第 33 卷第 10 期
2016 年 10 月

控 制 理 论 与 应 用
Control Theory & Applications

Vol. 33 No. 10
Oct. 2016

DOI: 10.7641/CTA.2016.15105

带有不稳定子系统的切换非线性系统的积分输入状态稳定

赵海娟, 宗广灯†

(曲阜师范大学 工学院, 山东 日照 276826)

摘要: 本文针对带有不稳定子系统的切换非线性系统研究了系统的积分输入状态稳定性问题. 应用导数不定的类Lyapunov函数得出切换非线性系统的积分输入状态稳定. 导数不定的类Lyapunov函数方法比传统的导数正定的Lyapunov函数的方法更具有一般性. 文中包含两种情况: 当所有子系统为积分输入状态稳定时, 切换非线性系统是积分输入状态稳定的; 当部分子系统为非积分输入状态稳定时, 本文证明了切换非线性系统的积分输入状态稳定. 最后应用一个仿真例子描述了所提结果的有效性.

关键词: 积分输入状态稳定; 切换系统; 导数不定的类Lyapunov函数; 不稳定子系统

中图分类号: TP273 **文献标识码:** A

Integral input-to-state stability of switched nonlinear systems with unstable subsystems

ZHAO Hai-juan, ZONG Guang-deng†

(School of Engineering, Qufu Normal University, Rizhao Shandong 276826, China)

Abstract: This paper deals with the problem of integral input-to-state stability (iISS) for switched nonlinear systems with unstable subsystems. A Lyapunov-like function with indefinite derivative is introduced to derive the iISS of switched nonlinear systems, which generalizes the classic Lyapunov function with positive definite derivative. Two cases are considered. A switched nonlinear system is iISS if all subsystems are iISS. Moreover, if some of subsystems are not iISS, the iISS property is shown for the switched nonlinear systems. An illustrative example is presented to demonstrate the effectiveness of the main results.

Key words: integral input-to-state stability; switched systems; Lyapunov-like function with indefinite derivative; unstable subsystems

1 Introduction

Switched systems, as an important class of hybrid systems, arise in situations where several dynamical subsystems are presented together with a switching signal specifying which subsystem is active. In recent years, different properties of switched systems, especially stability issues, are extensively studied in literature[1–3]. It is well-known that, in general, switched systems do not necessarily inherit the properties of the subsystems they are comprised of. For example, a switched system consisting of linear exponentially stable subsystems might become unstable[1].

When a control system is affected by an external input, it is important to guarantee the control system to be input-to-state stable. The notion of input-to-state stability (ISS) characterizes the continuity of state trajectories on the initial states and the external inputs[4]. As a very powerful analysis tool, ISS has got the attention of a large number of scholars and the past two decades have witnessed the development of ISS[5–10]. In [11], the authors proved the ISS property for the impulsive systems based on an ISS Lyapunov function with the fixed dwell-time condition, where the derivative of the Lyapunov function need to be negative definite. In [12], it provided partial Lyapunov characterizations for the ISS and iISS of the nonlinear systems. Notice that these results require the Lyapunov function has negative definite derivatives. In order to remove this restriction, some researchers studied the iISS problem for the nonlinear systems by Lyapunov-like function with indefinite derivative[13], but they did not consider the case of switched systems with unstable subsystem, which usually undergo complex behaviors. This motivates us to examine the iISS of the nonlinear switched systems.

In this paper, some effective results ensuring the iISS are presented for switched nonlinear systems with unstable subsystems by the Lyapunov-like function with indefinite derivative. First, the not integral input-to-state stability (non-iISS) is analysed for nonlinear systems based on Lyapunov-like function method.

Received 21 October 2015; accepted 15 June 2016.
†Corresponding author. E-mail: lovelyletian@gmail.com; Tel.: +86 63-33980488.
Recommended by Associate Editor: HONG Yiguang.
Supported by National Natural Science Foundation of China (61273123, 61304059), Program for New Century Excellent Talents in University (NCET–13–0878) and Program for Scientific Research Innovation Team in Colleges and Universities of Shandong Province.

Then, the sufficient conditions for switched nonlinear systems are shown with all subsystems are iISS. Furthermore, the iISS of switched nonlinear system is given if some subsystems are not iISS. Under this case, the upper bound on the total time length of the unstable subsystem activated is given. Based on the mentioned Lyapunov-like function with indefinite derivative, a dissipation-like sufficient condition for the iISS of switched systems is also derived. Finally, a numerical example is employed to demonstrate the effectiveness of the proposed method.

In summary, this work primarily has the following contributions: 1) The problem of the integral input-to-state stability for the switched nonlinear systems is investigated based on the Lyapunov-like function with indefinite derivative. The derivative of Lyapunov-like function may be indefinite, rather than negative definite, which relaxes the sufficient condition for the iISS of switched nonlinear systems. 2) The property of iISS for switched nonlinear systems is considered under average dwell time switching signals. Two cases are considered. i) When each of the constituent systems is iISS, the switched nonlinear system is iISS. ii) When only some of the constituent systems are iISS, the iISS of switched nonlinear systems can also be guaranteed, which extends available results on iISS for switched nonlinear systems whose constituent systems are all iISS. 3) Based on the Lyapunov-like function with indefinite derivative method and average dwell time switching signal technique, the sufficient condition of iISS is extended to a dissipation-like sufficient condition for the iISS of switched nonlinear system.

2 Problem formulation and preliminaries

Consider the following n-dimensional switched nonlinear system:

$$\dot{x}(t) = f_{\sigma(t)}(t, x(t), u(t)),\ t \geqslant t_0,\ x(t_0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state; $u \in \mathcal{L}_\infty^m$ is the system input; $x_0 \in \mathbb{R}^n$ is the initial state and $\sigma(\cdot) : [t_0, \infty) \to \mathcal{P} = \{1, 2, \cdots, N\}$ is a piecewise constant, right continuous function which specifies at each time t the index of the active system; $f_i(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz in (t, x) and uniformly continuous in u, and satisfy $f_i(\cdot, 0, 0) = 0, i \in \mathcal{P}$.

The definitions of ISS and iISS are given below.

Definition 1[14] A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s, and $\beta(r, s) \to 0$ as $s \to \infty$.

Definition 2[14] System (1) is said to be ISS, if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that, for any initial state x_0 and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geqslant t_0$ and satisfies

$$\|x(t)\| \leqslant \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \leqslant s \leqslant t} \|u(s)\|). \quad (2)$$

Definition 3[4] System (1) is said to be iISS, if there exist a $\mathcal{KL}$ function β, a $\mathcal{K}_\infty$ function α, and a $\mathcal{K}$ function γ, such that, for any initial state x_0 and any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \geqslant t_0$ and satisfies

$$\alpha(\|x(t)\|) \leqslant \beta(\|x_0\|, t - t_0) + \int_{t_0}^{t} \gamma(\|u(\tau)\|)\mathrm{d}\tau. \quad (3)$$

Definition 4 For any $t \geqslant t_0$ and any switched signal $\sigma(\varsigma)$, $t_0 \leqslant \varsigma < t$, let $N_\sigma(t_0, t)$ mean the number of switchings of $\sigma(\varsigma)$ during the interval $[t_0, t]$. If there exist $N_0 \geqslant 0$ and $\tau > 0$ such that $N_\sigma(t_0, t) \leqslant N_0 + (t - t_0)/\tau$, then τ and N_0 are called average dwell time and the chatter bound, respectively. And, $S_{\mathrm{ave}}[\tau, N_0]$ denotes the class of switching signals with average dwell time τ and chatter bound N_0.

In this following, two lemmas checking iISS and non-iISS properties are given for nonlinear system based on the Lyapunov-like function with indefinite derivative.

Consider the following n-dimensional nonlinear system:

$$\dot{x}(t) = f(t, x(t), u(t)),\ t \geqslant t_0,\ x(t_0) = x_0, \quad (4)$$

where $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is the state; $x_0 \in \mathbb{R}^n$ is the initial state; $u(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is the control input.

Lemma 1[13] Consider system (4). Assume that there exists a continuously differentiable function $V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a function $\eta \in \mathcal{K}$, a constant c and a continuous function $\phi(t) : \mathbb{R}^+ \to \mathbb{R}$, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $u \in \mathbb{R}^m$, such that

a) $\alpha_1(\|x\|) \leqslant V(t, x) \leqslant \alpha_2(\|x\|)$;

b) $\dot{V}(t, x) \leqslant \phi(t)V(t, x) + \eta(\|u\|)$;

c) $\int_{t_0}^{\infty} \phi^+(\tau)\mathrm{d}\tau \leqslant c < \infty$;

d) there exist two constants $\delta > 0$ and $T > t_0$ satisfying

$$\int_{t_0}^{t} \phi^-(\tau)\mathrm{d}\tau \geqslant \delta(t - t_0),\ t > T, \quad (5)$$

where

$$\phi^+(\tau) = \max\{\phi(\tau), 0\},\ \phi^-(\tau) = \max\{-\phi(\tau), 0\},$$

then system (4) is iISS with $\gamma(s) = C\eta(s)$, where

$$C = \mathrm{e}^{\int_0^\infty \phi^+(\tau)\mathrm{d}\tau} < \infty.$$

Lemma 2 Consider system (4). If the condition (a) in Lemma 1 holds, and there is a continuous function $\varphi(t) : \mathbb{R}^+ \to \mathbb{R}$ and a constant $\bar{c}$, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $u \in \mathbb{R}^m$, such that

e) $\dot{V}(t, x) \geqslant \varphi(t)V(t, x) + \eta(\|u\|)$;

f) $\int_{t_0}^{\infty} \varphi^-(\tau)\mathrm{d}\tau \leqslant \bar{c} < \infty$;

g) there exist two constants $\bar{\delta} > 0$ and $T > t_0$ satisfying

$$\int_{t_0}^{t} \varphi^+(\tau)\mathrm{d}\tau \geqslant \bar{\delta}(t - t_0),\ t > T, \tag{6}$$

where

$$\varphi^+(\tau) = \max\{\varphi(\tau), 0\},\ \varphi^-(\tau) = \max\{-\varphi(\tau), 0\},$$

then system (4) is non-iISS.

Proof We shall prove that the system is non-iISS by contradiction. From a) and e), one can get

$$V(t, x) \geqslant \mathrm{e}^{\int_{t_0}^{t} \varphi(\tau)\mathrm{d}\tau} V_0 + \int_{t_0}^{t} \eta(\|u(s)\|)\mathrm{e}^{\int_{s}^{t} \varphi(\tau)\mathrm{d}\tau}\mathrm{d}s, \tag{7}$$

where $V_0 = V(t_0, x(t_0))$, for all $t \geqslant t_0$. From f) and g), we obtain

$$\mathrm{e}^{\int_{t_0}^{t} \varphi(\tau)\mathrm{d}\tau} \geqslant \mathrm{e}^{-\bar{c}}\mathrm{e}^{\bar{\delta}(t-t_0)}.$$

This together with (7) and a) implies

$$V(t, x) \geqslant \mathrm{e}^{-\bar{c}}\mathrm{e}^{\bar{\delta}(t-t_0)}\alpha_1(\|x_0\|) + \mathrm{e}^{-\bar{c}} \int_{t_0}^{t} \eta(\|u(\tau)\|)\mathrm{d}\tau,$$

for all $t \geqslant T$, which combining the condition a) leads to

$$\|x(t)\| \geqslant \alpha_2^{-1}(\mathrm{e}^{-\bar{c}}\mathrm{e}^{\bar{\delta}(t-t_0)}\alpha_1(\|x_0\|) + \mathrm{e}^{-\bar{c}} \int_{t_0}^{t} \eta(\|u(\tau)\|)\mathrm{d}\tau) \tag{8}$$

for all $t \geqslant T$ with $T > t_0$ being a constant.

If the system is iISS, then there exists a class $\mathcal{KL}$ function β, a class $\mathcal{K}_\infty$ function α and $\mathcal{K}$ function γ such that

$$\alpha(\|x(t)\|) \leqslant \beta(\|x_0\|, t - t_0) + \int_{t_0}^{t} \gamma(\|u(\tau)\|)\mathrm{d}\tau \tag{9}$$

for all $t \geqslant t_0$, and $x_0 \in \mathbb{R}^n/\{0\}$. When the input $u(t) \equiv 0$, the above two formulas can derive

$$\|x(t)\| \geqslant \alpha_2^{-1}(\mathrm{e}^{-\bar{c}}\mathrm{e}^{\bar{\delta}(t-t_0)}\alpha_1(\|x_0\|)),\ \forall t \geqslant T, \tag{10}$$

$$\|x(t)\| \leqslant \alpha^{-1}(\beta(\|x_0\|, t - t_0)), \tag{11}$$

respectively. This leads to a contradiction by letting $t \to +\infty$ from (10) and (11). So system (4) is non-iISS and the proof is complete.

3 Main results

In this section, the main results are given. Two cases are considered. First, when all subsystems are iISS, we prove that system (1) keeps the iISS property under some restrictions. Second, when not all subsystems are iISS, we establish some criteria verifying the iISS property through the above lemmas. Further, we propose a dissipation-like sufficient condition proofing the iISS for system (1).

3.1 All subsystem iISS

Theorem 1 Consider system (1). If there exist a class of continuously differentiable functions $V_i(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, class $\mathcal{K}_\infty$ functions $\alpha_{1,i}, \alpha_{2,i}$, class $\mathcal{K}$ functions η_i, constants c_i and a constant $\mu > 1$, continuous functions $\phi_i(t) : \mathbb{R}^+ \to \mathbb{R}$, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$, such that

i) $\alpha_{1,i}(\|x\|) \leqslant V_i(t, x) \leqslant \alpha_{2,i}(\|x\|)$;

ii) $\dot{V}_i(t, x) \leqslant \phi_i(t)V_i(t, x) + \eta_i(\|u\|)$;

iii) $V_{i_k}(t_k, x(t_k)) \leqslant \mu V_{i_{k-1}}(t_k, x(t_k))$;

iv) $\int_{t_0}^{\infty} \phi_i^+(\tau)\mathrm{d}\tau \leqslant c_i < \infty$;

v) there exist constants $\delta_i > 0$ and $T > t_0$ satisfying

$$\int_{t_0}^{t} \phi_i^-(\tau)\mathrm{d}\tau \geqslant \delta_i(t - t_0),\ \forall t > T, \tag{12}$$

where

$$\phi_i^+(\tau) = \max\{\phi_i(\tau), 0\},\ \phi_i^-(\tau) = \max\{-\phi_i(\tau), 0\}.$$

If the switching signal satisfies the average dwell time $\tau > \tau^* = \dfrac{\ln \mu}{\delta}$, $\delta = \min\limits_{i\in\mathcal{P}}\{\delta_i\}$, then system (1) is iISS.

Proof We have the switching sequence $\Sigma = \{x_0 : (i_0, t_0), \cdots, (i_k, t_k), \cdots | i_k \in \mathcal{P}, k \in \mathbb{N}\}$, which means that the i_kth subsystem is active when $t \in [t_k, t_{k+1})$. From the condition ii), we have

$$V_{i_k}(t, x(t)) \leqslant \mathrm{e}^{\int_{t_k}^{t} \phi_{i_k}(\tau)\mathrm{d}\tau} V_{i_k}(t_k, x(t_k)) + \int_{t_k}^{t} \eta_{i_k}(\|u(s)\|)\mathrm{e}^{\int_{s}^{t} \phi_{i_k}(\tau)\mathrm{d}\tau}\mathrm{d}s$$

for all $t_k \leqslant t < t_{k+1}$. Taking iii) into account and by iteration, we get

$$\begin{aligned} &V_{i_k}(t, x(t)) \leqslant \\ &\mu\mathrm{e}^{\int_{t_k}^{t} \phi_{i_k}(\tau)\mathrm{d}\tau} V_{i_{k-1}}(t_k, x(t_k)) + \\ &\int_{t_k}^{t} \eta_{i_k}(\|u(s)\|)\mathrm{e}^{\int_{s}^{t} \phi_{i_k}(\tau)\mathrm{d}\tau}\mathrm{d}s \leqslant \\ &\mu^{N_\sigma}\mathrm{e}^{\int_{t_0}^{t} \bar{\phi}(\tau)\mathrm{d}\tau} V_{i_0}(t_0, x(t_0)) + \mathrm{e}^{c} \int_{t_0}^{t} \bar{\eta}(\|u(s)\|)\mathrm{d}s \cdot \\ &(\mu^{N_\sigma}\mathrm{e}^{\int_{t_1}^{t_2} \phi_{i_1}(\tau)\mathrm{d}\tau + \cdots + \int_{t_k}^{t} \phi_{i_k}(\tau)\mathrm{d}\tau} + \cdots + \\ &\mu\mathrm{e}^{\int_{t_k}^{t} \phi_{i_k}(\tau)\mathrm{d}\tau} + 1), \end{aligned} \tag{13}$$

where $\bar{\phi}(t) = \max\limits_{i\in\mathcal{P}}\{\phi_i(t)\}$, $\bar{\eta}(r) = \max\limits_{i\in\mathcal{P}}\{\eta_i(r)\}$, $c = \max\limits_{i\in\mathcal{P}}\{c_i\}$. Taking the following expression

$$\int_{t_0}^{t} \phi_i(\tau)\mathrm{d}\tau \leqslant c_i - \int_{t_0}^{t} \phi_i^-(\tau)\mathrm{d}\tau \tag{14}$$

into mind, (13) indicates

$$V_{i_k}(t,x(t)) \leqslant \mu^{N_\sigma} V_{i_0} \mathrm{e}^{c-\int_{t_0}^{t}\bar{\phi}^-(\tau)\mathrm{d}\tau} + \int_{t_0}^{t}\bar{\eta}(\|u(s)\|)\mathrm{d}s \cdot \mathrm{e}^{c}(\mu^{N_\sigma}\mathrm{e}^{\int_{t_1}^{t_2}\phi_{i_1}(\tau)\mathrm{d}\tau+\cdots+\int_{t_k}^{t}\phi_{i_k}(\tau)\mathrm{d}\tau}+\cdots+1). \tag{15}$$

For conciseness and without loss of generality, we suppose the i_m-subsystem is activated at time instant t_m in the following. By taking notice of (13), there holds

$$\mathrm{e}^{\int_{t_m}^{t_{m+1}}\phi_{i_m}(\tau)\mathrm{d}\tau+\cdots+\int_{t_k}^{t}\phi_{i_k}(\tau)\mathrm{d}\tau} \leqslant \mathrm{e}^{\int_{t_m}^{t}\bar{\phi}(\tau)\mathrm{d}\tau} \leqslant \mathrm{e}^{\mathrm{c}}. \tag{16}$$

Substituting (16) and i) into (15), we further derive

$$\alpha_{1,i_k}(\|x(t)\|) \leqslant \alpha_{2,i_0}(\|x(t_0)\|)\mathrm{e}^{\mathrm{c}}\mu^{N_\sigma}\mathrm{e}^{-\int_{t_0}^{t}\bar{\phi}^-(\tau)\mathrm{d}\tau} + \mathrm{e}^{2\mathrm{c}}\frac{(\mu^{N_\sigma+1}-1)}{\mu-1}\int_{t_0}^{t}\bar{\eta}(\|u(s)\|)\mathrm{d}s,\ \forall t \geqslant t_0.$$

Due to the arbitrariness of k, this further gives

$$\underline{\alpha}_1(\|x(t)\|) \leqslant \bar{\alpha}_2(\|x(t_0)\|)\mathrm{e}^{\frac{\ln\mu}{\tau}(t-t_0)-\int_{t_0}^{t}\bar{\phi}^-(\tau)\mathrm{d}\tau} \times \mathrm{e}^{\mathrm{c}}\mu^{N_0} + \mathrm{e}^{2\mathrm{c}}\frac{(\mu^{N_\sigma+1}-1)}{\mu-1}\int_{t_0}^{t}\bar{\eta}(\|u(s)\|)\mathrm{d}s, \tag{17}$$

where $\underline{\alpha}_1(r) = \min\limits_{i_k\in\mathcal{P}}\{\alpha_{1,i_k}(r)\}$, $\bar{\alpha}_2(r) = \max\limits_{i_k\in\mathcal{P}}\{\alpha_{2,i_k}(r)\}$. From (v), there exists a finite time instant $T \geqslant t_0$ such that

$$\int_{t_0}^{t}\phi_i^-(\tau)\mathrm{d}\tau \geqslant \delta_i(t-t_0),\ \forall t \geqslant T. \tag{18}$$

Construct a function with the form as

$$\beta(r,t) = \begin{cases} \mathrm{e}^{\mathrm{c}}\mu^{N_0}\bar{\alpha}_2(r), & t_0 \leqslant t \leqslant T, \\ \mathrm{e}^{\mathrm{c}}\mu^{N_0}\bar{\alpha}_2(r)\mathrm{e}^{(\frac{\ln\mu}{\tau}-\delta)t}, & t \geqslant T+1, \\ \mathrm{e}^{\mathrm{c}}\mu^{N_0}\bar{\alpha}_2(r)(T+1-t+ \\ (t-T)\times\mathrm{e}^{(\frac{\ln\mu}{\tau}-\delta)(T+1)}), & T < t < T+1, \end{cases} \tag{19}$$

where $\delta = \min\limits_{i\in\mathcal{P}}\{\delta_i\}$, and δ_i is ensured by (12). From (19), we know that $\beta(r,t)$ is a continuous function for a fixed r. On the other hand, letting $r := \delta - \dfrac{\ln\mu}{\tau}$ and the convexity of e^{-rt} yields

$$\mathrm{e}^{-rt} \leqslant T+1-t+\mathrm{e}^{-r(T+1)}(t-T)$$

for all $T < t < T+1$. Inequality (20) implies $\beta(r,t)$ decreases in t when r is fixed. Thus, $\beta(r,t)$ is a class $\mathcal{KL}$ function.

Now, for all $t \geqslant t_0$, we will prove that

$$\bar{\alpha}_2(\|x_0\|)\mathrm{e}^{\mathrm{c}}\mu^{N_0}\mathrm{e}^{\frac{\ln\mu}{\tau}(t-t_0)}\mathrm{e}^{-\int_{t_0}^{t}\bar{\phi}^-(\tau)\mathrm{d}\tau} \leqslant \beta(\|x_0\|, t-t_0). \tag{20}$$

It is clear from condition v) that there exists a $T > t_0$ such that (12) holds for all $t > T$. Inequality (20) indicates that

$$\bar{\alpha}_2(\|x_0\|)\mathrm{e}^{\mathrm{c}}\mu^{N_0}\mathrm{e}^{\frac{\ln\mu}{\tau}(t-t_0)}\mathrm{e}^{-\int_{t_0}^{t}\bar{\phi}^-(\tau)\mathrm{d}\tau} \leqslant \bar{\alpha}_2(\|x_0\|)\mathrm{e}^{\mathrm{c}}\mu^{N_0}\mathrm{e}^{(\frac{\ln\mu}{\tau}-\delta)(t-t_0)},\ \forall t > T. \tag{21}$$

Combining (20) and (21), we obtain

$$\bar{\alpha}_2(\|x_0\|)\mathrm{e}^{\mathrm{c}}\mu^{N_0}\mathrm{e}^{\frac{\ln\mu}{\tau}(t-t_0)}\mathrm{e}^{-\int_{t_0}^{t}\bar{\phi}^-(\tau)\mathrm{d}\tau} \leqslant \bar{\alpha}_2(\|x_0\|)\mathrm{e}^{\mathrm{c}}\mu^{N_0}(T+1-(t-t_0)+ \mathrm{e}^{-r(T+1)}(t-t_0-T)) = \beta(\|x_0\|, t-t_0) \tag{22}$$

for all $T < t-t_0 < T+1$. From (21) and (22), we find that (20) holds for $T < t-t_0 < T+1$. And it is trivial to show that (20) also holds for $0 \leqslant t-t_0 \leqslant T$ and $t-t_0 \geqslant T+1$. So, formula (20) is proved.

Obviously, from (17) and (20), we can get

$$\alpha(\|x(t)\|) \leqslant \beta(\|x_0\|, t-t_0) + \int_{t_0}^{t}\gamma(\|u(\tau)\|)\mathrm{d}\tau,$$

where $\alpha(r) = \underline{\alpha}_1(r)$, $\gamma(r) = \mathrm{e}^{2\mathrm{c}}\dfrac{(\mu^{N_\sigma+1}-1)}{\mu-1}\bar{\eta}(r)$. From Definition 2, we know that the above formula guarantees that system (1) is iISS. This completes the proof.

Remark 1 In Theorem 1, the switched system composed of all iISS subsystems is iISS through the use of Lyapunov-like function with indefinite derivative. When $u(t) = 0$, the iISS problem can be simplified to asymptotical stability analysis for system (1). Checking the asymptotic stability of system (1) using the traditional Lyapunov function theory, its derivative is usually required to be negative definite[14]. Note that here the derivative of the Lyapunov-like function is indefinite, which relaxes the Lyapunov-like condition to some extent compared with the existing results.

3.2 Some subsystem not iISS

Theorem 2 Consider switched system (1). Suppose that all conditions in Theorem 1 for the ith subsystems are satisfied. For the jth subsystems, if there exist a class of continuously differentiable functions $V_j(t,x): \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, $j \in \mathcal{P}$, class $\mathcal{K}_\infty$ functions $\alpha_{1,j}, \alpha_{2,j}$, functions $\eta_j, \bar{\eta}_j \in \mathcal{K}$, two continuous functions $\varphi_j(t)$ and $\phi_j(t): \mathbb{R}^+ \to \mathbb{R}$ and two constants $c_j, \bar{c}_j$, and a constant $\mu > 1$ such that

i) $\alpha_{1,j}(\|x\|) \leqslant V_j(t,x) \leqslant \alpha_{2,j}(\|x\|)$;

ii) $\varphi_j(t)V_j(t,x) + \bar{\eta}_j(\|u\|) \leqslant \dot{V}_j(t,x) \leqslant \phi_j(t)V_j(t,x) + \eta_j(\|u\|)$;

iii) $\displaystyle\int_{t_0}^{\infty}\varphi_j^-(\tau)\mathrm{d}\tau \leqslant \bar{c}_j < \infty$, $\displaystyle\int_{t_0}^{\infty}\phi_j^-(\tau)\mathrm{d}\tau \leqslant c_j < \infty$;

iv) there exist constants $\bar{\delta}_j, \delta_j > 0$ and $T > t_0$ satisfying

$$\int_{t_0}^{t}\varphi_j^+(\tau)\mathrm{d}\tau \geqslant \bar{\delta}_j(t-t_0),\ \int_{t_0}^{t}\phi_j^+(\tau)\mathrm{d}\tau \geqslant \delta_j(t-t_0)$$

for all $t \geqslant T$. If, at the switching instants t_k, $k = 0,1,2,3,\cdots$, there holds

$$V_{i_k}(t_k, x(t_k)) \leqslant \mu V_{i_{k-1}}(t_k, x(t_k)), \tag{23}$$

and there exists a constant δ with $0 < \delta < \delta_i$ satisfying

$$\frac{\varSigma_{\mathrm{us}}}{t-t_0} \leqslant \frac{\delta_i - \delta}{M}, \tag{24}$$

where $M = \sup\limits_{t\geqslant t_0}\{\phi_i^-(t) + \phi_j^+(t)\}$ is a finite constant, and $\varSigma_{us}$ denotes the length of all the time intervals when the jth subsystem is activated in $[t_0, t)$. Then switched system (1) is iISS under the switching signal satisfying condition (24) and the average dwell time $\tau > \tau^* = \dfrac{\ln\mu}{\delta}$.

Proof We have the switching sequence $\varSigma = \{x_0\colon (i_0,t_0),\cdots,(i_k,t_k),\cdots | i_k \in \mathcal{P}, k \in \mathbb{N}\}$, which means that the i_kth subsystem is activated at the time interval $[t_k, t_{k+1})$. For any $t \geqslant t_0$, there exists a finite positive integer k such that $t \in [t_k, t_{k+1})$. Without loss of generality, we let the ith subsystem be activated in $[t_k, t_{k+1})$ and $[t_0, t_1)$. From the condition ii) of Theorem 1, we have

$$\begin{aligned}&V_i(t,x(t)) \leqslant\\ &\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}V_i(t_k,x(t_k))+\\ &\int_{t_k}^{t}\eta_i(\|u(s)\|)\mathrm{e}^{\int_s^t\phi_i(\tau)\mathrm{d}\tau}\mathrm{d}s,\ t_k\leqslant t<t_{k+1}.\end{aligned}$$

Consider (23), it holds that

$$\begin{aligned}&V_i(t,x(t)) \leqslant\\ &\mu\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j(\tau)\mathrm{d}\tau+\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}\times V_j(t_{k-1},x(t_{k-1}))+\\ &\int_{t_{k-1}}^{t_k}\eta_j(\|u(s)\|)\mathrm{e}^{\int_s^{t_k}\phi_j(\tau)\mathrm{d}\tau}\mathrm{d}s\times\mu\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}+\\ &\int_{t_k}^{t}\eta_i(\|u(s)\|)\mathrm{e}^{\int_s^t\phi_i(\tau)\mathrm{d}\tau}\mathrm{d}s.\end{aligned} \tag{25}$$

By iteration, we get

$$\begin{aligned}&V_i(t,x(t)) \leqslant\\ &\mu^{N_\sigma}\mathrm{e}^{\int_{t_0}^{t_1}\phi_i(\tau)\mathrm{d}\tau}\cdots\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}V_0+\\ &\int_{t_0}^{t}\eta(\|u(s)\|)\mathrm{d}s(\mu^{N_\sigma}\mathrm{e}^{\int_{t_0}^{t_1}\phi_i^+(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}+\cdots+\\ &\mu\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j^+(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}+\mathrm{e}^{\int_{t_k}^{t}\phi_i^+(\tau)\mathrm{d}\tau}),\end{aligned} \tag{26}$$

where $\eta(r) = \max\limits_{i,j\in\mathcal{P}}\{\eta_i(r), \eta_j(r)\}$.

By the properties of $\phi_i(\tau)$ and $\phi_j(\tau)$, this further indicates

$$\begin{aligned}&\mathrm{e}^{\int_{t_0}^{t_1}\phi_i(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_1}^{t_2}\phi_j(\tau)\mathrm{d}\tau}\cdots\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}V_0\leqslant\\ &\mathrm{e}^{\int_{t_0}^{t}\phi_i^+(\tau)-\phi_i^-(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_1}^{t_2}+\cdots+\int_{t_{k-1}}^{t_k}(\phi_i^-(\tau)+\phi_j^+(\tau))\mathrm{d}\tau}V_0.\end{aligned}$$

Applying iv) in Theorem 1 and (24), we obtain

$$\begin{aligned}&\mu^{N_\sigma}\mathrm{e}^{\int_{t_0}^{t_1}\phi_i(\tau)\mathrm{d}\tau}\cdots\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}V_0\leqslant\\ &\alpha_{2,i}(\|x(t_0)\|)\mu^{N_0}\mathrm{e}^{c_i+(\frac{\ln\mu}{\tau}-\delta)(t-t_0)}\leqslant\\ &\alpha_{2,i}(\|x(t_0)\|)\mu^{N_0}\mathrm{e}^{c_i+\delta_i(T-t_0)}\mathrm{e}^{(\frac{\ln\mu}{\tau}-\delta)(t-t_0)}\end{aligned} \tag{27}$$

for all $t \geqslant T$, where $T > t_0$ is a finite constant.

To continue the analysis, we need to prove that the part of (26) in the bracket is bounded by a finite constant. We suppose the j-th subsystem is activated at time instant t_{m} in the following. By taking notice of (24), there holds

$$\begin{aligned}&\mathrm{e}^{\int_{t_{m-1}}^{t_m}\phi_i^+(\tau)\mathrm{d}\tau}\cdots\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}\leqslant\\ &\mathrm{e}^{c_i}\mathrm{e}^{-\delta_i(t-t_m)}\mathrm{e}^{\varSigma_{us}M}.\end{aligned} \tag{28}$$

From (24), we have

$$\frac{\varSigma_{\mathrm{us}}}{t_k-t_m} \leqslant \frac{\delta_i-\delta}{M}. \tag{29}$$

Based on this property, one obtains

$$\mathrm{e}^{\int_{t_{m-1}}^{t_m}\phi_i^+(\tau)\mathrm{d}\tau}\cdots\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}\leqslant\mathrm{e}^{c_i} \tag{30}$$

for $t \geqslant T$, where $T > t_0$ is finite. When $t < T$, it is easy to show that

$$\begin{cases}\mathrm{e}^{\int_{t_{m-1}}^{t_m}\phi_i(\tau)\mathrm{d}\tau}\cdots\mathrm{e}^{\int_{t_{k-1}}^{t_k}\phi_j(\tau)\mathrm{d}\tau}\mathrm{e}^{\int_{t_k}^{t}\phi_i(\tau)\mathrm{d}\tau}\leqslant\mathrm{e}^{c_i+MT},\\ m=1,2,\cdots,k+1.\end{cases} \tag{31}$$

Combining (27) with (31) and applying i) in Theorem 1, we can get

$$\begin{aligned}&\alpha_{1,i}(\|x(t)\|)\leqslant\\ &\alpha_{2,i}(\|x(t_0)\|)\mathrm{e}^{c_i+\delta_i(T-t_0)}\mathrm{e}^{(\frac{\ln\mu}{\tau}-\delta)(t-t_0)}\times\mu^{N_0}+\\ &\frac{(\mu^{N_\sigma+1}-1)}{\mu-1}\mathrm{e}^{c_i+MT}\int_{t_0}^{t}\eta(\|u(s)\|)\mathrm{d}s.\end{aligned} \tag{32}$$

Based on Theorem 1, we further derive that there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ functions γ, such that

$$\alpha(\|x(t)\|)\leqslant\beta(\|x_0\|,t-t_0)+\int_{t_0}^{t}\gamma(\|u(\tau)\|)\mathrm{d}\tau,$$

where $\beta(r,t) = \alpha_{2,i}(r)\mu^{N_0}\mathrm{e}^{c_i+\delta_i(T-t_0)}$ for all $t_0 \leqslant t \leqslant T$; $\beta(r,t) = \alpha_{2,i}(r)\mu^{N_0}\mathrm{e}^{c_i+\delta_i(T-t_0)}\mathrm{e}^{(\frac{\ln\mu}{\tau}-\delta)t}$ for all $t \geqslant T+1$; $\beta(r,t) = \alpha_{2,i}(r)\mu^{N_0}\mathrm{e}^{c_i+\delta_i(T-t_0)}(T+1-t+\mathrm{e}^{(\frac{\ln\mu}{\tau}-\delta)(T+1)}(t-T))$ for all $T < t < T+1$, and

$$\gamma(r) = \frac{(\mu^{N_\sigma+1}-1)}{\mu-1}\mathrm{e}^{c_i+MT}\eta(\|u(r)\|).$$

So the switched system is iISS by Definition 2. The proof is complete.

Remark 2 In Theorem 2, the positive constant δ can depict the convergence rate of the switched system to a great degree. The relationship $\delta < \delta_i$ shows that the switched system might be convergent with a slower speed than the stable subsystem, which results from the influence of the unstable dynamics.

Under the constraint of switch conditions, the iISS property of the switched systems with unstable subsystems is proved by using the indefinite derivative Lyapunov-like function technique. In the following, we will present a dissipation-like sufficient condition for the iISS of system (1).

Corollary 1 Suppose that conditions i) iii) iv) v) of Theorem 1 and conditions i) iii) iv) (23) of Theorem 2 hold. In addition, if there exist class $\mathcal{K}$ functions ρ_{q}^{i}, ρ_{q}^{j}, $\bar{\rho}_{\mathrm{q}}^{j}$, $q=\{1,2\}$, such that

$$\begin{aligned}&\dot{V}_i(t,x)\leqslant\\&(\rho_1^i(\|u\|)+\phi_i(t))V_i(t,x)+\rho_2^i(\|u\|),\end{aligned}\tag{33}$$

$$\begin{aligned}&(\bar{\rho}_1^j(\|u\|)+\varphi_j(t))V_j(t,x)+\bar{\rho}_2^j(\|u\|)\leqslant\\&\dot{V}_j(t,x)\leqslant\\&(\rho_1^j(\|u\|)+\phi_j(t))V_j(t,x)+\rho_2^j(\|u\|),\end{aligned}\tag{34}$$

then switched system (1) is iISS under the switching signal satisfying condition (24) and the average dwell time $\tau>\tau^*=\dfrac{\ln\mu}{\delta}$.

4 Numerical example

Consider the following switched nonlinear system

$$\dot{x}(t)=f_{\sigma(t)}(t,\ x(t))+g_{\sigma(t)}(t,x(t))u_{\sigma(t)}(t),\tag{35}$$

for $x(t)\in\mathbb{R}^2$, $t\geqslant t_0$, where $\sigma(t):[0,\infty)\to\{1,2\}$ is the switching signal. Let

$$f_1=\begin{bmatrix}(\dfrac{3}{1+t^2}-1)x_2-2x_1\\-2x_2\end{bmatrix},$$

$$f_2=\begin{bmatrix}(2-\dfrac{4}{1+t^2})x_2\\2x_1-2x_2\end{bmatrix},$$

and $g_1=[1\ \ 1]^{\mathrm{T}}$, $g_2=[0\ \ 4]^{\mathrm{T}}$.

Choosing function $V_1(t,x)=\dfrac{1}{2}(x_1^2+x_2^2)$, we have

$$\dot{V}_1(t,x)\leqslant(\frac{3}{1+t^2}-2)V_1(t,x)+\|u\|^2,$$

where $\phi_1(t)=\dfrac{3}{1+t^2}-2$ and $\eta_1(\|u\|)=\|u\|^2$. Then we get

$$\int_0^\infty\phi_1^+(\tau)\mathrm{d}\tau\leqslant\int_0^\infty\frac{3}{1+\tau^2}\mathrm{d}\tau\leqslant\frac{3\pi}{2}<\infty,$$

and when $t>5$, there holds

$$\int_0^t\phi_1^-(\tau)\mathrm{d}\tau\geqslant\frac{6}{5}t.$$

So the first subsystem is iISS with $\delta_1=\dfrac{6}{5}$. Similar to the above process, the candidate Lyapunov-like function for the second subsystem is chosen as $V_2(t,x)=x_1^2+x_2^2$. Then computing the derivative of V_2, we have

$$\dot{V}_2(t,x)\leqslant\phi_2(t,x)V_2(t,x)+\eta_2(\|u\|),$$

where $\phi_2(t,x)=3-\dfrac{4}{1+t^2}$ and $\eta_2(\|u\|)=4\|u\|^2$. Furthermore, one gets

$$\int_0^\infty\phi_2^-(\tau)\mathrm{d}\tau\leqslant2\pi,$$

and for $t>5$, there holds

$$\int_0^t\phi_2^+(\tau)\mathrm{d}\tau\geqslant\frac{9}{5}t,$$

where $\delta_2=\dfrac{9}{5}$.

When $u_1(t)=\mathrm{e}^{-t}$ and $x_0=[1\ \ -1]^{\mathrm{T}}$, we can get the subsystem 1 is iISS by Lemma 1, then the state trajectory of subsystem 1 is depicted in Fig.1. Similarly, let $u_2(t)=\mathrm{e}^{-t}$, $x_0=[1\ -1]^{\mathrm{T}}$, the state trajectory of the subsystem 2 is depicted in Fig.2, from which we can see that the subsystem 2 is non-iISS.

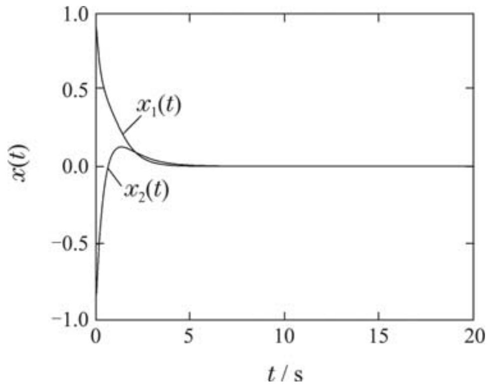

Fig. 1 The state trajectory of subsystem 1

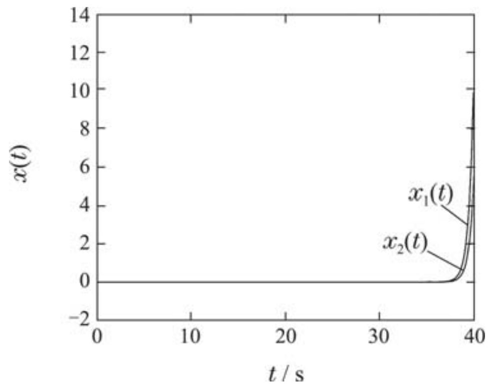

Fig. 2 The state trajectory of subsystem 2

From the above analysis and based on Theorem 2, we get $M=\sup\limits_{t\geqslant0}\{\phi_i^-(t)+\phi_j^+(t)\}=5$, $\mu=2$, and let $\delta=\dfrac{1}{5}<\dfrac{6}{5}=\delta_1$. The average dwell time scheme requires $\tau>\tau^*=\dfrac{\ln\mu}{\delta}=3.4657$. Then, the switching signal satisfies $\dfrac{\Sigma_{\mathrm{us}}}{t-t_0}\leqslant\dfrac{4}{5}$.

This switching signal meets the required condition $\dfrac{0.8}{4}=\dfrac{1}{5}\leqslant\dfrac{4}{5}$. By choosing the input $u(t)=\mathrm{e}^{-t}$ and the initial condition $x_0=[1\ \ -1]^{\mathrm{T}}$, one can get the trajectories of the system and the switching signal respectively as shown in Fig.3, which characterizes the iISS property of switched nonlinear system.

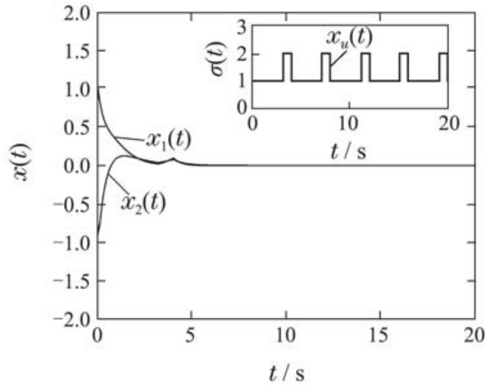

Fig. 3 The switching signal and the trajectory of the switched system

5 Conclusions

In this paper, we have addressed the iISS problem for the switched nonlinear systems. The existence of unstable modes in switched systems makes the work more challenging and meaningful. A Lyapunov-like function with indefinite derivative is introduced, which produces a relaxed iISS result. Using the average dwell time technique, some iISS conditions have been provided. An example of second-order switched nonlinear systems has been provided to show the effectiveness of the main results.

References:

[1] LIBERZON D. *Switching in Systems and Control* [M]. Boston, MA: Birkhäuser, 2003.

[2] LEE S H, KIM T H, LIM J T. A new stability analysis of switched systems [J]. *Automatica*, 2000, 36(6): 917 – 922.

[3] ALEKSANDROV A Y, CHEN Y, PLATONOV A V, et al. Stability analysis for a class of switched nonlinear systems [J]. *Automatica*, 2011, 47(10): 2286 – 2291.

[4] SONTAG E D. Smooth stabilization implies coprime factorization [J]. *IEEE Transactions on Automatic Control*, 1989, 34(4): 435 – 443.

[5] SONTAG E D. Comments on integral variants of ISS [J]. *Systems & Control Letters*, 1998, 34(1): 93 – 100.

[6] MULLER M A, LIBERZON D. Input/output-to-state stability and state norm estimators for switched nonlinear systems [J]. *Automatica*, 2012, 48(9): 2029 – 2039.

[7] ZONG G D, HOU L L, XU S Y. Robust $l_2 - l_\infty$ state feedback control for uncertain discrete-time switched systems with mode-dependent time-varying delays [J]. *Asian Journal of Control*, 2010, 12(4): 568 – 573.

[8] HOU L L, ZONG G D, WU Y Q, et al. Exponential $l_2 - l_\infty$ output tracking control for discrete-time switched system with time-varying delay [J]. *International Journal of Robust and Nonlinear Control*, 2012, 22(11): 1175 – 1194.

[9] WANG Y E, SUN X M, SHI P. Input-to-state stability of switched nonlinear systems with time delays under asynchronous switching [J]. *IEEE Transactions on Cybernetics*, 2013, 43(6): 2261 – 2265.

[10] SUN X M, DU S L, SHI P. Input-to-state stability for nonlinear systems with large delay periods based on switching techniques [J]. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 2014, 61(6): 1789 – 1800.

[11] DASHKOVSKIY S, MIRONCHENKO A. Input-to-state stability of nonlinear impulsive systems [J]. *SIAM Journal on Control and Optimization*, 2013, 51(3): 1962 – 1987.

[12] KELLETT C M, WIRTH F R, DOWER P M. Input-to-state stability, integral input-to-state stability, and unbounded level sets [J]. *IFAC Proceedings Volumes*, 2013, 46(23): 38 – 43.

[13] NING C Y, HE Y, WU M, et al. Input-to-state stability of nonlinear systems based on an indefinite Lyapunov function [J]. *System & Control Letters*, 2012, 61(12): 1254 – 1259.

[14] KHALIL H K. *Nonlinear Systems* [M]. 3rd ed. Upper Saddle River, NJ: Prentice-Hall, 2002.

作者简介:

赵海娟 (1990–), 女, 硕士研究生, 目前研究方向为切换非线性系统, E-mail: zhj52108@163.com;

宗广灯 (1976–), 男, 教授, 博士生导师, 目前研究方向为切换系统、输入状态稳定、网络控制、鲁棒控制及其应用等, E-mail: lovelyletian@gmail.com.