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Generalized Moment Theorem and Its Application to Distributed-Parameter Control Systems

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Abstract

In this paper, we generalize the original moment theorem to the problem of existence of a linear operator with minimum norm under some constraint conditions in an abstract normed space. The results are applied to a class of largest controllable set and optimal control problems for distributed parameter systems.

I. Introduction

The original moment problem in functional analysis was proposed by Banach in 1932 [1]. Then, Krein generalized Banach's results to the problem of existence of a linear functional with minimum norm under some constraint conditions in an abstract normed space. Krasovskii applied Krein's results to the time-optimal control problem for lumped-parameter systems in 1957. Subsequently, a series of papers have appeared[2][3][4][5][6][7][8].

II. Generalized Moment Theorem

We consider the following generalized moment problem: Given a linear operator ϕ from B-space U onto B-space X, a point $\tilde{x}\in X$ and a real number $L>0$, find a $u\in U$ satisfying

$$\|u\|_U\leq L \tag{2.1}$$

such that

$$\phi u=\tilde{x} \tag{2.2}$$

For this problem, we can prove the following basic theorem.

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Generalized moment theorem Assume that

1) ϕ is a bounded operator with domain U and range a closed subset of X;

2) $\phi^* x^* = 0_{U^*}$ implies $x^* = 0_{X^*}$, where 0_{X^*} and 0_{U^*} are the zero elements in B-space X^* and U^* respectively;

Then a necessary and sufficient condition for the existence of a solution to the generalized moment problem is that

$$\langle \tilde{x}, x^* \rangle \leq L \|\phi^* x^*\|_{U^*} \tag{2.3}$$

for all $x^* \in X^*$, where "$*$" denotes duality.

Proof Under assumptions 1) and 2), we have the following basic facts:

(i) ϕ^* is determined uniquely as a bounded linear operator with domain X^* by assumption 1).

(ii) the Fredholm Selection holds for ϕ by assumption 1)[1], that is

$$R(\phi) = (N(\phi^*))^{\perp}, \tag{2.4}$$

where $R(\phi)$ denotes the range of ϕ and

$$N(\phi^*) = \{ x^* \in X^*(\Omega): \phi^* x^* = 0_{U^*} \} \tag{2.5}$$

$$(N(\phi^*))^{\perp} = \{ x \in X(\Omega): \langle x, x^* \rangle = 0,\ x^* \in N(\phi^*) \}. \tag{2.6}$$

It follows from assumption 2) that $N(\phi^*)$ only includes the zero element 0_{X^*}.

So according to (2.4) and (2.6), $R(\phi)$ is the whole space X.

Now, let us consider the set

$$Q = \{ x \in X:\ x = \phi u,\ \|u\|_U \leq L,\ u \in U \}. \tag{2.7}$$

It is easy to verify that Q is a convex set with interior points by assumptions and dasic fact (ii). Now, we want to prove that Q is closed:

It is only necessary to prove that $\tilde{x} \in Q$ when any sequence $\{x_n\}$ in Q converges to $\tilde{x} \in X$. In fact, according to the definition of Q, we have $u_n \in S_U = \{ u \in U: \|u\|_U \leq L \}$ for every x_n: $n = 1, 2, \cdots$, such that $\phi u_n = x_n$. Since $x_n \to \tilde{x}$ as $n \to \infty$, so

$$\phi u_n \to \tilde{x} \text{ as } n \to \infty. \tag{2.8}$$

Since S_U is a bounded set, there exists a subsequence $\{u_{n_k}\}$ in $\{u_n\}$ which converges weakly to some $\tilde{u} \in U$. In addition, since S_U is a closed convex set, S_U is weak-closed, therfore $\tilde{u} \in S_U$. Hence, we have

$$\langle u_{n_k}, u^* \rangle \longrightarrow \langle \tilde{u}, u^* \rangle \qquad \text{as } K \longrightarrow \infty$$

for any $u^* \in U^*$. Evidently

$$\langle u_{n_k}, \phi^* x^* \rangle \longrightarrow \langle \tilde{u}, \phi^* x^* \rangle \qquad \text{as } K \longrightarrow \infty$$

for rny $x^* \in X^*$. Therefore

$$\langle \phi u_{n_k}, x^* \rangle \longrightarrow \langle \phi\tilde{u}, x^* \rangle \qquad \text{as } K \longrightarrow \infty$$

for auy $x^* \in X^*$. But

$$\phi u_{n_k} \longrightarrow \tilde{x} \qquad \text{as } K \longrightarrow \infty$$

by (2.8). So

$$\langle \tilde{x}, x^* \rangle = \langle \phi\tilde{u}, x^* \rangle$$

for any $x^* \in X^*$. Hence $\tilde{x} = \phi u$ and $\tilde{x} \in Q$ since $\tilde{u} \in S_U$.

Now let us couplete the proof of the theorem. We note that the existence of a solution to the problem is equivalent to $\tilde{x} \in Q$. In view of the fact that Q is a closed convex subset with interior points in X as shown above, it follows from the separation theorem [9][1], a necessary and sufficient condition for $\tilde{x} \in Q$ is that

$$\langle \tilde{x}, x^* \rangle \le \sup_{\phi u \in Q} \langle x^*, \phi u \rangle = \sup_{\|u\|_U \le L} \langle x^*, \phi u \rangle \quad \text{for any} \quad x^* \in X^*.$$

By basic fact (i), the above formulation can be rewritten as follows:

$$\sup_{\|u\|_U \le L} \langle x^*, \phi u \rangle = \sup_{\|u\|_U \le L} \langle \phi^* x^*, u \rangle \le \sup_{\|u\|_U \le L} \|u\|_U \|\phi^* x^*\| \le$$
$$\le L \|\phi^* x^*\|_{U^*}.$$

This means that (2.3) holds and the proof is complete.

Remark 2.1 Note that ϕ is a bounded operator with domain U, so $R(\phi)$ is closed $\longleftrightarrow R(\phi^*)$ is closed [1]. We also know that if there exists a real number $\alpha > 0$ such that $\|\phi^* x^*\| \ge \alpha \|x^*\|$ $\forall x^* \in X^*$, then $R(\phi^*)$ is closed [10].

The following theorem is useful in the application of the generalized moment theorem to distributed-parametr control systems.

Theorem 1 Assume that the assumptions of the generalized moment theorem hold. Then for any given $\tilde{x} \in X$, we have:

1) there exists at least one $u_0 \in U$ such that $\phi u_0 = \tilde{x}$ and

$$\|u_0\|_U = 1 / \min_{\langle \tilde{x}, x^* \rangle = 1} \|\phi^* x^*\|_{U^*} ; \tag{2.9}$$

2) the element u_0 in 1) is the solution with minimum norm. In other words, for any u which satisfies $\phi u=\tilde{x}$ we have $\|u_0\|_U \leq \|u\|_U$.

Proof According to Remark 2.1, the set

$$S=\{u^*:\phi^* x^*=u^*,\ x^*\in X^*\} \tag{2.10}$$

is closed. It is easy to verify that the set S is a nontrivial closed subspace in reflexive B-space U^* by assumption 2) of the generalized moment theorem and the assumptions of this theorem. Therefore S itself is a reflexive B-space. For any given $\tilde{x}\in X$, we can define a functional $\tilde{u}$ on S as follows

$$\tilde{u}(u^*)=\langle \tilde{x}, x^*\rangle,\ \phi^* x^*=u^*,\ \forall x^*\in X^* \tag{2.11}$$

since there is a one to one correspondence between x^* and u^* according to assumption 2) of the generalized moment theorem. Hence

$$\|\tilde{u}\|=\sup_{\|u^*\|_S=1}\langle \tilde{x}\in X^*\rangle, x^*\in X^*. \tag{2.12}$$

But, by the Hahn-Banach Theorem, we see that the continuous linear functional $\tilde{u}$ on the subspace S of U^* can be extended to a continuous linear functional u_0 on U^* with the same norm and $u_0\in U$ since U is a reflexive B-space. In other words, we have: a) $\|u_0\|=\|\tilde{u}\|$ and b) $u_0(u^*)\triangleq\langle u_0, u^*\rangle=\tilde{u}(u^*)\triangleq\langle \tilde{u}, u^*\rangle$, $u^*\in S$. By (2.11) and b) we have

$$\langle u_0, u^*\rangle=\langle \tilde{x}, x^*\rangle,\ \forall x^*\in X^*,\ u^*\in S.$$

It follows that

$$\langle u_0, \phi^* x^*\rangle=\langle \tilde{x}, x^*\rangle,\quad \langle \phi u_0, x^*\rangle=\langle \tilde{x}, x^*\rangle,\ \forall x^*\in X^*$$

hence,

$$\phi u_0=\tilde{x} \tag{2.13}$$

Therefore, by (2.12) and a), we obtain

$$\|u_0\|=\sup_{\|u^*\|S=1}\langle \tilde{x}, x^*\rangle,\ x^*\in X^* \tag{2.14}$$

Since the unit sphere of a reflexive B-space is weakly compact, we can change "sup" into "max" in (2.14), and moreover, (2.14) can be rewritten as follows

$$\|\tilde{u}\| = \max_{\phi^* x^*} \langle \tilde{u}, \phi^* x^* \rangle / \|\phi^* x^*\|_{U^*} = \max_{x^*} \langle \tilde{x}, x^* \rangle / \|\phi^* x^*\|_{U^*} =$$

$$\max_{\langle \tilde{x}, x^* \rangle = 1} \|\phi^* x^*\|^{-1} = 1 / \min_{\langle \tilde{x}, x^* \rangle = 1} \|\phi^* x^*\|_{U^*} . \qquad (2.15)$$

It follows from a), b), (2.13) and (2.15) that the conclusion 1) of the theorem holds. To prove conclusion 2) of the theorem, we note that for any $u \in U$ satisfying $\phi u = \tilde{x}$ and $x^* \in X^*$ we have

$$\langle \tilde{x}, x^* \rangle = \langle \phi u, x^* \rangle = \langle u, \phi^* x^* \rangle \leq \|u\|_U \|\phi^* x^*\|_{U^*} ,$$

namely $\|u\|_U \geq \langle \tilde{x}, x^* \rangle / \|\phi^* x^*\|_{U^*}$, $x^* \in X^*$.

So

$$\|u\|_U \geq \max_{\langle \tilde{x}, x^* \rangle = 1} \langle \tilde{x}, x^* \rangle / \|\phi^* x^*\|_{U^*} = 1 / \min_{\langle \tilde{x}, x^* \rangle = 1} \|\phi^* x^*\|_{U^*} = \|u_0\|_U .$$

Thus, the proof for the theorem is complete.

In fact, the assumed conditions of the generalized moment theorem can be relaxed. We have the following theorems and remarks.

Theorem 2 Assume that 1) X and U are reflexive B-space; 2) ϕ is a closed and densely defined linear operator on U and its range is a closed subset of X; 3) $\phi^* x^* = O_{U^*}$ implies $x^* = O_{X^*}$ then, for any given $\tilde{x} \in X$ we have

a) there exists a sequence $\{u_n\} \subset U$ such that $\phi u_n \to x$ weakly and

$$\lim_{n \to \infty} \|u_n\| = (1 / \min_{\langle \tilde{x}, x^* \rangle = 1} \|\phi^* x^*\|) \triangleq \lambda, \quad x^* \in D(\phi^*)$$

b) for any u which satisfies $\phi u = \tilde{x}$, we have $\|u\| \geq \lambda$.

Remark 2 In addition to the assumptions of theorem 2, if $R(\phi^*)$ is dense in U^* then we must have a unique solution u_0 satisfying $\phi u_0 = \tilde{x}$ and $\|u_0\| = \lambda$.

Theorem 3 Assume that U is a reflexive B-space and ϕ is a bounded linear operator with domain U and range a closed subset of B-space X. Then a necessary and sufficient condition for the existence of a solution to the generalized moment problem is that

$$\langle \tilde{x}, \hat{x}^* \rangle \leq L \|\phi^* x^*\|, \quad x^* \in X^*$$

for any $\hat{x}^* \in \hat{X}^* = (x^* + z : z \in N(\phi^*),\ x^* \in X^*) = X^* / N(\phi^*)$ —— the quotient space of X^* on $N(\phi^*)$.

Remark 3 Theorm 3 means that we can discuss the optimal control problem even when the system is not approximately controllable (see (3.2) below and [10]).

The proofs of theorems 2, 3 and remark 2 will be given in another paper.

Ⅲ. Discussion of Largest Null Controllable Set

In this section, we shall discuss the problem of the largest null (or weak null) controllable set for a class of distributed-parameter systems whose solutions can be expressed in the form of

$$x(t)=\phi_{t-t_0}x_0+\int_{t_0}^{t} H(t,s)u(s)ds,\quad t\in[t_0,T],\quad x(t_0)=x_0. \tag{3.1}$$

For any given $t_a\in[t_0,T]$, we have

$$\tilde{x}=x(t_a)-\phi_{t_a-t_0}x_0=\int_{t_0}^{t_a} H(t,s)u(s)ds\triangleq\phi u \tag{3.2}$$

According to definition of the largest null controllable set [12] and Q the largest null (or weak null) controllable set D^* is given by

$$D^*=\{\,x_0:-\phi_{t_a-t_0}x_0\in Q\,\} \tag{3.3}$$

Now we introduce some notation:

$$\Omega_0=\{\,\zeta\in X:\|\zeta\|_x=1\,\},K(\zeta)=\inf_{\|x^*\|_{X^*}=1} L\|\phi^*x^*\|_{U^*}/\langle x^*,\zeta\rangle. \tag{3.4}$$

Theorem 4 Assume that the assumptions of the generalized moment theorem hold and control constraint (2.1) is imposed. Then the set

$$Q=\{\tilde{x}:\|\tilde{x}\|_x\le K(\tilde{x}/\|\tilde{x}\|_x)\}+\{0_x\},\ \text{where } K(\cdot)\text{ is given by (3.4)}. \tag{3.5}$$

Proof According to the generalized moment theorem, a necessary and sufficient condition for $\tilde{x}\in Q$ is

$$\langle x^*,\tilde{x}\rangle\le L\|\phi^*x^*\|_{U^*},\quad x^*\in X^*. \tag{3.6}$$

Obviously, $\tilde{x}=0_x$ satisfies (3.6). After setting $\zeta_0=\tilde{x}/\|\tilde{x}\|_x\in\Omega_0$ for the case where $\tilde{x}\neq 0_x$, (3.6) can be rewritten as

$$\|\tilde{x}\|_x \langle x^*, \zeta_0 \rangle \le L\|\phi^* x^*\|_{U^*} \; x^* \in X^*, \text{ namely } \|\tilde{x}\|_x \le \inf_{\|x^*\|=1} L\|\phi^* x^*\|_{U^*}/\langle x^*, \zeta_0 \rangle. \tag{3.7}$$

Thus, the theorem is proved with (3.4) taken into account.

In what follows, we consider the particular case where X and U are H-space.

Theorem 5 Under the assumpptons of the generalized moment theorem and that X, U are real Hilbert spaces, the largest null (or weak null) controllable set with control constraint (2.1) is given by

$$D^* = \{x_0 : -\phi_{t_\alpha - t_0} x \in Q\}, \quad \text{where } Q = \{\tilde{x} \in X: \langle \tilde{x}, \mu \tilde{x} \rangle \le 1\}, \mu = (\phi\phi^*)^{-1}/L^2, \tag{3.8}$$

and in the case of (3.2)

$$\phi\phi^*(\cdot) = \int_{t_0}^{t_\alpha} H(t_\alpha, s) H^*(t_\alpha, s)(\cdot) ds. \tag{3.9}$$

Proof Obviously, by assumptions of the theorem, $(\phi\phi^*)$ is a self-adjoint bounded linear operator on X. We see that $(\phi\phi^*)^{-1}$exists and hence $\phi\phi^*$ and $(\phi\phi^*)^{-1}$ are positive operators on X [11]. It is easy to verify that

$$(\langle \tilde{x}, x \rangle)^2 = (\langle \mu\tilde{x}, \mu^{-1}x \rangle)^2 = (\langle \mu^{\frac{1}{2}} x, \mu^{-\frac{1}{2}} x \rangle)^2 \le$$

$$\le \|\mu^{\frac{1}{2}} \tilde{x}\|^2 \|\mu^{-\frac{1}{2}} x\|^2 = \langle \mu\tilde{x}, \tilde{x} \rangle \langle \mu^{-1}x, x \rangle, \quad \forall \tilde{x}, x \in X. \tag{3.10}$$

By the results in II, a necessary and sufficient condition for $\tilde{x} \in Q$ is that $|\langle x, \tilde{x} \rangle_x| \le L\|\phi^* x\|_U$. But $\|\phi^* x\|_U = (\langle x, \phi\phi^* x \rangle_x)^{\frac{1}{2}}$. Hence a necessary and sufficient condition for $\tilde{x} \in Q$ is

$$|\langle x, \tilde{x} \rangle_x| \le (\langle x, \mu^{-1}x \rangle_x)^{\frac{1}{2}}, \quad x \in X. \tag{3.11}$$

We can also prove that a necessary and sufficient condition for (3.11) to hold is

$$\langle \tilde{x}, \mu\tilde{x} \rangle_x \le 1. \tag{3.12}$$

In fact, we only need to consider the case where $\tilde{x} \ne 0_x$. When (3.11) holds, we can octain (3.12) by substituting $x = \mu\tilde{x}/\langle \tilde{x}, \mu\tilde{x} \rangle_x$

into (3.11), since μ is a positive operator. Conversely, when (3.12) holds, we can obtain (3.11) by (3.10). So we have (3.8) and the theorem is proved.

Ⅳ. Examples

In this section, we shall apply the foregoing results to two specific optimal control problems.

Example 1 Consider the system described by

$$z_{tt}=z_{\xi\xi}+u,\ z(0,t)=z(1,t)=0,\ z(\xi,0)=z_0(\xi),\ \partial z(\xi,0)/\partial t=z_1(\xi), \tag{4.1}$$

where $0\leq t\leq 1$, $0\leq\xi\leq 1$. Let $U=L_2([0,1],\ L_2[0,1])$. The optimal control problem is to find $u\in U$ such that the corresponding solution z_u of (4.1) satifies $z_u(\xi,1)=0$, $\partial z_u(\xi,1)/\partial t=0$, and minimizes the functional $J(u)=\|u\|_U$.

Setting $z_t=y$, $A=\partial^2/\partial^2\xi$ with domain $D(A)=(H^2[0,1]\cap H_0{}^1[0,1])\subset L_2[0,1]$, (4.1) can be rewritten as

$$W_t=\mathbb{A}W+Bu,\ W=(w_1,w_2)^T=(z,y)^T,\ \mathbb{A}=\begin{pmatrix}0 & I\\ A & 0\end{pmatrix},B=\begin{pmatrix}0\\ I\end{pmatrix}$$

I – identity operator. (4.2)

Let us introduce a Hilbert space $H=D(A^{\frac{1}{2}})\times L_2[0,1]$ so that for w, $\overline{w}\in H$,

$$\langle w,\overline{w}\rangle_H=\langle A^{\frac{1}{2}}w_1,\ A^{\frac{1}{2}}\overline{w}_1\rangle_{L_2}+\langle w_2,\overline{w}_2\rangle_{L_2}=\int_0^1 w_{1\xi}\,\overline{w}_{1\xi}d\xi$$
$$+\int_0^1 w_2\overline{w}_2 d\xi,\quad w=(w_1,w_2)^T,$$

$\overline{w}=(\overline{w}_1,\overline{w}_2)^T$. (4.3)

Operator $\mathbb{A}$ with domain $D(\mathbb{A})=D(A)\times D(A^{\frac{1}{2}})$ generates a strongly continuous semi-group $\{T_t\}$ on H [10]:

$$T_t w=\begin{pmatrix}\sum\limits_1^{\infty}2(\langle w_1,\phi_n\rangle_{L_2}\cos n\pi t+\frac{1}{n\pi}\langle w_2,\phi_n\rangle_{L_2}\sin n\pi t)\phi_n\\ \sum\limits_1^{\infty}2(-n\pi\langle w_1,\phi_n\rangle_{L_2}\sin n\pi t+\langle w_2,\phi_n\rangle_{L_2}\cos n\pi t)\phi_n\end{pmatrix},$$

$$w=\begin{pmatrix}w_1\\ w_2\end{pmatrix},\quad \phi_n=\sin n\pi\xi. \tag{4.4}$$

Therfore, (4.2) has a solution of the form for $w_0=(z_0(\cdot),\ z(\cdot))^T$, $w(1)=(z(\cdot,1),y(\cdot,1))^T$:

$$w(t)=T_t w_0+\int_0^t T_{t-\tau}Bu(\tau)d\tau,\quad 0\le t\le 1$$

$$\text{and } w(1)=T_1 w_0+\int_0^1 T_{1-\tau}Bu(\tau)d\tau. \tag{4.5}$$

When $z_0\in D(A)$ and $z_1\in H_0^1[0,1]$, (4.5) is the strong solution. So the optimal control problem is equivalent to finding the solution u_0 with minimum norm of the generalized moment prodlem for

$$\tilde{w}=-T_1 w_0=\int_0^1 T_{1-\tau}Bu(\tau)d\tau\triangleq\phi u. \tag{4.6}$$

We can see that $U=U^*$, $T_t^*=T_{-t}$ and $B^*=[0,I]$. So it can be readily shown that

$$u^*=\phi^* w^*=B^*T_{1-t}^* w=\sum_1^\infty 2((-1)^{n+1}n\pi\ \langle w_1^*,\ \phi_n\rangle_{L_2}\sin n\pi t$$

$$+(-1)^n\langle w_2^*,\phi_n\rangle_{L_2}\cos n\pi t)\phi_n, \tag{4.7}$$

$$\|\phi^* w^*\|_{U^*}^2\ \|=\|\phi^* w\|_U=\sum_1^\infty(n^2\pi^2\langle w_1^*,\phi_n\rangle_{L_2}^2+\langle w_2^*,\phi_n\rangle_{L_2}^2)=\|w^*\|_H^2. \tag{4.8}$$

Since $\{T_t\}$ is a strongly continuous semigroup and by (4.8) and Remark 2.1, we can see that the assumptons of the generalized moment theorem hold.

Noting that $\tilde{w}=-T_1w_0$, $w_0=(z_0,z_1)^T$, and by (4.4), we can verify that the minimum element $w_0^*=(w_{01}^*,\ w_{02}^*)^T$ satisfies

$$2\sum_1^\infty(-1)^{n+1}n^2\pi^2\langle w_{01}^*,\phi_n\rangle_{L_2}\langle z_0,\phi_n\rangle_{L_2}$$

$$+2\sum_1^\infty(-1)^{n+1}\langle w_{02}^*,\phi_n\rangle_{L_2}\langle z_1,\phi_n\rangle_{L_2}=1. \tag{4.9}$$

$$\left.\begin{aligned}&\sum_{1}^{\infty} n^2\pi^2 \langle w_{01}^*, \phi_n \rangle_{L_2} \phi_n + \beta \sum_{1}^{\infty} (-1)^{n+1} n^2\pi^2 \langle z_0, \phi_n \rangle_{L_2} \phi_n = 0,\\ &\sum_{1}^{\infty} \langle w_{02}^*, \phi_n \rangle_{L_2} \phi_n + \beta \sum_{1}^{\infty} (-1)^{n+1} \langle z_1, \phi_n \rangle_{L_2} \phi_n = 0.\end{aligned}\right\} \tag{4.10}$$

where β is the Lagrange Multiplier. By (4.9), (4.10), we can obtain

$$\beta = -1/a, \quad a = 2 \sum_{1}^{\infty} (n^2\pi^2 \langle z_0, \phi_n \rangle_{L_2}^2 + \langle z_1, \phi_n \rangle_{L_2}^2) \tag{4.11}$$

and

$$u_0^* = (2/a) \sum_{1}^{\infty} \Big(n\pi \langle z_0, \phi_n \rangle_{L_2} \sin n\pi t - \langle z_1, \phi_n \rangle_{L_2} \cos n\pi t \Big)\phi_n,$$

$$\|u_0^*\|_{U^*}^2 = 1/2a. \tag{4.12}$$

Citing the results in [5], the unique optimal control is given by

$$u_0 = (u_0^*/\|u_0^*\|) = 4 \sum_{1}^{\infty} \Big(n\pi \langle z_0, \phi_n \rangle_{L_2} \sin n\pi t - \langle z_1, \phi_n \rangle_{L_2} \cos n\pi t \Big) \sin n\pi\xi$$

and

$$J(u_0) = \| u_0 \|_U = 1/\|u_0^*\| = \sqrt{2a}\,. \tag{4.13}$$

Remark 4.1 Now, for $z_0 \in D(A)$ and $z_1 \in H_0{}^1[0,1]$, let us find the largest null controllable set $D^*(0,1,L)$ with control constrint (2.1). we can verify that

$$\phi\phi^*(w^*) = \int_0^1 \begin{pmatrix} \displaystyle\sum_{1}^{\infty} (2/n\pi) \langle u^*, \phi_n \rangle_{L_2} \sin n\pi\tau \; \phi_n \\ \displaystyle\sum_{1}^{\infty} - \langle u^*, \phi_n \rangle_{L_2} \cos n\pi\tau \; \phi_n \end{pmatrix} d\tau =$$

$$=\begin{bmatrix} \sum_{1}^{\infty}(-1)^{n+1}\langle w_1^*, \phi_n\rangle_{L_2}\phi_n \\ \sum_{1}^{\infty}(-1)^{n+1}\langle w_2^*, \phi_n\rangle_{L_2}\phi_n \end{bmatrix},$$

$$\triangleq Y=(y_1,y_2)^T=\left(2\sum_{1}^{\infty}\langle y_1,\phi_n\rangle_{L_2}\phi_n,\ 2\sum_{1}^{\infty}\langle y_2,\phi_n\rangle_{L_2}\phi_n\right)^T, \tag{4.14}$$

where u^* is expressed by (4.7). According to (4.14), we can define $(\phi\phi^*)^{-1}$ and obtain

$$(\phi\phi^*)^{-1}\tilde{w}$$

$$=\left(\sum_{1}^{\infty}4(-1)^{n+1}\langle\tilde{w}_1,\phi_n\rangle_{L_2}\phi_u,\ \sum_{1}^{\infty}4(-1)^{n+1}\langle\tilde{w}_2,\phi_n\rangle_{L_2}\phi_n\right)^T.$$

So

$$\langle(\phi\phi^*)^{-1}\tilde{w},\ \tilde{w}\rangle_H$$

$$=\int_0^1\sum_{1}^{\infty}8(n^2\pi^2(-1)^{n+1}\langle\tilde{w}_1,\phi_n\rangle_{L_2}^2(\cos n\pi\xi)^2$$

$$+(-1)^{n+1}\langle\tilde{w}_2,\phi_n\rangle\phi_n^2)d\xi$$

$$=\sum_{1}^{\infty}4(n^2\pi^2(-1)^{n+1}\langle\tilde{w}_1,\phi_n\rangle_{L_2}^2+(-1)^{n+1}\langle\tilde{w}_2,\phi_n\rangle_{L_2}^2). \tag{4.15}$$

Noting that $\tilde{w}=-T_1w_0$ with (4.15) taken into account and using the results of Theorem 5, we obtain

$$D^*(0,1,L)=\{w_0=(z_0,z_1)^T:4\sum_{1}^{\infty}(n^2\pi^2\langle z_0,\phi_n\rangle_{L_2}^2+\langle z_1,\phi_n\rangle_{L_2}^2\le L^2\}. \tag{4.16}$$

This means that: If the optimal control problem is to find a control minimizing some cost functional $J_1(u)$ with the control constraint $\|u\|_U \leq L$, then for given z_0 and z_1, it follows from (4.11) and (4.16) that in order to have an optimal solution, L, z_0, and z_1 must satisfy $L \geq \sqrt{2a}$. This conclusion is the same as that from (4.13). But now we obtain the result without finding the optimal control with cost functional $\|u\|_U$.

Example 2 Consider the system given by

$$z_t = z_{\xi\xi} + u, \quad z(0, t) = z(1, t) = 0, \quad z(\xi, 0) = z_0(\xi), \quad 0 \leq \xi \leq 1, \quad 0 \leq t \leq 1. \tag{4.17}$$

Let $U = L_2([0, 1], L_2[0, 1])$. The optimal control problem is to find a $u \in U$ such that the corresponding solution z_u of (4.17) satisfies $z_u(\xi, 1) = z_1(\xi)$ and minimizes the functional $J(u) = \|u\|_U$. (4.17) can be rewritten as follows

$$z_t = Az + Bz, \quad A = \partial^2/\partial\xi^2, \quad D(A) = H_0^1[0, 1] \cap H^2[0, 1], \quad z \in D(A), \quad B = I. \tag{4.18}$$

Operator A generates a strongly continuous semigroup $\{T_t\}$ on $L_2[0,1]$ [10]:

$$(T_t z)(\xi) = \sum_1^\infty 2e^{-n^2\pi^2 t} \sin n\pi\xi \int_0^1 \sin n\pi y \, z(y) dy$$

and (4.18) has a solution of the form:

$$z(\xi, t) = T_t z_0 + \int_0^t T_{t-\tau} Bu(\tau) d\tau, \quad \text{or} \quad z_1(\xi) = z(\xi, 1) = T_1 z_0 + \int_0^1 T_{1-\tau} Bu(\tau) d\tau.$$

Hence the optimal control problem is equivalent to finding the solution u_0 with minimum norm of the generalized moment problem. In order to satisfy the assumptions of the generalized moment theorem and Theorem 1, we consider the subspace $V \subset H_0^1[0, 1]$ in $L_2[0, 1]$ and endow the norm in V as follows

$$\|z\|_V^2 = \sum_1^\infty n^2\pi^2 \left(\int_0^1 \sin n\pi y \, z(y) dy \right)^2, \quad z \in V,$$

so $$\|z^*\|_{V^*}^2 = \sum_1^\infty (1/n^2\pi^2) \left(\int_0^1 \sin n\pi y \, z^*(y) dy \right)^2, \quad z^* \in V^*.$$

Hence, we obtain

$$u^* = \phi^* z^* = B^* T_{1-t}^* z^* = T_{1-t}^* z^* = \sum_{1}^{\infty} 2e^{-n\pi(1-t)} \langle z^*, \phi_n \rangle_{L_2} \phi_n, \ \phi_n = \sin n\pi\xi ,$$

$$\|\phi^* z^*\|_{U^*}^2 \geq (1 - e^{-2\pi^2}) \|z^*\|_{Y^*}^2 .$$

Analogous to what we did in Example I we can obtain

$$u_0 = (1/ab) \sum_{1}^{\infty} (n^2\pi^2 e^{-n\pi(1-t)} / 1 - e^{2n^2\pi^2}) \langle \tilde{z}, \phi_n \rangle_{L_2} \phi_n, \ J(u_0) = 1/\sqrt{b} ,$$

$$a = \sum_{1}^{\infty} (n^2\pi^2 / 1 - e^{-2n^2\pi^2}) \langle \tilde{z}, \phi_n \rangle_{L_2}^2 ,$$

$$b = (1/4a^2) \sum_{1}^{\infty} (n^3\pi^3 (1 - e^{-2n\pi}) / (1 - e^{-2n^2\pi^2})^2) \langle \tilde{z}, \phi_n \rangle_{L_2}^2 .$$

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