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周期时变布尔网络的完全同步化

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摘要: 本文主要研究了驱动–响应结构下的布尔网络的完全同步化, 其中驱动系统是一个周期性时变的布尔网络. 对于上述问题, 本文基于逻辑系统的代数形式下分两种情况讨论. 对于每种情况, 都将给出一个完全同步的充要条件. 相应地, 提出了两个响应布尔网络的同步化方案. 最后, 通过一些数值例子来说明本文结果的有效性.

关键词: 布尔网络; 周期性时变布尔网络; 完全同步

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Complete synchronization of the periodically time-variant Boolean networks

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Abstract: This paper mainly investigates the complete synchronization of two Boolean networks (BNs) coupled unidirectionally in the drive-response configuration, where the drive BN is periodically time-variant. We discuss it in two cases under the algebraic framework of logical systems. For each case, a necessary and sufficient criterion for complete synchronization is presented. Accordingly, two general design approaches to a synchronizing response BN are developed. Finally, numerical examples are given in order to illustrate the effectiveness of our results.

Key words: Boolean network (BN); periodically time-variant Boolean network (PTVBN); complete synchronization

1 Introduction

Boolean network was first introduced by Kauffman in 1969. It has been widely used in modeling complex systems such as gene regulatory networks, biological evolution models and neural networks. At each discrete time point, every node in a BN can take one of two binary values, 1 or 0, where 1 means that the node is active and 0 inactive. Every node updates state on the basis of a logical relationship with the form of a Boolean function. Although the models of BNs are very simple, they can describe basic dynamic behavior of many real systems and provide useful information. Therefore, lots of issues about BNs have been investigated such as stability, controllability, optimal control and so on[1–6].

As we know, the synchronization problem of systems has been attracting many researchers from various fields, see [7–9]. The complete synchronization of BNs which can be equivalently transformed into the linear discrete-time systems by utilizing the semi-tensor product (STP)[10] also caused a lot of attention and many results were provided. For example, a necessary and sufficient criterion for complete synchronization of BNs was provided in [11], a general approach for the design of a response BN was given in [12] and the synchronization problem of BNs with some special cases such as different update, time-delay and output-coupled were investigated in [13–16], respectively, and so on.

Most of researchers just considered the situation that the BNs were time-invariant Boolean networks (TIBNs), while the synchronization problem of time-variant Boolean networks (TVBNs) was rarely studied. TVBNs have many essential properties different from TIBNs. As a special case of TVBNs, periodically time-variant Boolean networks (PTVBNs) can be used to describe many systems such as the switched BNs which have periodical switching signals[17], the Boolean con-

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trol networks (BCNs) which have dynamical controllers[18], the perturbed BNs which have periodical function perturbations[19], and so on. Therefore, the study of synchronization of PTVBNs is significant. Although the problem of PTVBNs have been studied in few papers[20–21], the problem of complete synchronization of PTVBNs has not been solved well. This is the motivation of our paper.

In this paper, we investigate a special case of the complete synchronization problem for two BNs that the drive BN is periodically time-variant. At first, we study the cycles of PTVBNs and provide some special properties which never exist in the cycles of TIBNs. Then, according to these properties, the PTVBNs are divided into two categories. After that, the synchronization problems are solved for both two categories by utilizing STP and the results are turned out to be quite different from each other. At last, two examples for both categories are provided to support our points. The major contributions of this paper: 1) provide two necessary and sufficient criteria for complete synchronization of PTVBNs; 2) make two design approaches to a synchronizing response BN, which can complete synchronize with the drive BN in finite steps.

This paper is organized as follows. Problem formulation and some preliminaries are given in Section 2. In Section 3, the periodically time-variant drive Boolean network is discussed in two cases and some criteria for complete synchronization and some general approaches for the design of a synchronizing response BN are provided. Two numerical examples are given in Section 4 and a conclusion is drawn in Section 5.

2 Problem formulation and preliminaries

First, we give some notations.

1) $\mathbb{N}$ represents the set of non-negative integers, $\mathbb{N}^+$ represents the set of positive integers and $\mathbb{N}_{[\lambda_1,\lambda_2]}$ represents the set of integers from λ_1 to λ_2 which means that $\mathbb{N}_{[\lambda_1,\lambda_2]}=\{\lambda_1,\lambda_1+1,\cdots,\lambda_2\}$.

2) $I_n\in\mathbb{R}_{n\times n}$ represents the identity matrix, δ_n^i represents the i-th column of I_n.

3) $\varDelta_n$ represents the set of δ_n^i, where $i=1,2,\cdots,n$.

4) $\varepsilon(x)$ represents a bijection from $\varDelta_{2^n}$ to $\mathbb{N}_{[1,2^n]}$ such that $x=\delta_{2^n}^{\varepsilon(x)}$.

5) $\mathrm{Col}(A)$ represents the set of all columns of A, where $A\in\mathbb{R}_{m\times n}$.

6) $B=[\delta_m^{i_1},\delta_m^{i_2},\cdots,\delta_m^{i_n}]$ is called a logical matrix and its shorthand form is $B=\delta_m[i_1,i_2,\cdots,i_n]$.

7) $\varPhi_n$ represents a logical matrix such that $\varPhi_n=\delta_{2^{2n}}[1,2^n+2,2\cdot2^n+3,\cdots,(2^n-2)2^n+2^n-1,2^{2n}]$.

Consider two BNs with n nodes respectively coupled unidirectionally in the drive-response configuration

$$x_i(t+1)=f_i^{\sigma(t)}(x_1(t),x_2(t),\cdots,x_n(t)),\quad(1)$$

$$y_i(t+1)=g_i^{\phi(t)}(x_1(t),x_2(t),\cdots,x_n(t),y_1(t),y_2(t),\cdots,y_n(t)),\quad(2)$$

where $\sigma(t)=t\%p+1$ is a periodic function with period p, $t\%p$ represents the remainder of t/p, $\phi(t)$ is a function from $\mathbb{N}$ to $\mathbb{N}^+$, the x_i and y_i represent the nodes of the drive BN and response BN, respectively, $f_i^{j_1}$ and $g_i^{j_2}$ represent the Boolean functions from $\{1,0\}^n$ to $\{1,0\}$ and from $\{1,0\}^{2n}$ to $\{1,0\}$ for every $i\in\mathbb{N}_{[1,n]}$, $j_1\in\mathbb{N}_{[1,p]}$ and $j_2\in\mathbb{N}^+$, respectively.

The main mathematical tool of this paper is the STP, which is a generation of conventional matrix product. Since most properties of the conventional matrix product remain true under STP, the product of this paper is usually considered as STP if there is no confusion. By [22], the Boolean values 1 and 0 can be equivalently transformed into the vectors $\delta_2^1=(1,0)^{\mathrm{T}}$ and $\delta_2^2=(0,1)^{\mathrm{T}}$, respectively. Take $x(t)=x_1(t)x_2(t)\cdots x_n(t)$ and $y(t)=y_1(t)y_2(t)\cdots y_n(t)$, then BNs (1) and (2) are equivalent to the following discrete-time systems:

$$x(t+1)=F_{\sigma(t)}x(t),\quad(3)$$

$$y(t+1)=G_{\phi(t)}x(t)y(t),\quad(4)$$

where $F_{\sigma(t)}\in L_{2^n\times2^n}$ and $G_{\phi(t)}\in L_{2^n\times2^{2n}}$ are the structure matrices of BNs (1) and (2), respectively. For convenience, let $x(t,x(0))$ represent the trajectory of the drive BN (1) with initial state $x(0)\in\{1,0\}^n$ and $y(t,x(0),y(0))$ be the trajectory of the response BN (2) under the drive trajectory $x(t,x(0))$ with the initial state $y(0)\in\{1,0\}^{2n}$.

Let the columns of $F_{\sigma(t)}$ and $G_{\phi(t)}$ be $\delta_{2^n}^{\alpha_1^{\sigma(t)}},\delta_{2^n}^{\alpha_2^{\sigma(t)}},\cdots,\delta_{2^n}^{\alpha_{2^n}^{\sigma(t)}}$ and $\delta_{2^n}^{\beta_1^{\phi(t)}},\delta_{2^n}^{\beta_2^{\phi(t)}},\cdots,\delta_{2^n}^{\beta_{2^{2n}}^{\phi(t)}}$, respectively, which means that

$$F_{\sigma(t)}=\delta_{2^n}[\alpha_1^{\sigma(t)}\ \ \alpha_2^{\sigma(t)}\ \ \cdots\ \ \alpha_{2^n}^{\sigma(t)}],\quad(5)$$

$$G_{\phi(t)}=\delta_{2^n}[\beta_1^{\phi(t)}\ \ \beta_2^{\phi(t)}\ \ \cdots\ \ \beta_{2^{2n}}^{\phi(t)}].\quad(6)$$

The complete synchronization of BNs (1) and (2) is defined as follows.

Definition 1 BNs (1) and (2) are completely synchronized if there is a positive integer k such that $y(t,x(0),y(0))=x(t,x(0))$ for all $x(0),y(0)\in\varDelta_{2^n}$ and any $t\geqslant k$.

Because the complete synchronization for BNs is closely related to the attractors (including the fixed point and cycles), we provide an introduction to the cycles of PTVBNs.

Definition 2[20] A state $x_0\in\varDelta_{2^n}$ is called a fixed point of BN (1) if $F_{\sigma(t)}x_0 = x_0$ for any $t \geqslant 0$. A sequence $\{x(0), x(1), \cdots, x(t), x(t+1), \cdots\}$ is called a cycle of BN (1) with length l if: 1) $x(t+l) = x(t)$ for any $t \geqslant 0$; 2) for any $0 < T < l$, there is a positive integer $\widetilde{t}$ such that $x(\widetilde{t}+T) \neq x(\widetilde{t})$.

In this paper, the periodic sequence $\{x(0), x(1), \cdots, x(t), x(t+1), \cdots\}$ which is a cycle of BN (1) with length l is denoted by

$$\overline{\{\tilde{x}(0), \tilde{x}(1), \cdots, \tilde{x}(l-1), \tilde{x}(l)\}},$$

where $\tilde{x}(l) = x(0)$ and $\tilde{x}(i) = x(i)$ for every $i \in N_{[0,l-1]}$. For any $t \geqslant 0$, we call $x(t)$ in this cycle if and only if $x(t) = \tilde{x}(t\%l)$.

For the cycles of PTVBNs, the following Lemma can be obtained.

Lemma 1 For BN (1), there is a positive integer k such that $x(k)$ is in one of the cycles of BN (1) for every $x(0) \in \Delta_{2^n}$ and any $t \geqslant k$.

Proof Take $F = F_{\sigma(p-1)}F_{\sigma(p-2)} \cdots F_{\sigma(0)}$, we have

$$x((t+1)p) = Fx(tp) \tag{7}$$

as a subsystem of BN (3). Because of the infiniteness of the state space, for any $x(0) \in \Delta_{2^n}$, there are positive integers k and l such that

$$\{x(t'p), x((t'+1)p), \cdots, x(t'+l)p\} \tag{8}$$

is one cycle of BN (7) for any $t' \geqslant k$. It can be found that

$$\begin{aligned}&\{\cdots, x(t'p), x(t'p+1), \cdots, x((t'+1)p),\\ &x((t'+1)p+1), x((t'+1)p+2), \cdots,\\ &x((t'+2)p), x((t'+2)p+1), \cdots,\\ &x((t'+l)p-1), x((t'+l)p), \cdots\}\end{aligned} \tag{9}$$

is a cycle of BN (3). So the proof is completed.

3 Main results

Suppose that BN (1) has s cycles represented as

$$\overline{\{\tilde{x}_i(0), \tilde{x}_i(1), \cdots, \tilde{x}_i(l_i)\}},\ i \in \mathbb{N}_{[1,s]}. \tag{10}$$

Accordingly, define a set

$$\begin{aligned}\mathcal{C} = \{x | x \in \Delta_{2^n},\ &\text{there exist } i \in \mathbb{N}_{[1,s]} \text{ and}\\ &j \in \mathbb{N}_{[0,l_i-1]} \text{ such that } \tilde{x}_i(j) = x\}.\end{aligned} \tag{11}$$

About the cycles of PTVBNs, two cases need to be introduced.

1) There is $x \in \mathcal{C}$ which appears several times in a period of a cycle. For example, consider

$$x(t+1) = F_{\sigma(t)}x(t) \tag{12}$$

a PTVBN with period 2 and $F_1 = \delta_2[1,2], F_2 = \delta_2[2,1]$. Then $\overline{\{\delta_2^1, \delta_2^1, \delta_2^2, \delta_2^2, \delta_2^1\}}$ is a cycle with length 4, it can be found that δ_2^1 and δ_2^2 both appear twice in a period of this cycle.

2) There is $x \in \mathcal{C}$ which is in more than one cycle. For example, let

$$x(t+1) = F_{\sigma(t)}x(t) \tag{13}$$

be a PTVBN with period 2 and $F_1 = \delta_4[2,3,1,2]$, $F_2 = \delta_4[2,4,3,1]$. Then δ_4^2 is in two cycles of this BN: $\overline{\{\delta_4^2, \delta_4^3, \delta_4^3, \delta_4^1, \delta_4^2\}}$ and $\overline{\{\delta_4^4, \delta_4^2, \delta_4^4\}}$.

Define a subset of $\mathcal{C}$ as follows:

$$\begin{aligned}\mathcal{C}' = \{\, x | x \in \mathcal{C}, &\text{there are } i_1, i_2 \in \mathbb{N}_{[1,s]},\\ &j_1 \in \mathbb{N}_{[0,l_{i_1}-1]} \text{ and } j_2 \in \mathbb{N}_{[0,l_{i_2}-1]},\\ &\text{such that } \tilde{x}_{i_1}(j_1) = \tilde{x}_{i_2}(j_2) = x \text{ and}\\ &\tilde{x}_{i_1}(j_1+1) \neq \tilde{x}_{i_2}(j_2+1)\}.\end{aligned} \tag{14}$$

It can be found that $\mathcal{C}' = \Delta_2$ in BN (12) and $\mathcal{C}' = \{\delta_4^2, \delta_4^3\}$ in BN (13), respectively. Therefore, $\mathcal{C}'$ is not always an empty set. So we divide complete synchronization problem for BNs (1) and (2) into two cases: 1) $\mathcal{C}' = \varnothing$; 2) $\mathcal{C}' \neq \varnothing$.

3.1 Case 1: $\mathcal{C}' = \varnothing$

Because $\mathcal{C}' = \varnothing$, consider the candidate response BN (2) as a TIBN. Then equation (6) is rewritten as

$$G_{\phi(t)} \equiv G = \delta_{2^n}[\beta_1\ \beta_2\ \cdots\ \beta_{2^{2n}}]. \tag{15}$$

Therefore, the candidate response BN can be described as follows:

$$y(t+1) = Gx(t)y(t). \tag{16}$$

Then by equations (3)(5)(15) and (16), we have

$$\begin{aligned}&x(t+1)y(t+1) =\\ &F_{\sigma(t)}x(t)Gx(t)y(t) =\\ &(F_{\sigma(t)} \otimes I_{2^n})(I_{2^n} \otimes G)(\varPhi_n \otimes I_{2^n})x(t)y(t) =\\ &\varTheta_{\sigma(t)}x(t)y(t),\end{aligned} \tag{17}$$

where

$$\begin{aligned}&\varTheta_{\sigma(t)} =\\ &\delta_{2^{2n}}[(\alpha_1^{\sigma(t)}-1)2^n + \beta_1, (\alpha_1^{\sigma(t)}-1)2^n +\\ &\beta_2, \cdots, (\alpha_1^{\sigma(t)}-1)2^n + \beta_{2^n},\\ &(\alpha_2^{\sigma(t)}-1)2^n + \beta_{2^n+1},\\ &(\alpha_2^{\sigma(t)}-1)2^n + \beta_{2^n+2}, \cdots,\\ &(\alpha_2^{\sigma(t)}-1)2^n + \beta_{2\cdot 2^n}, \cdots,\\ &(\alpha_{2^n}^{\sigma(t)}-1)2^n + \beta_{(2^n-1)2^n+1},\\ &(\alpha_{2^n}^{\sigma(t)}-1)2^n + \beta_{(2^n-1)2^n+2}, \cdots,\\ &(\alpha_{2^n}^{\sigma(t)}-1)2^n + \beta_{2^{2n}}].\end{aligned} \tag{18}$$

Then the following theorem can be obtained.

Theorem 1 Assume $\mathcal{C}' = \varnothing$, where $\mathcal{C}'$ is defined in (14). BNs (1) and (2) are completely synchronized if and only if there is a non-negative integer $k \leqslant 2^{2n}$ such that

$$\mathrm{Col}(\Theta_{[kp+i]}) \subseteq \{\, \delta_{2^{2n}}^{1}, \delta_{2^{2n}}^{2^n+2}, \cdots, \delta_{2^{2n}}^{2^{2n}} \},\quad i \in \mathbb{N}_{[1,p]}, \tag{19}$$

where p is the period of BN (1), $\Theta_{[0]} = I_{2^{2n}}$ and $\Theta_{[t]} = \Theta_{\sigma(t-1)}\Theta_{\sigma(t-2)}\cdots\Theta_{\sigma(0)}$ for any $t > 0$.

Proof Sufficiency. According to equation (17), we have

$$x(t)y(t) = \Theta_{[t]}x(0)y(0). \tag{20}$$

By Theorem 2 in [13], $x(t) = y(t)$ for every $x(0), y(0) \in \Delta_{2^n}$ if and only if $\mathrm{Col}(\Theta_{[t]}) \subseteq \{\delta_{2^{2n}}^{1}, \delta_{2^{2n}}^{2^n+2}, \cdots, \delta_{2^{2n}}^{2^{2n}}\}$. For any $t \geqslant kp+1$, there exist $j \in \mathbb{N}_{[1,p]}$ and $k' \in \mathbb{N}$ such that $t = kp+j+k'p$. Meanwhile, it can be found that $\Theta_{[k'p]} = \Theta_\theta^{k'}$, where $\Theta_\theta = \Theta_p \Theta_{p-1}\cdots\Theta_1(\Theta_{[0]} = \Theta_\theta^0 = I_{2^{2n}})$. Then we have

$$\Theta_{[t]} = \Theta_{[kp+j]}\Theta_\theta^{k'}. \tag{21}$$

Obviously, the following equation is also true:

$$\mathrm{Col}(\Theta_{[kp+j]}\Theta_\theta^{k'}) \subseteq \mathrm{Col}(\Theta_{[kp+j]}). \tag{22}$$

By utilizing equations (19)(21) and (22), we have

$$\mathrm{Col}(\Theta_{[t]}) \subseteq \{\delta_{2^{2n}}^{1}, \delta_{2^{2n}}^{2^n+2}, \cdots, \delta_{2^{2n}}^{2^{2n}}\}.$$

Therefore, $x(t) = y(t)$ for every $x(0), y(0) \in \Delta_{2^n}$ and any $t \geqslant kp+1$. This means the complete synchronization of BNs (1) and (2).

Necessity. By Definition 1, if BNs (1) and (2) are completely synchronized, there is a positive integer k' such that $x(t) = y(t)$ for every $x(0), y(0) \in \Delta_{2^n}$ and any $t \geqslant k'$. Hence, one can obtain

$$\mathrm{Col}(\Theta_{[t]}) \subseteq \{\delta_{2^{2n}}^{1}, \delta_{2^{2n}}^{2^n+2}, \cdots, \delta_{2^{2n}}^{2^{2n}}\},\ t \geqslant k'. \tag{23}$$

Then, by equation (23), take a positive integer k such that $kp+1 \geqslant k'$, then equation (19) holds.

By [22], there exist positive integers $r_1 < r_2 \leqslant 2^{2n}$ such that $\Theta_\theta^{r_1} = \Theta_\theta^{r_2}$, which means that $\Theta_{[(r_1+\tau)p]} = \Theta_{[(r_2+\tau)p]}$ for any $\tau \geqslant 0$. So the BNs (1) and (2) can not be completely synchronized if there exist integers $k' > 2^{2n}p$ such that $\mathrm{Col}(\Theta_{[k']}) \subsetneq \{\delta_{2^{2n}}(1), \delta_{2^{2n}}(2^n+2), \cdots, \delta_{2^{2n}}(2^{2n})\}$. Therefore, we have $k' \leqslant 2^{2n}p$ and $k \leqslant 2^{2n}$. The proof is completed.

For case 1, let us provide an approach for the design of a synchronizing response BN.

Theorem 2 Assume $\mathcal{C}' = \varnothing$, the response BN (2) completely synchronizes with the drive BN (1) if

$$\beta_{(\varepsilon(\tilde{x}_i(j))-1)2^n+d} = \varepsilon(\tilde{x}_i(j+1)),\quad d \in \mathbb{N}_{[1,2^n]},\ i \in \mathbb{N}_{[1,s]},\ j \in \mathbb{N}_{[0,l_i-1]}. \tag{24}$$

Proof By Lemma 1, there is a positive integer k such that $x(t) = \tilde{x}_i(j)$ is in one cycle of BN (1) for every $x(0) \in \Delta_{2^n}$ and any $t \geqslant k$. By equations (16)−(18), $y(t+1) \in \{\delta_{2^n}^{r} |\ r = \beta_{(\varepsilon(\tilde{x}_i(j))-1)2^n+d},\ d \in \mathbb{N}_{[1,2^n]}\}$ if $x(t) = \tilde{x}_i(j)$ for every $y(0) \in \Delta_{2^n}$. By (24), we have $y(t+1) = \tilde{x}_i(j+1) = x(t+1)$. Therefore, $y(t) = x(t)$ for every $x(0), y(0) \in \Delta_{2^n}$ and any $t > k$. The proof is completely.

3.2 Case 2: $\mathcal{C}' \neq \varnothing$

In order to get the main results, we give a lemma first.

Lemma 2 There is no TIBN such that it completely synchronizes with BN (1) if $\mathcal{C}' \neq \varnothing$.

Proof We prove it by reduction to absurdity. Assume that BN (2) is a TIBN and completely synchronizes with BN (1). By Definition 1, there is a positive integer k_1 such that $y(t, x(0), y(0)) = x(t, x(0))$ for every $x(0), y(0) \in \Delta_{2^n}$ and any $t \geqslant k_1$. Because $\mathcal{C}' \neq \varnothing$, there exists $x_0 \in \mathcal{C}$ such that $\tilde{x}_{i_1}(j_1) = \tilde{x}_{i_2}(j_2) = x_0$ and $\tilde{x}_{i_1}(j_1+1) \neq \tilde{x}_{i_2}(j_2+1)$ for some $i_1, i_2 \in \mathbb{N}_{[1,s]}$, $j_1 \in \mathbb{N}_{[0,l_{i_1}-1]}$ and $j_2 \in \mathbb{N}_{[0,l_{i_2}-1]}$. Take $x(0) = \tilde{x}_{i_1}(0)$, we have

$$x(k_2l_{i_1}+j_1) = \tilde{x}_{i_1}(j_1), \tag{25}$$

$$x(k_2l_{i_1}+j_1+1) = \tilde{x}_{i_1}(j_1+1) \tag{26}$$

for any $k_2 \geqslant 0$. Take k_2 such that $k_2l_{i_1} \geqslant k_1$, we can obtain

$$y(k_2l_{i_1}+j_1) = x(k_2l_{i_1}+j_1) = \tilde{x}_{i_1}(j_1), \tag{27}$$

$$y(k_2l_{i_1}+j_1+1) = x(k_2l_{i_1}+j_1+1) = \tilde{x}_{i_1}(j_1+1). \tag{28}$$

By equation (18), this means that

$$\beta_{(\varepsilon(\tilde{x}_{i_1}(j_1))-1)2^n+\varepsilon(\tilde{x}_{i_1}(j_1))} = \varepsilon(\tilde{x}_{i_1}(j_1+1)). \tag{29}$$

Similarly, take $x(0) = \tilde{x}_{i_2}(0)$, we can get

$$\beta_{(\varepsilon(\tilde{x}_{i_2}(j_2))-1)2^n+\varepsilon(\tilde{x}_{i_2}(j_2))} = \varepsilon(\tilde{x}_{i_2}(j_2+1)). \tag{30}$$

Because $\varepsilon(x)$ is a bijection, we have $\varepsilon(\tilde{x}_{i_1}(j_1)) = \varepsilon(\tilde{x}_{i_2}(j_2))$ and $\varepsilon(\tilde{x}_{i_1}(j_1+1)) \neq \varepsilon(\tilde{x}_{i_2}(j_2+1))$. Then the following equations hold:

$$(\varepsilon(\tilde{x}_{i_1}(j_1))-1)2^n+\varepsilon(\tilde{x}_{i_1}(j_1)) = (\varepsilon(\tilde{x}_{i_2}(j_2))-1)2^n+\varepsilon(\tilde{x}_{i_2}(j_2)), \tag{31}$$

$$\beta_{(\varepsilon(\tilde{x}_{i_1}(j_1))-1)2^n+\varepsilon(\tilde{x}_{i_1}(j_1))} \neq \beta_{(\varepsilon(\tilde{x}_{i_2}(j_2))-1)2^n+\varepsilon(\tilde{x}_{i_2}(j_2))}. \tag{32}$$

Equations (31) and (32) lead to a contradiction. Therefore, the proof is completed.

By Lemma 2, only TVBN is the candidate that can completely synchronize with BN (1) in Case 2. By equations (3)−(6) we have

$$x(t+1)y(t+1) = \Theta'_{\varphi(t)}x(t)y(t), \tag{33}$$

where

$$\Theta'_{\varphi(t)} = \delta_{2^{2n}}[(\alpha_1^{\sigma(t)}-1)2^n+\beta_1^{\phi(t)}, (\alpha_1^{\sigma(t)}-1)2^n+\beta_2^{\phi(t)}, \cdots, (\alpha_1^{\sigma(t)}-1)2^n+\beta_{2^n}^{\phi(t)}, (\alpha_2^{\sigma(t)}-1)2^n+\beta_{2^n+1}^{\phi(t)},$$

$(\alpha_2^{\sigma(t)}-1)2^n+\beta_{2^n+2}^{\phi(t)},\cdots,(\alpha_2^{\sigma(t)}-1)2^n+\beta_{2\cdot 2^n}^{\phi(t)},\cdots,(\alpha_{2^n}^{\sigma(t)}-1)2^n+\beta_{(2^n-1)2^n+1}^{\phi(t)},(\alpha_{2^n}^{\sigma(t)}-1)2^n+\beta_{(2^n-1)2^n+2}^{\phi(t)},\cdots,(\alpha_{2^n}^{\sigma(t)}-1)2^n+\beta_{2^{2n}}^{\phi(t)}].$ (34)

Remark 1 If $\phi(t)$ is not a periodic function, $\Theta'_{\varphi(t)}$ may not be periodically time-variant. In this case, $\Theta'_{\varphi(t)}$ becomes quite complex and we can hardly provide a regular conclusion. Therefore, $\phi(t)$ is just considered as a periodic function.

Remark 2 By Definition 1 and Lemma 1, if BNs (1) and (2) are completely synchronized, it must satisfy

$$\beta_{(\varepsilon(\tilde{x}_i(j))-1)2^n+\varepsilon(\tilde{x}_i(j))}^{\phi(j)}=\varepsilon(\tilde{x}_i(j+1)),\quad i\in\mathbb{N}_{[1,s]},\ j\in\mathbb{N}_{[0,l_i-1]}. \tag{35}$$

Because $\mathcal{C}'\neq\varnothing$, there exist $\tilde{x}_{i_1}(j_1)$, $\tilde{x}_{i_2}(j_2)$ in the cycles of BNs (1) such that $\tilde{x}_{i_1}(j_1)=\tilde{x}_{i_2}(j_2)$, $\tilde{x}_{i_1}(j_1+1)\neq\tilde{x}_{i_2}(j_2+1)$ and $\sigma(j_1)\neq\sigma(j_2)$. Therefore, if the equality (35) holds for BN (1) in case 2, it must satisfy condition (1): for any $j_1,j_2\in\mathbb{N}$, if $\sigma(j_1)\neq\sigma(j_2)$, then $\phi(j_1)\neq\phi(j_2)$. By Remark 1, take $\phi(t)=t\%q+1$, where q is the period of $\phi(t)$. Then condition (1) is equivalent to condition (2): for any $j_1,j_2\in\mathbb{N}$, if $j_1\%p\neq j_2\%p$, then $j_1\%q\neq j_2\%q$. It can be found that condition (2) holds if and only if $q=mp$, where $m\in\mathbb{N}^+$. This conclusion can be proved by utilizing the properties of integer and Euclidean algorithm.

By Remark 1 and Remark 2, we just consider $\phi(t)=t\%(mp)+1$ in Case 2. Then the following theorem can be obtained.

Theorem 3 Assume $\mathcal{C}'\neq\varnothing$, BNs (1) and (2) are completely synchronized if and only if there is a non-negative integer $k\leqslant 2^{2n}$ such that

$$\mathrm{Col}(\Theta_{[k(mp)+i]})\subseteq\{\delta_{2^{2n}}^1,\delta_{2^{2n}}^{2^n+2},\cdots,\delta_{2^{2n}}^{2^{2n}}\},\quad i\in\mathbb{N}_{[1,mp]}, \tag{36}$$

where mp is the period of BNs (2), $\Theta'_{[0]}=I_{2^{2n}}$ and $\Theta'_{[t]}=\Theta'_{\varphi(t-1)}\Theta'_{\varphi(t-2)}\cdots\Theta'_{\varphi(0)}$ for any $t>0$.

The proof is similar to Theorem 1.

We provide an approach for the design of a synchronizing response BN.

Theorem 4 BNs (1) and (2) are completely synchronized in Case 2 if

$$\beta_{(\varepsilon(\tilde{x}_i(j))-1)2^n+d}^{\phi(j+gp)}=\varepsilon(\tilde{x}_i(j+1)),\quad d\in\mathbb{N}_{[1,2^n]},\ i\in\mathbb{N}_{[1,s]},\ j\in\mathbb{N}_{[0,l_i-1]},\ g\in\mathbb{N}_{[0,m-1]}. \tag{37}$$

The proof is similar to Theorem 2.

Remark 3 Obviously, a TIBN can be treated as a PTVBN with period mp such that $G_{\phi(t)}\equiv G$. Therefore, no matter which case PTVBN (1) with the period of p belongs to, there exists a PTVBN with the period of mp to completely synchronize with it for any $m\in\mathbb{N}^+$.

4 Numerical examples

4.1 Example for Case 1

Give a drive BN as follows:

$$\begin{cases} x_1(t+1)=\begin{cases}\neg x_1(t)\wedge x_2(t), & \text{when } \sigma(t)=1,\\ x_1(t)\vee x_2(t), & \text{when } \sigma(t)=2,\end{cases}\\ x_2(t+1)=\begin{cases} x_1(t)\leftrightarrow x_2(t), & \text{when } \sigma(t)=1,\\ \neg(x_1(t)\leftrightarrow x_2(t)), & \text{when } \sigma(t)=2.\end{cases}\end{cases} \tag{38}$$

Now, let us design a response BN to completely synchronize with BN (38) by using Theorem 2. It can be obtained that $F_1=\delta_4[3,4,2,3]$, $F_2=\delta_4[2,1,1,4]$ and the period is $p=2$. The cycle of (38) is C: $\overline{\{\delta_4^1,\delta_4^3,\delta_4^1\}}$. According to Theorem 2, take

$$\beta_1=\beta_2=\beta_3=\beta_4=3,\quad \beta_9=\beta_{10}=\beta_{11}=\beta_{12}=1,$$

and $\beta_5,\beta_6,\beta_7,\beta_8,\beta_{13},\beta_{14},\beta_{15},\beta_{16}$ can be chosen arbitrarily. For example, take

$$\beta_5=\beta_7=1,\ \beta_6=\beta_8=4,\quad \beta_{13}=\beta_{15}=2,\ \beta_{14}=\beta_{16}=3.$$

Therefore, the structure matrix of the designed response BN is

$$G=\delta_4[3,3,3,3,1,4,1,4,1,1,1,1,2,3,2,3].$$

By using the technique provided in [22], the logical equations for the designed response BN are

$$y_1(t+1)=(\neg x_1(t)\wedge x_2(t))\vee(\neg x_2(t)\wedge y_2(t)),$$
$$y_2(t+1)=x_2(t)\vee(x_1(t)\leftrightarrow y_2(t)).$$

Take $x(0)=(1,0)$ and $y(0)=(0,0)$. The Hamming distance $H(t)=|x_1(0)-y_1(0)|+|x_2(0)-y_2(0)|$ versus the time t is plotted in Fig 1. By Fig 1, these two BNs are completely synchronized from the fourth step.

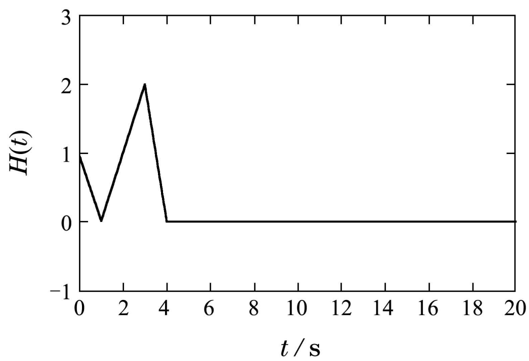

Fig. 1 Hamming distance versus time t with initial values $x(0)=(1,0)$ and $y(0)=(0,0)$

4.2 Example for Case 2

Consider a drive BN as

$$\begin{cases} x_1(t+1)=\begin{cases} x_2(t), & \text{when } \sigma(t)=1,\\ \neg x_1(t)\vee x_2(t), & \text{when } \sigma(t)=2, \end{cases}\\ x_2(t+1)=\begin{cases} x_1(t)\leftrightarrow x_2(t), & \text{when } \sigma(t)=1,\\ \neg(x_1(t)\leftrightarrow x_2(t)), & \text{when } \sigma(t)=2. \end{cases} \end{cases} \tag{39}$$

Let us provide the response BNs to completely synchronize with BN (39) by utilizing Theorem 4. It can be obtained that $F_1 = \delta_4[1,4,2,3]$, $F_2 = \delta_4[2,3,1,2]$ and the period is $p = 2$. The cycles of (39) are $C_1 : \overline{\{\delta_4^2, \delta_4^4, \delta_4^2\}}$ and $C_2 : \overline{\{\delta_4^3, \delta_4^2, \delta_4^3\}}$.

1) Consider the period of the response BN $q = p = 2$. By Theorem 4, take

$$\begin{aligned} &\beta_5^1 = \beta_6^1 = \beta_7^1 = \beta_8^1 = 4,\\ &\beta_9^1 = \beta_{10}^1 = \beta_{11}^1 = \beta_{12}^1 = 2,\\ &\beta_5^2 = \beta_6^2 = \beta_7^2 = \beta_8^2 = 3,\\ &\beta_{13}^2 = \beta_{14}^2 = \beta_{15}^2 = \beta_{16}^2 = 2, \end{aligned}$$

and β_1^1, β_2^1, β_3^1, β_4^1, β_{13}^1, β_{14}^1, β_{15}^1, β_{16}^1, β_1^2, β_2^2, β_3^2, β_4^2, β_9^2, β_{10}^2, β_{11}^2, β_{12}^2 can be chosen arbitrarily in {1,2,3,4}. For example, take $\beta_1^1 = \beta_2^1 = \beta_1^2 = \beta_2^2 = 1$, $\beta_3^1 = \beta_4^1 = \beta_9^2 = \beta_{10}^2 = 2$, $\beta_{13}^1 = \beta_{14}^1 = \beta_3^2 = \beta_4^2 = 3$, $\beta_{15}^1 = \beta_{16}^1 = \beta_{11}^2 = \beta_{12}^2 = 4$.

Then we can obtain the structure matrix of the designed response BN

$$\begin{aligned} &G_1 = \delta_4[1,1,2,2,4,4,4,4,2,2,2,2,3,3,4,4],\\ &G_2 = \delta_4[1,1,3,3,3,3,3,3,2,2,4,4,2,2,2,2], \end{aligned}$$

and the logical equations for the designed response BN are

$$y_1(t+1) = \begin{cases} x_2(t), \ \text{when } \sigma(t) = 1,\\ (\neg x_1(t) \wedge \neg x_2(t)) \vee (x_2(t) \wedge y_1(t)),\\ \qquad \text{when } \sigma(t) = 2, \end{cases}$$

$$y_2(t+1) = \begin{cases} (x_1(t) \leftrightarrow x_2(t)) \wedge y_1(t),\\ \qquad \text{when } \sigma(t) = 1,\\ x_1(t), \ \text{when } \sigma(t) = 2. \end{cases}$$

Take $x(0) = (0,0)$ and $y(0) = (0,1)$. The Hamming distance $H(t) = |x_1(0) - y_1(0)| + |x_2(0) - y_2(0)|$ versus the time t is plotted in Fig 2. By Fig 2, these two BNs are completely synchronized from the fifth step.

2) Consider the period of the response BN $q = 2p = 4$. By Theorem 4, take

$$\begin{aligned} &\beta_5^1 = \beta_6^1 = \beta_7^1 = \beta_8^1 = 4,\\ &\beta_9^1 = \beta_{10}^1 = \beta_{11}^1 = \beta_{12}^1 = 2,\\ &\beta_5^2 = \beta_6^2 = \beta_7^2 = \beta_8^2 = 3,\\ &\beta_{13}^2 = \beta_{14}^2 = \beta_{15}^2 = \beta_{16}^2 = 2,\\ &\beta_5^3 = \beta_6^3 = \beta_7^3 = \beta_8^3 = 4,\\ &\beta_9^3 = \beta_{10}^3 = \beta_{11}^3 = \beta_{12}^3 = 2,\\ &\beta_5^4 = \beta_6^4 = \beta_7^4 = \beta_8^4 = 3,\\ &\beta_{13}^4 = \beta_{14}^4 = \beta_{15}^4 = \beta_{16}^4 = 2, \end{aligned}$$

and β_1^1, β_2^1, β_3^1, β_4^1, β_{13}^1, β_{14}^1, β_{15}^1, β_{16}^1, β_1^2, β_2^2, β_3^2, β_4^2, β_9^2, β_{10}^2, β_{11}^2, β_{12}^2, β_1^3, β_2^3, β_3^3, β_4^3, β_{13}^3, β_{14}^3, β_{15}^3, β_{16}^3, β_1^4, β_2^4, β_3^4, β_4^4, β_9^4, β_{10}^4, β_{11}^4, β_{12}^4 can be chosen arbitrarily in {1,2,3,4}. For example, take $\beta_1^1 = \beta_2^1 = \beta_1^2 = \beta_2^2 = \beta_3^3 = \beta_4^3 = \beta_3^4 = \beta_4^4 = 1$, $\beta_3^1 = \beta_4^1 = \beta_9^2 = \beta_{10}^2 = \beta_1^3 = \beta_2^3 = \beta_1^4 = \beta_2^4 = 2$, $\beta_{13}^1 = \beta_{14}^1 = \beta_3^2 = \beta_4^2 = \beta_{15}^3 = \beta_{16}^3 = \beta_{11}^4 = \beta_{12}^4 = 3$, $\beta_{15}^1 = \beta_{16}^1 = \beta_{11}^2 = \beta_{12}^2 = \beta_{13}^3 = \beta_{14}^3 = \beta_9^4 = \beta_{10}^4 = 4$.

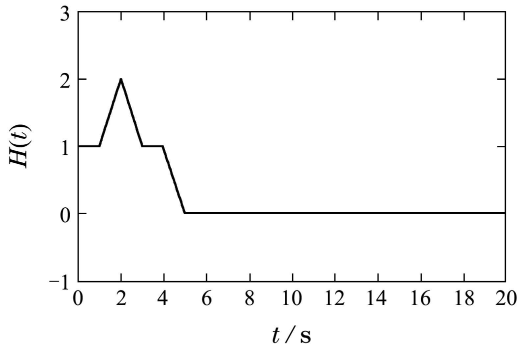

Fig. 2 Hamming distance versus time t with initial values $x(0) = (0,0)$ and $y(0) = (0,1)$

Then we can obtain the structure matrix of the designed response BN

$$\begin{aligned} &G_1 = \delta_4[1,1,2,2,4,4,4,4,2,2,2,2,3,3,4,4],\\ &G_2 = \delta_4[1,1,3,3,3,3,3,3,2,2,4,4,2,2,2,2],\\ &G_3 = \delta_4[2,2,1,1,4,4,4,4,2,2,2,2,4,4,3,3],\\ &G_4 = \delta_4[2,2,1,1,3,3,3,3,4,4,3,3,2,2,2,2], \end{aligned}$$

and the logical equations for the designed response BN are

$$y_1(t+1)=\begin{cases} x_2(t), & \text{when } \sigma(t) = 1,\\ (\neg x_1(t) \wedge \neg x_2(t)) \vee (x_2(t) \wedge y_1(t)), & \\ & \text{when } \sigma(t) = 2,\\ x_2(t), & \text{when } \sigma(t) = 3,\\ x_1(t) \leftrightarrow x_2(t), & \text{when } \sigma(t) = 4, \end{cases}$$

$$y_2(t+1)=\begin{cases} (x_1(t) \leftrightarrow x_2(t)) \wedge y_1(t),\\ \qquad \text{when } \sigma(t) = 1,\\ x_1(t), \ \text{when } \sigma(t) = 2,\\ (x_1(t) \leftrightarrow x_2(t)) \wedge \neg y_1(t),\\ \qquad \text{when } \sigma(t) = 3,\\ (x_1(t) \wedge \neg x_2(t)) \vee (x_2(t) \wedge \neg y_1(t)),\\ \qquad \text{when } \sigma(t) = 4. \end{cases}$$

Take $x(0) = (0,0)$ and $y(0) = (0,1)$. The Hamming distance $H(t) = |x_1(0) - y_1(0)| + |x_2(0) - y_2(0)|$ versus the time t is plotted in Fig 3. By Fig 3, these two BNs are completely synchronized from the third step.

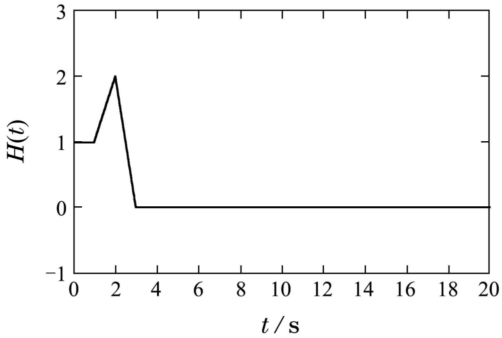

Fig. 3 Hamming distance versus time t with initial values $x(0) = (0, 0)$ and $y(0) = (0, 1)$

5 Conclusions

In this paper, we have studied the problem of complete synchronization for two BNs, which are coupled unidirectionally in a drive-response configuration and the drive BN is a PTVBN. Because the structure of PTVBN is more complex than TIBN's. The drive B-N has been considered in two different kinds. We have provided the necessary and sufficiency criteria for complete synchronization, and general approaches for the design of a response BN for both kinds, respectively. Some numerical examples have been given to support these viewpoints. In the future work, we will investigate the complete synchronization of PTVBNs with time-delay and output-coupled.

References:

[1] LI H, WANG Y, LIU Z. Stability analysis for switched Boolean networks under arbitrary switching signals [J]. *IEEE Transactions on Automatic Control*, 2014, 59(7): 1978 – 1982.

[2] LI F, SUN J. Controllability and optimal control of a temporal Boolean network [J]. *Neural Networks–the Official Journal of the International Neural Network Society*, 2012, 34(4): 10 – 17.

[3] CHEN H, LI X, SUN J. Stabilization, controllability and optimal control of Boolean networks with impulsive effects and state constraints [J]. *IEEE Transactions on Automaic Control*, 2015, 60(3): 806 – 811.

[4] ZHAO Y, LI Z, CHENG D. Optimal control of logical control networks [J]. *IEEE Transactions on Automatic Control*, 2011, 56(8): 1766 – 1776.

[5] FORNASINI E, VALCHER M E. Optimal control of Boolean control networks [J]. *IEEE Transactions on Automatic Control*, 2014, 59(5): 1258 –1270.

[6] TIAN H, WANG Z, HOU Y, et al. State feedback controller design for synchronization of master-slave Boolean networks based on core input-state cycles [J]. *Neurocomputing*, 2016, 174(22): 1031 – 1037.

[7] ZHANG H, ZHANG J, YANG G, et al. Leader-based optimal coordination control for the consensus problem of multi-agent differential games via fuzzy adaptive dynamic programming [J]. *IEEE Transactions on Fuzzy Systems*, 2015, 23(1): 152 – 163.

[8] ZHANG H, MA T, HUANG G, et al. Robust global exponential syncrhonization of uncertain chaotic delayed neural networks via dual-stage impulsive control [J]. *IEEE Transactions on Systems Man and Cybernetics Part B-Cybernetics*, 2010, 40(3): 831 – 844.

[9] ZHANG H, HUANG W, WANG Z, et al. Adaptive synchronization between two different chaotic systems with unknown parameters [J]. *Physics letter A*, 2006, 350(5/6): 363 – 366.

[10] QI H, CHENG D. Analysis and control of Boolean networks: a semi-tensor product approach [C] //*Asian Control Conference*. Hong Kong: IEEE, 2009, 8: 1352 – 1356.

[11] LI R, CHU T. Complete synchronization of Boolean networks [J]. *IEEE Transactions on Neural Networks and Learning Systems*, 2012, 23(5): 840 – 846.

[12] LI R, YANG M, CHU T. Synchronization design of Boolean networks via the semi-tensor product method [J]. *IEEE Transactions on Neural Networks and Learning Systems*, 2013, 24(6): 996 – 1001.

[13] ZHANG H, WANG X, LIN X. Synchronization of Boolean networks with different update schemes [J]. *IEEE Transactions on Computational Biology and Bionformatics*, 2014, 11(5): 965 – 972.

[14] LI R, YANG M, CHU T. Synchronization of Boolean networks with time-delays [J]. *Applied Mathematics and Computation*, 2012, 219(3): 917 – 927.

[15] ZHONG J, LU J, LIU Y, et al. Synchronization in an arrary of output-coupled Boolean networks with time delay [J]. *IEEE Transactions on Neural Networks and Learning Systems*, 2014, 25(12): 2288 – 2294.

[16] LI R, CHU T. Synchronization in an arrary of coupled Boolean networks [J]. *Physics Letters A*, 2012, 376(45): 3071 – 3075.

[17] LI H, WANG Y. On reachability and controllability of switched Boolean control networks [J]. *Automatica*, 2012, 48(11): 2917 – 2922.

[18] CHENG D. Input-state approach to Boolean networks [J]. *IEEE Transactions on Neural Networks*, 2009, 20(3): 512 –521.

[19] XIAO Y, DOUGHERTY E R. The impact of function perturbations in Boolean networks [J]. *Bioinformatics*, 2007, 23(10): 1265 – 1273.

[20] ZOU Y, ZHU J. Cycles of periodically time-variant Boolean networks [J]. *Automatica*, 2015, 51: 175 –179.

[21] ZHANG H, TIAN H, WANG Z. Synchronization analysis and design of coupled Boolean networks based on periodic switching sequences [J]. *IEEE Transactions on Neural Networks and Learning Systems*. 2015, DOI: 10.1109/TNNLS. 2015.2499446.

[22] CHENG D, QI H. A linear representation of dynamics of Boolean networks [J]. *IEEE Transactions on Automatic Control*, 2010, 55(10): 2251 – 2258.

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