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau. \tag{21}$$

The start time point of the system (1) is t_0, we get $y_{\mathrm{d}}(t_i) = 0$ from Definition 3, then $\boldsymbol{\xi}_{\mathrm{d}}(t_i) = \mathbf{0}$ (seeing Eq. (20)). So we get $\boldsymbol{Y}_{\mathrm{d}}(t) = \mathbf{0}$ for $t \leqslant t_i$ from Eq. (11), then $\boldsymbol{\eta}_{\mathrm{ds}}(t_i) = \mathbf{0}$ (seeing Eq. (21)). In a word, the outer state and the stable inner state are all zero vectors.

The unstable inner state at time t_i is

$$\boldsymbol{\eta}_{\mathrm{du}}(t_i) = -\int_{t_i}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t_i)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau. \tag{22}$$

The outer state and inner state in the ideal situation are obtained in advance. As seen from the known conditions

$$\boldsymbol{\xi}(t_i) = \boldsymbol{\xi}_{\mathrm{d}}(t_i),\ \boldsymbol{\eta}_{\mathrm{s}}(t_i) = \boldsymbol{\eta}_{\mathrm{ds}}(t_i),\ \boldsymbol{\eta}_{\mathrm{u}}(t_i) = \boldsymbol{\eta}_{\mathrm{du}}(t_i),$$

for $t \geqslant t_i$. So Stable Initial State can be synthesized as

$$\boldsymbol{x}(t_i) = T_1^{-1}\begin{bmatrix} \mathbf{0}_{r\times 1} \\ T_2^{-1}\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\eta}_{\mathrm{u}}(t_i)\end{bmatrix}\end{bmatrix}_{n\times 1},$$

$$\boldsymbol{\eta}_{\mathrm{u}}(t_i) = -\int_{t_i}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t_i)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau.$$

The necessity condition has been proved.

Sufficiency As seen from the known conditions, we get the actual state components

$$\boldsymbol{x}(t_i) = T_1^{-1}\begin{bmatrix} \mathbf{0}_{r\times 1} \\ T_2^{-1}\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\eta}_{\mathrm{u}}(t_i)\end{bmatrix}\end{bmatrix}_{n\times 1},$$

$$\boldsymbol{\eta}_{\mathrm{u}}(t_i) = -\int_{t_i}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t_i)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau.$$

The desired state components are obtained from the above analysis in the necessity part of the proof. So we get $\boldsymbol{\xi}_{\mathrm{d}}(t_i) = \mathbf{0}$, $\boldsymbol{\eta}_{\mathrm{ds}}(t_i) = \mathbf{0}$. The calculation way of the outer state and the stable inner state is forward causal integration. Then we get $\boldsymbol{\xi}(t) = \boldsymbol{\xi}_{\mathrm{d}}(t)$ and $\boldsymbol{\eta}_{\mathrm{s}}(t) = \boldsymbol{\eta}_{\mathrm{ds}}(t)$ for $t \geqslant t_i$.

The unstable inner state in the actual situation can be obtained by forward integration.

$$\begin{aligned}
&\boldsymbol{\eta}_{\mathrm{u}}(t) = \\
&\mathrm{e}^{\tilde{A}_{\eta\mathrm{u}}(t-t_i)}\boldsymbol{\eta}(t_i) + \int_{t_i}^{t} \mathrm{e}^{\tilde{A}_{\eta\mathrm{u}}(t-\tau)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau = \\
&-\mathrm{e}^{\tilde{A}_{\eta\mathrm{u}}(t-t_i)}\int_{t_i}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t_i)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau + \\
&\int_{t_i}^{t} \mathrm{e}^{\tilde{A}_{\eta\mathrm{u}}(t-\tau)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau = \\
&-\int_{t}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau.
\end{aligned} \tag{23}$$

The unstable inner state in the ideal situation is obtained from Eq.(16) as

$$\boldsymbol{\eta}_{\mathrm{du}}(t_i) = -\int_{t_i}^{\infty} \mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t_i)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau.$$

Thus $\boldsymbol{\eta}_{\mathrm{u}}(t) = \boldsymbol{\eta}_{\mathrm{du}}(t)$, system state $\boldsymbol{x}(t)$ in the actual situation is the same as $\boldsymbol{x}_{\mathrm{d}}(t)$ in the ideal situation for $t \geqslant t_i$. The sufficiency has been proved.

Remark 2 Stable Initial State is defined in the actual situation. The specific form is given in Theorem 1, and the proven process displays the calculation method.

4 Pre-action

Definition 5 The system dynamic process before time t_i is defined as the pre-action process and t_i is the beginning point of the original reference trajectory from Definition 3. The pre-action time interval is $(-\infty, t_i]$ in the ideal situation but $[t_0, t_i]$ in practice.

In the ideal situation from Definition 1, when input functions from time negative infinity to time t_0. Then the system state changes from zero to the specified state at the same time. That's the pre-action process, which is a necessary procedure since the unstable internal dynamic is computed by backward time integration.

But in the actual situation, pre-action time is finite. Because the input of pre-action decays over time and becomes zero when time decreases to negative infinity in the limiting case. So the pre-action process can be truncated at time t_0 when the input signal is sufficient small.

Owning to the non-causality, pre-action is a necessary process which is done by non-causal stable inversion technique.

Pre-action process is the side-effect of the stable inversion, its time interval is roughly judged by method of trial according to Theorem 2.

Theorem 2 Let all the eigenvalues of coefficient matrix $\tilde{A}_{\eta\mathrm{u}}$ extracted from Eq.(15) lie to the right of the line $\mathrm{Re}(s)=\alpha$ in the complex plane for positive α. Let $\|\widetilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\cdot)\|<\beta$ and there exists matrix M to satisfy the formula $\|u_{\mathrm{d}}(t)\|_{\infty}<M\mathrm{e}^{\alpha(t-t_i)}$ for $\forall t\leqslant t_i$.

Proof Positive constant matrix M_{u} is existed according to the conditions of Theorem 2 to satisfy[8]

$$\|\mathrm{e}^{\tilde{A}_{\eta\mathrm{u}}(t-\tau)}\|_{\infty}<M_{\mathrm{u}}\mathrm{e}^{\alpha(t-\tau)},\ \forall t<\tau. \tag{24}$$

From Definition 3, $y_{\mathrm{d}}^{(r)}(t)\equiv\mathbf{0},\boldsymbol{\xi}_{\mathrm{d}}\equiv\mathbf{0},\boldsymbol{\eta}_{\mathrm{ds}}(t)\equiv\mathbf{0}$ for $\forall t\leqslant t_i$ can be obtained. Thus the input $u_{\mathrm{d}}(t)$ is only related to the unstable inner state $\boldsymbol{\eta}_{\mathrm{du}}(t)$. Let $A_{\eta}T_2^{-1}=\left[A_{\eta\mathrm{s}}\ A_{\eta\mathrm{u}}\right]$ and apply Eq.(9) and Eq.(16), when $t\leqslant t_i$ we get

$$\begin{aligned}
&\|u_{\mathrm{d}}(t)\|_{\infty}=\\
&\|B_{\mathrm{y}}^{-1}[y_{\mathrm{d}}^{(r)}(t)-A_{\xi}\boldsymbol{\xi}_{\mathrm{d}}(t)-A_{\eta}\boldsymbol{\eta}_{\mathrm{d}}(t)]\|_{\infty}=\\
&\|B_{\mathrm{y}}^{-1}[y_{\mathrm{d}}^{(r)}(t)-A_{\xi}\boldsymbol{\xi}_{\mathrm{d}}(t)-A_{\eta\mathrm{s}}\boldsymbol{\eta}_{\mathrm{ds}}(t)-\\
&A_{\eta\mathrm{u}}\boldsymbol{\eta}_{\mathrm{du}}(t)]\|_{\infty}=\\
&\|B_{\mathrm{y}}^{-1}A_{\eta\mathrm{u}}\boldsymbol{\eta}_{\mathrm{du}}(t)\|_{\infty}\leqslant\\
&\|B_{\mathrm{y}}^{-1}\|_{\infty}\|A_{\eta\mathrm{u}}\|_{\infty}\times\\
&\|\int_t^{+\infty}\mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau\|_{\infty}<\\
&\beta M_{\mathrm{u}}\|B_{\mathrm{y}}^{-1}\|_{\infty}\|A_{\eta\mathrm{u}}\|_{\infty}\int_{t_i}^{+\infty}\mathrm{e}^{\alpha(t-\tau)}d\tau=\\
&\frac{\beta M_{\mathrm{u}}}{\alpha}\|B_{\mathrm{y}}^{-1}\|_{\infty}\|A_{\eta\mathrm{u}}\|_{\infty}\mathrm{e}^{\alpha(t-t_i)}:=\\
&M\mathrm{e}^{\alpha(t-t_i)}.
\end{aligned} \tag{25}$$

The proof is done as we let

$$M=\frac{\beta M_{\mathrm{u}}}{\alpha}\|B_{\mathrm{y}}^{-1}\|_{\infty}\|A_{\eta\mathrm{u}}\|_{\infty}.$$

Let ε be a sufficient small positive real number, and $\|u_{\mathrm{d}}(t)\|_{\infty}<M\mathrm{e}^{\alpha(t-t_i)}<\varepsilon$.

Then the time interval of pre-action $\triangle t=t_i-t$ can be obtained by method of trial.

Generally, the initial state $\boldsymbol{x}(t_0)$ is zero. The time interval of the pre-action process will be larger if $\boldsymbol{x}(t_0)$ is nonzero. Because the system state error will converge to zero by a large section of time as the coefficient matrix A is Hurwitz. Unfortunately, neither $\boldsymbol{x}(t_0)$ is zero nor nonzero, bad tracking effect will appear when the time interval of pre-action is not large enough. Because large system state error at t_0 will seriously influence the tracking effect in the following trajectory.

5 Optimal state to state transition

As to pre-action, the reference trajectory has been extended from $[t_i,t_{\mathrm{f}}]$ to $(-\infty,+\infty)$ along the real axis in the complex plane according to Definition 3. The time interval of pre-action needs to be chosen large enough, otherwise, bad tracking effect will appear in $[t_i,t_{\mathrm{f}}]$. Furthermore, the time interval can not be chosen flexibly. Those are two disadvantages existed in the classical stable inversion method.

In order to overcome the two defects, OST based stable inversion technique is proposed here. In Section 3, the proposed Stable Initial State is an important condition to ensure precise tracking in the following trajectory. In this Section, We will use the optimal state to state transition technique to transit the initial state $\boldsymbol{x}(t_0)$ to Stable Initial State defined at t_i. Then precise tracking in $[t_i,t_{\mathrm{f}}]$ can be obtained. Furthermore, the time interval of the pre-action process can be designed flexibly.

Definition 6 Under the condition of finite time interval, bounded input u_{ref} and state $\boldsymbol{x}_{\mathrm{ref}}$ are obtained to transit system state from $\boldsymbol{x}(t_1)$ to $\boldsymbol{x}(t_2)$ with minimum input energy. The problem above is called the optimal state to state transition problem.

The input energy function transiting $\boldsymbol{x}(t_1)$ to $\boldsymbol{x}(t_2)$ is defined as

$$J(t_1,t_2,u)=\int_{t_1}^{t_2}u^{\mathrm{T}}Ru\mathrm{d}t. \tag{26}$$

The solution of Eq.(26) is solved in paper[17] as

$$\begin{aligned}
&u(t)=\\
&R^{-1}\boldsymbol{B}^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}(t_2-t)}G_{(t_1,t_2)}^{-1}[\boldsymbol{x}(t_2)-\mathrm{e}^{A(t_2-t_1)}\boldsymbol{x}(t_1)].
\end{aligned} \tag{27}$$

R is an positive definite symmetric real matrix. A and B are given parameters. $G_{(t_1,t_2)}$ is a controllable and invertible Gramian matrix.

$$G_{(t_1,t_2)}=\int_{t_1}^{t_2}\mathrm{e}^{A(t_2-\tau)}\boldsymbol{B}R^{-1}\boldsymbol{B}^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}(t_2-\tau)}\mathrm{d}\tau. \tag{28}$$

6 Simulation example

Consider the system (1) with the coefficient matrixes[15] chosen as

$$A=\begin{bmatrix}0&1&0&0\\-3.656&-0.436&3.573&-0.091\\0&0&0&1\\3.245&-0.126&-3.259&-0.076\end{bmatrix},$$

$$\boldsymbol{B}=\begin{bmatrix}0\\21.9027\\0\\3.588\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}0&1&0&1\end{bmatrix},\ D=0.$$

The original reference trajectory[11] is chosen as

$$y=0.1t^2(6-t)\sin(0.5\pi t),\ \text{for } t\in[0,6]. \tag{29}$$

6.1 Calculate stable initial state

According to Theorem 1, apart from $\boldsymbol{\eta}_{\mathrm{u}}(t_i)$, the other parts of the system state are zero vectors, so Stable Initial State is seen as follows:

$$\boldsymbol{x}(t_i)=T_1^{-1}\begin{bmatrix}\boldsymbol{0}_{r\times1}\\T_2^{-1}\begin{bmatrix}\boldsymbol{0}\\\boldsymbol{\eta}_{\mathrm{u}}(t_i)\end{bmatrix}\end{bmatrix}_{n\times1}, \tag{30}$$

$$\boldsymbol{\eta}_{\mathrm{u}}(t_i)=-\int_{t_i}^{\infty}\mathrm{e}^{-\tilde{A}_{\eta\mathrm{u}}(\tau-t)}\tilde{B}_{\eta\mathrm{u}}\boldsymbol{Y}_{\mathrm{d}}(\tau)\mathrm{d}\tau. \tag{31}$$

As seen from above equations, only three conditions including coefficient matrixes, reference trajectory and time t_i are needed to calculate Stable Initial State. Let the initial time $t_0=-2$. Y_{d} defined in Eq.(11) is obtained from Eq.(29). Then we can get the key coefficient matrixes in Eq.(15) as

$$\tilde{A}_{\eta\mathrm{u}}=\begin{bmatrix}0&1.0000\\-23.4649&0.3332\end{bmatrix},$$

$$\widetilde{B}_{\eta\mathrm{u}}=\begin{bmatrix}0&0&0\\23.4673&0.3729&6.1044\end{bmatrix}.$$

Then we can get the Stable Initial State from Eqs.(30)–(31) as follows:

$$\eta_{\mathrm{u}}(t_i)=\begin{bmatrix}0.8015\\6.4655\end{bmatrix},$$

$$\boldsymbol{x}(t_i)=T_1^{-1}\begin{bmatrix}\boldsymbol{0}_{r\times1}\\T_2^{-1}\begin{bmatrix}0\\0.8015\\6.4655\end{bmatrix}\end{bmatrix}_{n\times1}. \tag{32}$$

Eq.(32) is the Stable Initial State.

6.2 Simulation under zero and nonzero initial state

As seen from Figs.1–2, the solid line denotes the actual output trajectory starting from $t=-2$ and the dashed line denotes the reference trajectory. In the time interval $[-2,0]$, the optimal state to state transition technique functions well transiting the initial state at t_0 to Stable Initial State at t_i. In the time interval $[0,6]$, we get the desirable output trajectory.

Figure 1 denotes the output tracking at zero initial state $\boldsymbol{x}(t_0)=\begin{bmatrix}0&0&0&0\end{bmatrix}^{\mathrm{T}}$ while Fig.2 denotes the output tracking at a nonzero initial state

$$\boldsymbol{x}(t_0)=\begin{bmatrix}1.5&1.2&1.1&0.3\end{bmatrix}^{\mathrm{T}}.$$

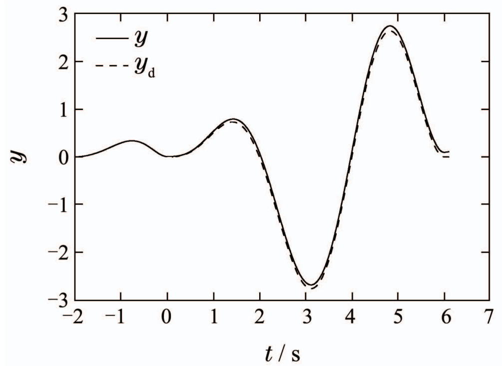


Fig. 1 Output tracking results of the proposed method with zero initial state case

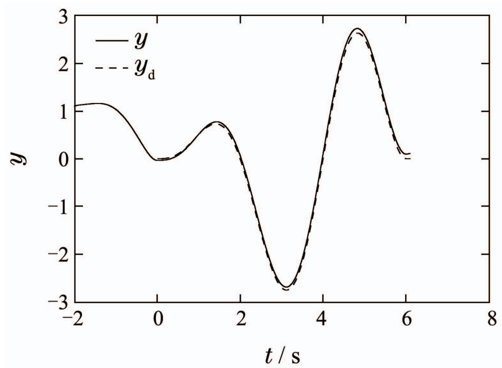


Fig. 2 Output tracking results of the proposed method with nonzero initial state case

Figure 3 denotes the input signal which is discontinuous at $t_0=0$. In the time interval $[-2,0]$, we get the minimum input energy and it is the optimal state to state transition process. But in the time interval $[0,6]$, we get the desired input by non-causal stable inversion technique.

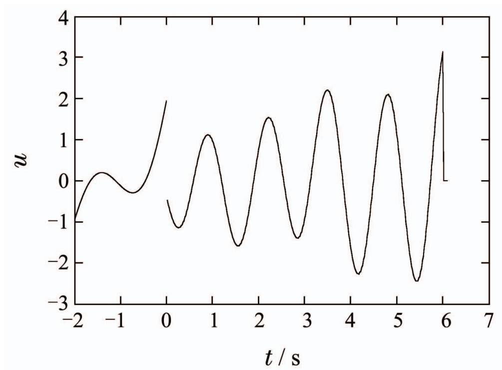


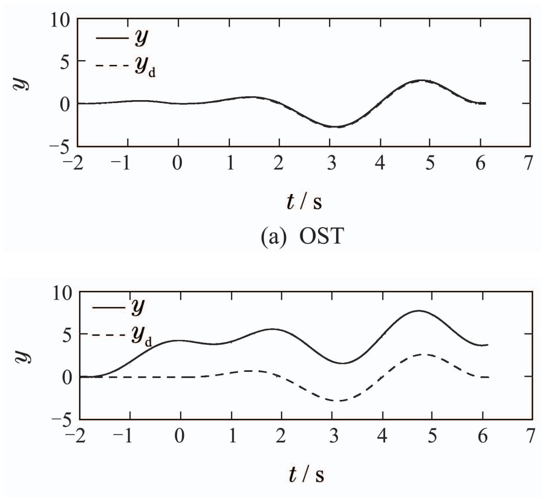
Fig. 3 Input trajectory of the proposed method with zero initial state

6.3 Comparison

In this section, the classical stable inversion method is compared with the proposed stable inversion method. The classical method includes the pre-action process displaying in the bottom parts of the following figures leading to approximate tracking and the proposed method includes the OST technique showing in the top parts of the following figures. The solid line denotes the actual output trajectory while the dashed line denotes the desired output trajectory. Pre-action or OST method functions before $t_0 = 0$ while non-causal stable inversion functions in the time interval $[0, 6]$.

6.3.1 Comparison under uniform extended time interval

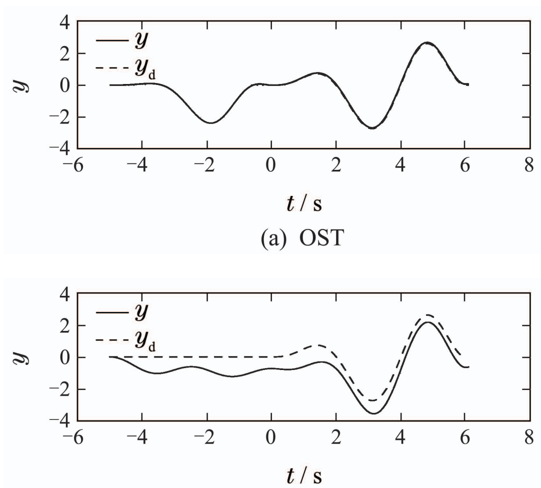
As seen from the bottom parts of Figs.4–6, the tracking effect will be better and better when the extended time becomes larger and larger. In Fig.4, the tracking effect is deteriorative when the extended time is 2 s, but grows better as the extended time grows. However, precise tracking in $[0, 6]$ failed owning to the fact that the extended time is finite rather than infinity.



(a) OST

(b) Pre-action

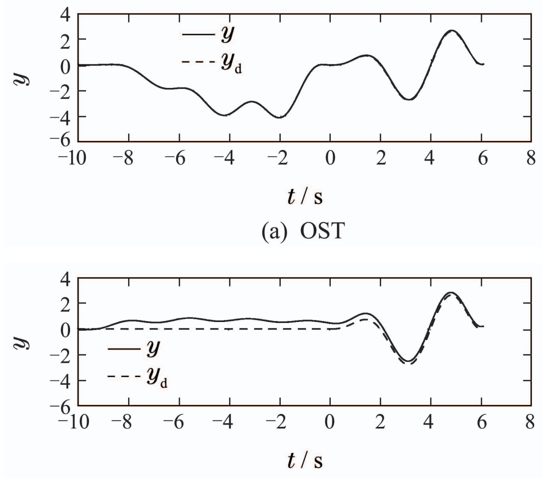
Fig. 4 Comparison of the two methods about output tracking when the extended time is 2 s



(a) OST

(b) Pre-action

Fig. 5 Comparison of the two methods about output tracking when the extended time is 5 s



(a) OST

(b) Pre-action

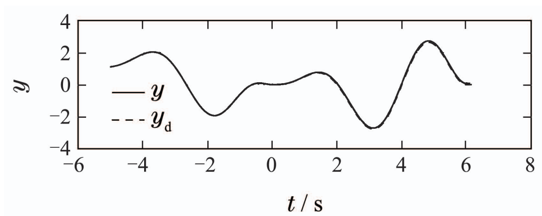
Fig. 6 Comparison of the two methods about output tracking when the extended time is 10 s

In the top parts of Figs.4–6, however, precise tracking in $[0, 6]$ is obtained regardless of the extended time. Furthermore, the output frequency and amplitude fluctuate smaller and smaller before $t_0 = 0$ when the extended time named as the transition time becomes shorter and shorter. As seen from the top parts of the above figures, little frequency and amplitude fluctuation appear with a appropriate transition time like Fig.4.

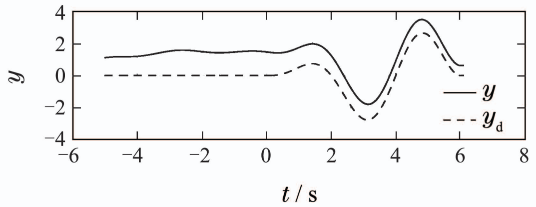
In a word, when considering the time interval $[0, 6]$, OST based stable inversion always gets the precise tracking effect but stable inversion with pre-action needs a large enough extended time. What's more, OST based stable inversion gets a better overall tracking effect defined in the whole time interval when the extended time is short to some extent.

In Fig.7, the comparison between the two methods when the initial state is nonzero is considered. Regardless of the initial state and extended time, precise tracking is always obtained in $[0, 6]$. But the classical method gets a worse tracking effect in $[0, 6]$ in Fig.5 and Fig.7.

Further more, OST based stable inversion also gets better overall tracking performance compared with the classical method when the overall time interval is $[-5, 6]$. And the overall tracking performance is defined by a norm of tracking error.



(a) OST



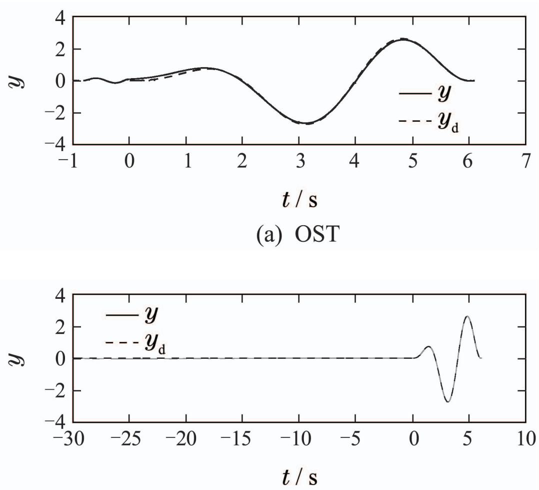
(b) Pre-action

Fig. 7 Comparison of the two methods about output tracking under nonzero initial state when the extended time is 5 s

6.3.2 Comparison under uniform tracking error interval

In this section, we'll compare the extended time of the two methods with the same tracking error interval. The reference trajectory before $t = 0$ is extended as zero output trajectory. So the whole tracking error is the absolute value of $(y - y_{\mathrm{d}})$.

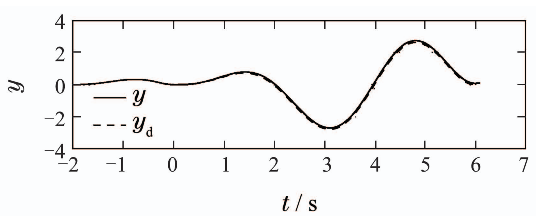
As seen from Figs.8–10 and Table 1, OST based stable inversion has a shorter extended time under uniform tracking error interval. When the whole tracking error limits to the interval $(22, 24)$, OST based stable inversion needs only 0.8 s extended time while the classical method needs 30 s extended time. A shorter extended time under condition of uniform tracking performance is obtained.



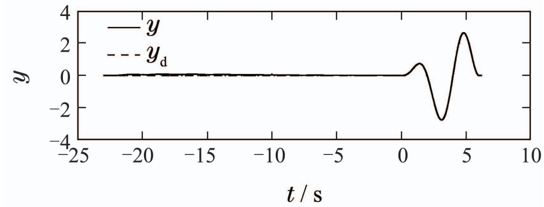
(a) OST

(b) Pre-action

Fig. 8 Comparison of the extended time between the proposed method with error of 22.7003 and time of 0.8 s and the classical method with error of 23.9856 and time of 30 s

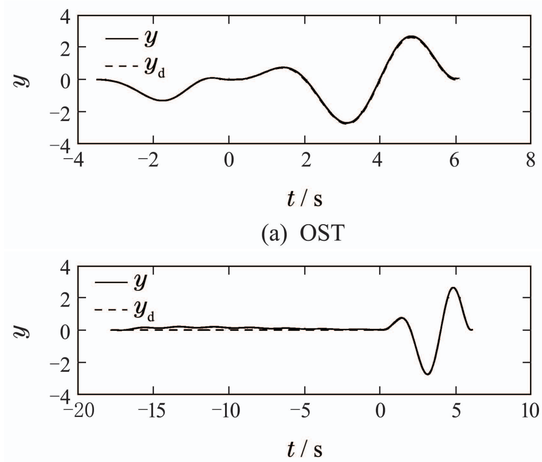


(a) OST



(b) Pre-action

Fig. 9 Comparison of the extended time between the proposed method with error of 39.8824 and time of 2 s and the classical method with error of 39.7489 and time of 23 s



(a) OST

(b) Pre-action

Fig. 10 Comparison of the extended time between the proposed method with error of 105.0993 and time of 3.5 s and the classical method with error of 100.4227 and time of 17.8 s

Table 1 Extended time under uniform tracking error

method	error interval		
	$(22, 24)$	$(39, 40)$	$(100, 106)$
OST	0.8 s	2 s	3.5 s
Pre-action	30 s	23 s	17.8 s

7 Conclusions

The proposed OST based non-causal stable inversion method can enable precise tracking come true under condition of an arbitrary extended time or an arbitrary initial state in the actual situation. The pre-action based classical stable inversion achieves precise tracking in the ideal situation and achieves desirable tracking in the actual situation with an appropriate extended time. However, the proposed method can cut down the extended time with the same overall tracking error or tracking performance.

References:

[1] BRÜLS O, BASTOS G J, SEIFRIED R. A stable inversion method for feedforward control of constrained flexible multibody systems [J]. *Journal of Computational and Nonlinear Dynamics*, 2014, 9(1): 1 – 9.

[2] ZOU Q, DEVASIA S. Precision preview-based stable-inversion for nonlinear nonminimum-phase systems: the VTOL example [J]. *Automatic*, 2007, 43(1): 117 – 127.

[3] DING J, MARCASSA F, TOMIZUKA M. Short seeking control with minimum jerk trajectories for dual actuator hard disk drive systems [C] //*Proceedings of American Control Conference*. Boston: ACC, 2004: 529 – 534.

[4] BENOSMAN M, VEY G L. Stable inversion of SISO nonminimum phase linear systems through output planning: an experimental application to the one-link flexible manipulator [J]. *IEEE Transactions on Control Systems Technology*, 2003, 11(4): 588 – 597.

[5] DEWEY J S, LEANG K, DEVASIA S. Experimental and theoretical results in output-trajectory redesign for flexible structures [J]. *Journal of Dynamic Systems, Measurement, and Control*, 1998, 12(4): 456 – 461.

[6] DEVASIA S, CHEN D, PADEN B. Nonlinear inversion-based output tracking [J]. *IEEE Transactions on Automatic Control*, 1996, 41(7): 930 – 942.

[7] CHEN D, PADEN B. Stable inversion of nonlinear non-minimum phase systems [J]. *International Journal of Control*, 1996, 64(1): 81 – 97.

[8] DEVASIA S. Output tracking with nonhyperbolic and near nonhyperbolic internal dynamics: Helicopter hover control [J]. *Journal of Guidance, Control and Dynamics*, 1997, 20(3): 573 – 580.

[9] DEVASIA S, PADEN B. Exact output tracking for nonlinear time-varying systems [C] //*Proceedings of the 33rd Conference on Decision and Control*. Lake Buena Vista: CDC, 1994: 2346 – 2355.

[10] SOGO T. On the equivalence between stable inversion for nonminimum phase systems and reciprocal transfer functions defined by the two-sided Laplace transform [J]. *Automatica*, 2010, 46(1): 122 – 126.

[11] LIU Shan, LIU Jie. Extended Laguerre basis function based iterative learning control for non-minimum phase systems [J]. *Control Theory & Applications*, 2012, 29(8): 985 – 992.
(刘山, 刘杰. 非最小相位系统的扩展Laguerre基函数迭代学习控制 [J]. 控制理论与应用, 2012, 29(8): 985 – 992.)

[12] TAYLOR D G, LI S. Stable inversion of continuous-time nonlinear systems by finite-difference methods [J]. *IEEE Transactions on Automatic Control*, 2002, 47(3): 537 – 542.

[13] HUNT L R, MEYER G. Stable inversion for nonlinear systems [J]. *Automatica*, 1997, 33(8): 1549 – 1554.

[14] LIU Shan, WU Tiejun. Stable inversion based iterative learning control for non-minimum phase plants [J]. *Control Theory & Applications*, 2003, 20(6): 831 – 837.
(刘山, 吴铁军. 基于稳定逆的非最小相位系统的迭代学习控制 [J]. 控制理论与应用, 2003, 20(6): 831 – 837.)

[15] ZOU Q, DEVASIA S. Preview-based stable inversion for output tracking of linear systems [J]. *Journal of Dynamic Systems, Measurement, and Control*, 1999, 121(4): 625 – 630.

[16] ZOU Q. Optimal preview-based stable-inversion for output tracking of nominimum-phase linear systems [J]. *Automatica*, 2009, 45(1): 230 – 237.

[17] LEWIS F L, VRABIE D L, SYRMOS V L. *Optimal Control* [M]. 3rd Edition. New Jersey: John Wiley & Sons, Inc., 2012: 138 – 140.

[18] IAMRATANAKUL D, JORDAN B, LEANG K K, et al. Optimal output transitions for dual-stage systems [J]. *IEEE Transactions on Control Systems Technology*, 2008, 16(5): 869 – 881.

[19] DEVASIA S. Time-optimal control with pre/post actuation for dual-stage systems [J]. *IEEE Transactions on Control Systems Technology*, 2012, 20(2): 323 – 334.

[20] DEVASIA S. Nonlinear minimum-time control with pre- and post-actuation [J]. *Automatica*, 2011, 47(7): 1379 – 1387.

[21] IAMRATANAKUL D, PEREZ H, DEVASIA S. Feedforward trajectory design for output transitions in discrete-time systems: disk-drive example [C] //*Proceedings of American Control Conference*. Denver: ACC, 2003: 3142 – 3147.

[22] WANG H M, ZOU Q Z, XU H B. Inversion-based optimal output tracking-transition switching with preview for nonmimimum-phase linear systems [J]. *Automatica*, 2012, 48(7): 1364 – 1371.

[23] ZHANG Li, LIU Shan. Basis function based adaptive iterative learning control for non-minimum phase systems [J]. *Acta Automatic Sinica*, 2014, 40(12): 2716 – 2725.
(张黎, 刘山. 非最小相位系统的基函数型自适应迭代学习控制 [J]. 自动化学报, 2014, 40(12): 2716 – 2725.)

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