Optimal Local Output Feedback Control in Discrete-time Decentralized Systems

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Abstract

In this paper, the derivation of optimal output feedback control for the discrete, time-invariant decentralized system is discussed. A numerical example illustrates the application of the theory.

1. Introduction

There has been an increasing interest in the study of decentralized control systems recently[1-5]. When control theory is applied to solve problems of large scale systems, an important feature called decentralization arises, i. e. many small stations which may be widely distributed over a large geographic area making the centralized control impossible. Each station is allowed to observe only local system outputs and control only local inputs. Since each local station is interconnected with each other, all the local controllers are involved in controlling the same large scale system. In this paper. the derivation of optimal output feedback control with respect to infinite-time quadratic performance criterion for the discrete time-invariant decentralized spstem is discussed. One convergent computation algorithm is proved and proposed, and a numerical example is illustrated.

2. The Statement of the problem and the Analysis

Consider the discrete, time-invariant decentralized control system described by

$$x(k+1) = Ax(k) + \sum_{i=1}^{N} B_i u_i(k) \tag{1}$$

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$$y_i(k) = C_i x(k) \qquad i=1,\ 2,\ \cdots,\ N$$

where

$$x(k) \in R^n,\ y_i(k) \in R^{m_i},\ u_i(k) \in R^{r_i},\ r = \sum_{i=1}^{N} r_i,\ m = \sum_{i=1}^{N} m_i,$$

and A, B_i, C_i are constant matrices with appropriate dimensions. The control $u_i(k)$, which minimizes the performance index,

$$J = \frac{1}{2} E\left\{ \sum_{k=0}^{\infty} [x^T(k+1)Qx(k+1) + \sum_{i=1}^{N} u_i^T(k) R_i u_i(k)] \right\} \tag{2}$$

where

Q is symmetric positive semidefinite,

R_i, $i=1,2,\cdots,N$, is symmetric positive definite,

is restricted to be the constant local output feedback form:

$$u_i(k) = -F_i\, y_i(k) = -F_i C_i x(k) \qquad i=1,2,\cdots,N \tag{3}$$

where $F_i \in R^{r_i \times m_i}$, is the feedback gain matrix to be determined to achieve optimization.

Let

$$B \triangleq [B_1,\ B_2,\ \cdots,\ B_N], \qquad C \triangleq [C_1^T,\ C_2^T,\ \cdots,\ C_N^T]^T$$

$$P \triangleq \text{block diag } [F_1, F_2, \cdots, F_N]$$

$$R \triangleq \text{block diag } [R_1, R_2, \cdots, R_N];$$

then (1) can be rewritten as

$$x(k+1) = (A - BFC)x(k), \tag{4}$$

Let $\phi(k)$ be the fundamental transmission matrix of (4), then

$$x(k) = \phi(k)x(0) \tag{5}$$

where

$$\phi(k) = (A - BFC)^k,\ \phi(0) = I$$

substituting (3) and (4) into (2),

$$J(F) = \frac{1}{2} E\left\{ x^T(0) \left[\sum_{k=0}^{\infty} \phi^T(k)(Q + C^T E^T RFC)\phi(k) - Q \right] x(0) \right\}. \tag{6}$$

To remove the dependence of the solution on the initial condition, we use the assumption that $x(0)$ is uniformly distributed on the surface of n-dimensional unit spheres[6]. Then (6) yields

$$J(F)=\frac{1}{2n}\,\mathrm{tr}\left\{\left[\sum_{k=0}^{\infty}\phi^T(k)(Q+C^TF^TRFC)\phi(k)\right]-Q\right\}. \tag{7}$$

Let

$$S\triangleq\{F:|\lambda(A-BFC)|\leq 1-\varepsilon,\ \varepsilon\text{ is fininte and positive}\} \tag{8}$$

where $\lambda(A-BFC)$ denotes the eigenvalues of $(A-BFC)$. If (A, B, C) is output stabilizable, then there always exists an $\varepsilon>0$ such that S is not an empty set. Although S does not include all the stable output feedback gain matrices of system (1), ε may be selected to be small enough to insure that S includes almost every stable output feedback gain matrix to be considered. On the other hand, if ε is finite and extremely small, the stable output feedback matrices, which are not in S, may be regarded as critically stable because the system eigenvalues are too close to the unit circle. Hence, the optimization problem can be posed as

Find $F^*\in S$ such that $J(F^*)\leq J(F)$, $\forall F\in S$. (9)

3. Main Results

Let

$$P_0(F)\triangleq Q+C^TF^TRFC \tag{10}$$

$$P_j(F)=\sum_{k=0}^{j}\phi^T(k)P_0\phi(k)=(A-BFC)^TP_{j-1}(F)(A-BFC)+P_0(F) \tag{11}$$

It con be easily proved that the sequence $\{P_i(F),\ j\geq 0\}$ is convergent if $(A-BFC)$ is stable. Hence, (7) can be rewritten as (the constant $\frac{1}{n}$ has been dropped for simplicity, and $P_j(F)\rightarrow P(F)$ is assumed)

$$J(F)=\frac{1}{2}\mathrm{tr}\{P(F)-Q\} \tag{12}$$

where

$$P(F)=(A-BFC)^TP(F)(A-BFC)+P_0 \tag{13}$$

and

$$F\in S.$$

Theorem 1

Let (A,B,C) be output stabilizable. If Q is positive semidefinite and R is positive definite, then $J(F)$ has a minimum in S.

Proof:

By assumption, $Q\geq 0$ and $R>0$, we have

$$P_0(F)\geq 0$$

and

$$P(F)=\sum_{k=0}^{\infty}[(A-BFC)^K]^T P_0(F)(A-BFC)^K\geq 0 \quad \forall F\in S.$$

It implies $J(F)\geq 0$, $\forall F\in S$. Let $J^*\triangleq \inf_{F\in S} J(F)$, thus $J^*\geq 0$. Then there exists a sequence $\{F^j, j\geq 0\}$ in S such that $J(F^i)\geq J(F^j)$, $i\leq J$, and $J(F^j)\to J(F^*)$. Now, let F^0 be the first term of $\{F^j, j\geq 0\}$ and

$$S_0\triangleq\{F|F\in S, \text{ and } 0\leq J(F)\leq J(F^0)\}. \tag{14}$$

Thus, $\{F^j, j\geq 0\}$ is a subset of nonempty set S_0. Because S is closed and $J(F)$ is continuous over S, S_0 is a closed subset of S[7]. Suppose S_0 is not bounded, then there exists an unbounuded sequence $\{F^{\hat{j}}, \hat{j}\geq 0\}\subset S_0$ such that for some $\hat{j}$, $\|F^{\hat{j}}\|\to\infty$. By (10), (12) and (13), the following equation

$$J(F^j)\geq\frac{1}{2}\text{ tr }\{C^T F^{j^T} RF^jC\}$$

implies $J(F^j)$ is not bounded, which contradicts (14). Therefore S_0 is bounded.

By theorem 3-40 of [7], every infinite subset, say $\{F^j, j\geq 0\}$, of a closed bounded set, say S_0, has a limit in S_0. Therefore, $\exists F^*\in S_0$ such that

$$J(F^*)=\inf_{F\in S} J(F)\leq J(F) \quad \forall F\in S. \qquad \text{Q. E. D.}$$

Theorem 2

Let $F^*\triangleq$ block diag $[F_1^*, F_2^*, \cdots, F_N^*]$ be a $n\times m$ matrix of real constant and $(A-BF^*C)$ be stable, i.e. $F\in S$. If $J(F^*)$ is minimal over S, then

$$F_i^*=R_i^{-1}B_i^TP^*K^*L^*C_i^T(C_iL^*C_i^T)^{-1} \quad i=1,2,\cdots,N \tag{15}$$

where

$$K^*=A-BF^*C \tag{16}$$

$$P^*=K^{*^T}P^*K^*+C^TF^{*^T}RF^*C+Q \tag{17}$$

$$L^*=K^*L^*K^{*^T}+I \tag{18}$$

Proof:

Assume F and $F+\varepsilon\Delta F$ be in S. By (13)

$$\Delta P(F)=P(F+\varepsilon\Delta F)-P(F)$$

The infinite series solution of $\Delta P(F)$ is

$$\Delta P(F)=\sum_{k=0}^{\infty}[(A-BFC)^K]^T\{\varepsilon C^T\Delta F^T[RFC-B^TP(F+\varepsilon\Delta F)(A-BFC)]+$$
$$+\varepsilon[RFC-B^TP(F+\varepsilon\Delta F)(A-BFC)]^T\Delta FC+\varepsilon^2C^T\Delta F^T[B^TP(F+\varepsilon\Delta F)B$$
$$+R]\Delta FC\}(A-BFC)^K \tag{19}$$

and

$$\Delta J(F)\triangleq J(F+\varepsilon\Delta F)-J(F)=\frac{1}{2}\operatorname{tr}\ \{\Delta P(F)\}. \tag{20}$$

Substituting (19) into (20) and using identities on the trace yields

$$\Delta J(F)=\frac{1}{2}\operatorname{tr}\{2\,\varepsilon\, CL(F)[RFC-B^TP(F+\varepsilon\Delta F)(A-BFC)]^T\Delta F+$$
$$+\varepsilon^2\Delta F^T[B^TP(F+\varepsilon\Delta F)B+R]\Delta F[CL(F)C^T]\} \tag{21}$$

where
$$L(F)=\sum_{k=0}^{\infty}(A-BFC)^k[(A-BFC)^k]^T$$

or
$$L(F)=(A-BFC)L(F)(A-BFC)^T+I. \tag{22}$$

Then, as $\varepsilon\to 0$, accurate to the first order in ε,(21) can be written as

$$\Delta J(F)=\varepsilon\ \operatorname{tr}\{CL(F)\{RFC-B^TP(F)(A-BFC)]^T\Delta F\}. \tag{23}$$

Hence[8],

$$\frac{\partial J}{\partial F}=[RFC-B^TP(F)(A-BFC)]L(F)C^T. \tag{24}$$

Since F is restricted to be block diagonal, we have only the corresponding block diagonal entries in (24), i.e.

$$\left.\frac{\partial J}{\partial F}\right|_{\text{diag }F}\triangleq\text{block diag}\left[\frac{\partial J}{\partial F_1},\ \frac{\partial J}{\partial F^2}\cdots\frac{\partial J}{\partial F_N}\right] \tag{25}$$

where
$$\frac{\partial J}{\partial F_i}\triangleq[R_iF_iC_i-B_i^TP(F)(A-BFC)]\,L(F)C_i^T. \tag{26}$$

Setting
$$\left.\frac{\partial J}{\partial F}\right|_{\text{diag }F^*}=0$$

we can obtain the results. Q. E. D.

Theorem 3

Let $Q\geq 0$, $R>0$, $F^0\in S_0$ where S_0 is defined in (14), and

$$d(F)=-\left.\frac{\partial J}{\partial F}\right|_{\text{diag } F} \tag{27}$$

then $\exists\ \bar{a}>0$ such that

$$S_{0a}\triangleq\{F+\alpha d(F)\mid F\in S_0,\ \alpha\in[0,\bar{a}]\}\subset S. \tag{28}$$

Proof:

We prove it by contradiction.

Suppose $\forall a>0$ $S_{0a}\not\subset S$, then it is possibie to construct a monotone decreasing sequence a_j which converges of zero, and a sequence $\{F^j, j\geq 0\}$ such that

$$\mathrm{Max}|\lambda[A-B(F^j+a_jd(F^j))C]|\geq 1.$$

Since S_0 is closed and bounded (see the proof in theorem 1) $\{F^j, j\geq 0\}$ has a limit in S_0, say $\bar{F}$. By (13) and (22), $P(F^j)$, $L(F^j)$ are bounded and continuous on S_0. Subsequently, $d(F^j)$ exists and is continuous on S_0. Because $F^j+a_jd(F^j)\to\bar{F}$,

$$\mathrm{Max}|\lambda[A-B(F^j+a_jd(F^j))C]|\to\mathrm{Max}|\lambda(A-B\bar{F}C)|\geq 1$$

which contradicts $\bar{F}\in S_0$. Q. E. D.

Theorem 4

Let $Q\geq 0$, $R>0$ and $F^0\in S_0$, where S_0 is defined in (14), then there exists a sequence $\alpha_j\in(0,1)$, $j\geq 0$, and $F^*\in S$ such that $J(F^j)$ is monotonically decreasing and converges to $J(F^*)$ and

$$\left.\frac{\partial J}{\partial F}\right|_{\text{diag } F^j}\longrightarrow\left.\frac{\partial J}{\partial F}\right|_{\text{diag } F^*}=0$$

where

$$F^j=\text{block diag }[F_1^j,\ F_2^j,\cdots,F_N^j]\in S.$$

Furthermore,

$$F^{j+1}=F^j+\alpha_jd(F^j)\qquad j\geq 0$$

$$d(F^j)=-\left.\frac{\partial J}{\partial F}\right|_{\text{diag } F^j}.$$

Proof:

From theorem 3, $\forall F^j\in S_0$, $\exists a>0$, and $\alpha_j\in[0\ a]$ such that $F^j+\alpha_jd(F^j)\in S$. Substituting $(\varepsilon\Delta F)$ by $\alpha_jd(F^j)$ in equation (21) and using identities on the trace, we have

$$\Delta J(F^j) \triangleq J(F^j + \alpha_j d(F^j)) - J(F^j)$$

$$= \frac{1}{2}[-2\alpha_j A(F^j) + \alpha_j^2 B(F^j) + \alpha_j^2 C(F^j, \alpha_j)] \tag{29}$$

where

$$A(F^j) = \mathrm{tr}\ \{d^T(F^j) d(F^j)\} \tag{30}$$

$$B(F^j) = \mathrm{tr}\ \{d^T(F^j)[B^T P(F^j) B + R] d(F^j)(CLC^T)\} \tag{31}$$

$$C(F^j, \alpha_j) = \mathrm{tr}\{d^T(F^j)[B^T \Delta P(F^j) B] d(F^j)(CLC^T) + \frac{2}{\alpha_j} CL[-B^T \Delta P(F^j)(A - BF^j C)]^T d(F^j)\} \tag{32}$$

and

$$\Delta P(F^j) = P(F^j + \alpha_j d(F^j)) - P(F^j), \tag{33}$$

From (30), $\exists\ 0 < M_1 < \infty$ such that

$$A(F^j) \geq M_1 \| d(F^j) \|^2 \geq 0 \tag{34}$$

From (31), $\exists\ 0 < M_2 < \infty$ such that

$$B(F^j) \leq M_2 \| d(F^j) \|^2 \leq \frac{M_2}{M_1} A(F^j) = M_3 A(F^j) \tag{35}$$

From (32), $\exists\ 0 < M_4 < \infty$ such that

$$C(F^j, \alpha_j) \leq M_4 \| d(F^j) \|^2 \leq \frac{M_4}{M_1} A(F^j) = M_5 A(F^j). \tag{36}$$

Substituting (34), (35) and (36) into (29), giives

$$\Delta J(F^j) \leq \frac{\alpha_j}{2}[-2 + \alpha_j(M_3 + M_5)] A(F^j). \tag{37}$$

Selecting $0 < \alpha_j < 2/(M_3 + M_5)$ assures that $J(F^j) \leq 0$. We may select α_j such that $0 < \alpha_j \leq \mathrm{Min}\ \{2/(M_3 + M_5),\ a,\ 1\}$ to assure $J(F^j) \leq 0$; then the sequence $\{F^j, j \geq 0\}$ is a subset of S_0. Since S_0 is closed and bounded, $\exists F^* \in S_0$ such that the sequence $\{F^j,\ j \geq 0\}$ converges to F^*[7]. It implies $\Delta J(F^j) \to 0$. From (30), (34) and (37),

$$0 \leq M_1 \| d(F^j) \|^2 \leq A(F^j) \leq \frac{2\Delta J(F^j)}{\alpha_j[-2 + \alpha_j(M_3 + M_5)]}. \tag{38}$$

Hence,

$$d(F^j) = -\frac{\partial J}{\partial F}\bigg|_{\mathrm{diag}\ F^j} \longrightarrow -\frac{\partial J}{\partial F}\bigg|_{\mathrm{diag}\ F^*} = 0. \qquad \text{Q. E. D.}$$

4. Example

Consider the following discrete-time decentralized system

$$x(k+1)=\begin{pmatrix}1 & 0 & 0.5\\ 0.5 & 0 & 0.5\\ 0 & -0.5 & 0\end{pmatrix}x(k)+\begin{pmatrix}1\\0\\0\end{pmatrix}u_1(k)+\begin{pmatrix}0\\0\\1\end{pmatrix}u_2(k)$$

$$y_1(k)=[1\quad 0\quad 0]x(k)$$

$$y_2(k)=[0\quad 0\quad 1]x(k)$$

with

$$Q=I_3,\ R=I_2.$$

Based on the above theorems, we use the following computation algorithm.

Step 1 Set a finte $\varepsilon>0$ for S. Select $F^0\in S$ to be the initial feedback matrix. Set $j=0$, go to step 2.

Step 2 Solve the following equations for $P(F^j)$ and $L(F^j)$.

$$P(F^j)=(A-BF^jC)^TP(F^j)(A-BF^jC)+C^T(F^j)^TRF^jC+Q$$

$$L(F^j)=(A-BF^jC)L(F^j)(A-BF^jC)^T+I.$$

Step 3 Evaluate the performance index $J(F^j)$.

$$J(F^j)=\frac{1}{2}\operatorname{tr}\ \{P(F^j)-Q\}$$

Step 4 Calculate the gradient cost

$$\left.\frac{\partial J}{\partial F}\right|_{\text{diag }F^j}=\operatorname{diag}\left[\frac{\partial J}{\partial F_1},\ \frac{\partial J}{\partial F_2},\ \cdots,\ \frac{\partial J}{\partial F_N}\right]$$

and

$$\frac{\partial J}{\partial F_i}=[R_iF_iC_i-B_i^TP(F^j)(A-BF^jC)]L(F^j)C_i^T\qquad i=1,2,\cdots,N$$

If $\left.\frac{\partial J}{\partial F}\right|_{\text{diag }F^j}<\in$ is satisfied, stop; othewise, go to step 5.

Step 5 Choose $0<\alpha_j<1$ and let $d(F^j)=-\left.\frac{\partial J}{\partial F}\right|_{\text{diag}F^j}$, set $k=0$ go to step 6.

Step 6 Calculate $F_{new}=F^j+\alpha_j^k\ d(F^j)$. If $F_{new}\in S$, go to step 7, otherwise $k=k+1$ and repeat step 6.

Step 7 Calculate $J(F_{new})$. If $J(F_{new})<J(F^j)$, go to step 8, otherwise $k=k+1$, go to step 6.

Step 8 $j=j+1$, $F^j=F_{new}$, go to step 2

In this example, set $\varepsilon = 10^{-10}$ and select $F^0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, then $J(F^0) = 1.344$. The calculation converges at $j = 7$, with $J(F^7) = J(F^*) = 1.092$, where

$$F^7 \doteq F^* = \begin{bmatrix} 0.5670 & 0 \\ 0 & 0.1499 \end{bmatrix} \in S$$

and

$$\left.\frac{\partial J}{\partial F}\right|_{F^7} \doteq \left.\frac{\partial J}{\partial F}\right|_{F^*} = \begin{bmatrix} -1.584 \times 10^{-6} & 0 \\ 0 & 5.138 \times 10^{-6} \end{bmatrix}.$$

5. Conclusion

In this paper, the derivation of optimal local output feedback control with respect to infinite time quadratic criterion for the discrete, time-invariant decentralized system is discussed. A numerical example illustrates the application of the theory. The design methodology has direct application to many problems for the decentralized control systems.

References

[1] N. R. Sandell, P. P. Vaiaiya, M. Athans, M. G. Safonov, Survey of Decentralized Control Methods for Large-scal Systems, IEEE Trans. Automatic Control, Vol. AC-23, pp. 108-128, 1978.

[2] R. Saeks, On the Decentralized Control of Interconnected Dynamical Systems, IEEE Trans. Automatic Control, Vol. AC-24, pp. 269-271, 1979.

[3] J. P. Corfmat, A. S. Morse, Decentralized Control of Linear Multivariable Systems, Automatica, Vol. 12, No. 5, pp. 479-496, 1976.

[4] S. H. Wang, Stabilization of Decentralized Control Systems via Timevarying Controllers, IEEE Trans. Automatic Control, Vol. AC-27, pp. 741-744, 1982.

[5] B. D. O. Anderson and J. B. Moore, Time-varying Feedback Law for Decentralized Control, IEEE Trans. Automatic Control, Vol. AC-26, pp. 1133-1139, 1981.

[6] W. S. Levine and M. Athans, On the Determination of the

Optimal Constant Control, IEEE Trans. Automatic Control, Vol. AC-15, No. 1, Feb,, 1970.

[7] T. M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, 1957.

[8] D. L. Kleinman, Suboptimal Design of Linear Regulator Systems Subject to Computer Storage Limitations, MIT Electronics Systems Lab., Combridge, MA., Tech. Rept ESL-R-297, 1967.

分散系统的最优局部输出反馈控制

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摘　要

本文对所导出的离散型分散系统最优输出反馈控制进行了讨论，提出和建议一种收敛的计算法并用数值例子加以说明。