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Improved Measures of Stability Robustness for Linear State Space Models*

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Abstract

In this paper, new improved perturbations bounds for the robust stability of linear time invariant systems are given. The case of structural perturbations is considered and it is shown that the bounds are superior to those reported in literature [1]. The bounds are easy to compute numerically. Several examples are given to demonstrate the new bounds and compare them with results previously reported.

Key words——Time domain; Perturbation bounds; Robust stability; Lineartime-invariant systems.

1. Introduction

One aspect of current development of multivariable feedback system theory has been concerned with stability robustness to modeling uncertainties and large parameter variations in the system dynamics. As known, a principal reason for using feedback rather than open loop control is the presence of modeling uncertainties and that robustness characterization of dynamic systems, subject to modeling perturbations is very important in the everyday life of engineers. There are three main approaches which have been applied

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to this problem in the literature:

(i) the time-domain approach. e. g., [2]-[5], which is based on a state space representation of a system;

(ii) the frequency-domain approach, e.g, [6]-[9], which is based on the transfer function representation, and

(iii) frequency domain approach which uses a state space representation of the system [1],[10].

In [1], new bounds on linear time-invariant perturbations which do not destabilize the system were given for both unstructured perturbations (when only a bound on the perturbation matrix is given) and structured perturbations (when the structure of perturbations is specified and the bounds on the structured perturbations are given). It was shown that these bounds are superior to time-domain stability robustness criteria reported in the recent literature in two senses: (i) they are less conservative and (ii) they can be applied to a more general class of systems and perturbations.

In this paper, we present new time-domain stability robustness criteria for linear state space models. A new algorithm is proposed which leads to new improved measures of stability robustness. The bounds are superior to those based on frequency domain approach, reported in [1] and can be applied to the same class of systems and perturbations. It is also shown that a similar algorithm can be developed which leads to improved frequency domain measures for stability robustness analysis.

2. Main Results

Assume that a linear time-invariant model of a physical system is described by the following state equation

$$(S_0): \qquad \dot{x}(t) = (A + \Delta A)x(t) \tag{1}$$

where $x \in R^n$ is a state vector, $A \in R^{n \times n}$ is the nominal closed-loop matrix, which is assumed to be asymptotically stable and ΔA is a perturbation matrix. In other words, all parameter variations and modelling uncertainties, which are time-invariant, are lumped into the matrix ΔA.

As in [1] assume that ΔA has the structure

$$\Delta A = S_1 \Delta E S_2 \tag{2}$$

where $S_1 \in R^{n\times p}$, $\Delta E \in R^{p\times q}$, $S_2 \in R^{q\times n}$, $p \leqslant n$, $q \leqslant n$, and S_1 and S_2 are known constant matrices. With no loss of generality, assume that rank $S_1 = p$ or/and rank $S_2 = q$, and let the elements of the perturbation matrix be denoted by $\{\Delta E_{ij}\}$,

$$|\Delta E_{ij}| \leqslant \varepsilon_{ij}\varepsilon \tag{3}$$

where $\varepsilon_{ij} > 0$ are given, and $\varepsilon > 0$ is unknown. For example, perturbations of sensors/actuators of a closed-loop control system can be represented in the form (3). For an alternative approach to sensors/actuators perturbations see [4].

The following bound on ε such that $A + \Delta A$ remains stable for all perturbations ΔA of the type (2) was recently obtained.

Theoremt 1 [1] Given the class of perturbations ΔA described by (2) and (3), then $A + \Delta A$ is stable if

$$\varepsilon < \frac{1}{\sup\limits_{\omega \geqslant 0} \pi[|S_2(j\omega - A)^{-1}S_1|U]} \triangleq \mu_Q \tag{4}$$

where $U \in R^{p\times q}$ is a matrix with elements given by $u_{ij} = \varepsilon_{ij}$ $|[\cdot]|$ denotes modulus matrix, i. e., matrix formed by taking moduli of elements of $[\cdot]$. and π denotes perron eigenvalue of a non-negative square matrix.

As pointed out in [1], the bound (4) is a significant improvement over recent ones reported, and applies to more general situations. For example, perturbations of sensors/actuators of a closed loop system can be represented in the form of (2).

The introduction of structured perturbations considerably overcomes the conservatism in the robustness tests, comparing to the unstructured perturbations. Still, one of the most basic need is for more refined tests and measures of robustness. In what follows we propose new time domain stability robustness tests for multivariable control systems.

Let the perturbation matrix ΔE be defined by

$$\Delta E = e(t)E \tag{5}$$

where the matrix E is given and $e(t)$ is unknown time-varying scalar function.

The following theorem gives the sector(e_{min}, e_{max}) i.e., the bounds on the scalar function $e(t)$ such that the perturbed system remains stable for all perturbations ΔA of the type (2), with ΔE defined with (5).

Theorem 2 If the following inequalities are satisfied

$$\lambda_{\min}^{-1}(W)=e_{\min}<e(t)<e_{\max}<\lambda_{\max}^{-1}(W) \quad (6)$$

where e(t) is memoryless, time-varying nonlinearity, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denotes the smallest and the largest eigenvalue of (•), and

$$W=Q^{-1}((S_1ES_2)^TP+P(S_1ES_2)) \quad (7)$$

where the matrix P is positive definite solution of the Lyapunov matrix equation

$$A^TP+PA+Q=0, \qquad Q>0 \quad (8)$$

for all $t\in[0,\infty)$, then the perturbed system remains asymptotically stable.

Proof: See Appendix 1.

In essence, Theorem 2 shows that if $e(t)\in(e_{\min},e_{\max})$ for all $t\in[0,\infty)$, then the perturbed system remains asymptotically stable. Hence, the direction

$$\{eE : e\in(e_{\min},e_{\max})\} \quad (9)$$

is termed stability direction.

The next corollary is easy to prove, but has an interesting interpretation.

Corollary 1 If $\lambda_{\min}(W)$ is not negative then the bound $e_{\min}$ ceases to exist and

$$e\in(-\infty,e_{\max}) \quad (10)$$

If $\lambda_{\max}(W)$ is not positive, then the bound $e_{\max}$ ceases to exist and

$$e\in(e_{\min},\infty) \quad (11)$$

The results of Theorem 2 will be adapted to provide an algorithm for determining computationally the largest positive number $\bar{e}_{\max}$ and the smallest negative number $\underline{e}_{\min}$, such that the perturbed system (1) With ΔE defined with (5) remains stable if

$$e(t)\in(\underline{e}_{\min},\bar{e}_{\max}), \text{ for all } t\in[0,\infty) \quad (12)$$

We review the procedure:

Step 1 Using the results of Theorem 2 determine $(e_{\min})_0$ and $(e_{\max})_0$.

Step 2 Consider the perturbed system ($\underline{S}_j$) as unperturbed system, i. e.

$$A=A+(e_{\min})_{j-1}S_1ES_2 \quad j=1,2,\cdots \quad (13)$$

and determine $(e_{min})_j$, using the results of Theorem 2.

Step 3 Check that the closed loop system $(\underline{S_j})$ is stable.

If this is the case, return to step 2. If not

$$\underline{e}_{min} = \sum_{p=0}^{j-1} (e_{min})_p \tag{14}$$

Step 4 Consider the perturbed system $(\overline{S_j})$ as unperturbed system, i.e.

$$A = A + (e_{max})_{j-1} S_1 E S_2 \quad j = 1, 2, \cdots \tag{15}$$

and determine $(e_{max})_j$ using the results of Theorem 2.

Step 5 Check that the closed loop system $(\overline{S_j})$ is stable.

If this is the case, return to step 4. If not

$$\bar{e}_{max} = \sum_{p=0}^{j-1} (e_{max})_p \tag{16}$$

(end of procedure).

Therefore, using this procedure, we obtain sequences $(\underline{S_p})$ and $(\overline{S_p})$ of closed loop perturbed systems and sequences $((e_{min})_p)$ and $((e_{max})_p)$ of scalars. For each p, the eigenvalues of the corresponding perturbed systems $(\underline{S_p})$, i.e. $(\overline{S_p})$ have negative real parts. If for any j there is an eigenvalue with zero real part, we will not be able to apply Theorem 2, and we shall have

$$\underline{e}_{min} = \sum_{p=0}^{j-1} (e_{min})_p \text{ or } \bar{e}_{max} = \sum_{p=0}^{j-1} (e_{max})_p \tag{17}$$

If $p \to \infty$, then the Lyapunov equation (8) becomes progresivelly more ill-conditioned as $p \to \infty$ and the process will have to stop for some finite values of $\bar{e}_{max}$ and $\underline{e}_{min}$.

Notice that the problem of the selection of an initial value matrix Q_0 for determining $(e_{min})_0$ and $(e_{max})_0$ has not been discussed. An obvious possibility is to select Q_0 as $Q_0 = 2I$, where I is an identity matrix, and to use the same value of Q_0 in all iterations.

3. Numerical Examples

Example 1 The following simple second order linear time invariant system was considered in [1]

$$\dot{x}(t) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x(t) \qquad (18)$$

Table 1 gives bounds of allowable perturbations which do not disturb system stability, applying the frequency domain criterion (4) and time domain criterion (6), when the perturbed ellements of E have different combinations. In this case it was assumed that $S_1 = S_2 = I_{2\times 2}$. Table 1 also gives the bounds of allowable perturbations when the iterative algorithm, eqns. (13)—(16), is applied. The results show that the new bounds are a significant improvement over the bounds based on the frequency domain criterion. It should be pointed out that the time domain approach involves checking of only two inequalities, eqn. (6), while the frequency domain methodology requires that the criterion (4) be satisfied over the whole range of frequences.

Table 1 also gives the "exact" bounds which provide necessary and sufficient conditions for stability robustness [1]. These "exact" bounds are difficult to compute generally, but in the 2×2 case, they can be obtained by observation. However, it should be pointed out that these "explicit" bounds correspond to the "worst" case and they do not distinguish possible directions of perturbations with the same structure. This explains why some of the bounds determined by the iterative algorithm are larger than the "exact" bounds given in [1].

Table 2 gives the bounds on the allowable perturbations for perturbations with the same structure but with different directions. As can be seen, the frequency domain criterion does not distinguish perturbations with the same structure, but with different directions, in all cases the bound μ_Q has the same value, while the time domain criterion gives different bounds for different directions.

Example 2 The following example was also given in [1]. The closed loop stable matrix is deffined by

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \qquad (19)$$

while the perturbation matrix ΔA is defined by

$$S_1 = \begin{bmatrix} 7 & 8 \\ 12 & 14 \end{bmatrix}; \quad S_2 = \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix}; \quad \Delta E = e\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \qquad (20)$$

For this example

$$\mu_Q = 0.08160793$$

$$\underline{e}_{min} \to -\infty$$

$$\bar{e}_{max} = 0.2263783$$

Therefore, the results of Example 2 lead to the similar conclusion as the results of Example 1.

Table 1 Comparison of Stability Robustness Bounds for Example 1

Perturbed Elements of A	a_{11} a_{12} a_{21} a_{22}	a_{11} a_{21}	a_{21}	a_{12} a_{21}	a_{21} a_{22}	a_{11} a_{21} a_{22}
	1 1 1 1	1 0 1 0	0 0 1 0	0 1 1 0	0 0 1 1	1 0 1 1
e_{min}	−4.236068	−0.9249506	−0.9758431	−1	−3.302776	−6.495898
e_{max}	0.236068	0.4805062	0.6558431	0.5	0.3027756	0.2736769
μ	0.3295388	0.9150402	1	0.8107933	0.4	0.3713509
$\underline{e}_{min}$	−∞	−0.9999991	−0.9999999	−∞	−∞	−∞
Number of iterations		7	5			
$\bar{e}_{max}$	0.9999172	2.999936	∞	1.999989	1.999516	1.499948
Number of iterations	78	29		18	178	44
"exact" bounds	0.3333	1	1	1	0.4	0.3723

Table 2 Comparation of Stability Robustness Bounds for Perturbations with the Same Structure but Different Directions

Perturbed Elements of A			a_{11} a_{12} a_{21} a_{22}				
	1 1 1 1	−1 1 −1 1	−1 −1 1 1	1 −1 1 −1	1 1 −1 −1	−1 1 1 1	1 −1 1 1
e_{min}	−4.236068	−0.8090171	−2.414214	−0.3090170	−0.4142136	−1.609476	−12.32455
e_{max}	0.236068	0.3090170	0.4142136	0.8090171	0.2414214	0.2761424	0.3245553
μ_Q	0.3295388	0.3295388	0.3295388	0.3295388	0.3295388	0.3295388	0.3295388
$\underline{e}_{min}$	−∞	−∞	−∞	−0.3333333	−∞	−1.618034	−8
$\bar{e}_{max}$	0.9999172	0.3333333	+∞	+∞	+∞	0.6180287	1.499985

4. Conclusions

A computationally efficient method for time domain robustness evaluation in linear multivariable control systems has been proposed. Bounds on structured perturbations in the state space models have been established such that stability of the dynamic system is assured. The bounds are superior to those reported in the recent literature. Several numerical examples have been used to demonstrate the new bounds and compare them with results previously reported.

Appendix: Proof. of Theorem 2.

The proof. proceeds by utilizing arguments of Lyapunov theory. Choose the positive definite Lyapunov function as

$$V(x) = x^T(t)Px(t) \tag{A.1}$$

where the matrix P is positive definite solution of (8).

$$\dot{V}(x)|_{(1)} = x^T(t)(P(A+e(t)(S_1ES_2)) + (A+e(t)(S_1ES_2))^T P)x(t) \tag{A.2}$$

$$= -x^T(t)(Q - e(t)(P(S_1ES_2) + (S_1ES_2)^T P))x(t) \tag{A.3}$$

Asymptotic stability follows if $\dot{V}(x)|_{(1)}$ is negative definite, which follows if

$$Q - e(t)(P(S_1ES_2) + (S_1ES_2)^T P) > 0 \tag{A.4}$$

To prove the conditions (6) recall the following lemma.

Lemma 1[11] If R and S are symmetric matrices and R is positive definite, there exists a nonsingular matrix Y such that

$$Y^T(R+S)Y = I + G \tag{A.5}$$

where the matix G is a diagonal matrix whose elements are eigenvalues of $R^{-1}S$.

Therefore, using the results of Lemma 1, it can be easily concluded that the perturbed system (S_0), eqns. (1), (2) and (5), will remain asymptotically stable if the following inequality is satisfied,

$$1 - e(t)\lambda_j(Q^{-1}(P(S_1ES_2) + (S_1ES_2)^T P)) > 0 \quad j = 1,2,\cdots,n \tag{A.6}$$

i.e.

$$1 - e(t)\lambda_i(W) > 0, \ j = 1,2,\cdots,n \tag{A.7}$$

where the matrix W is defined by (7).

Now under the assumption that $\lambda_{\max}(W) > 0$, which is an usual case, it follows that

$$e_{\max} = \lambda_{\max}^{-1}(W) \tag{A.8}$$

In a similar way, if $\lambda_{\min}(W) < 0$ then

$$e_{\min} = \lambda_{\min}^{-1}(W) \tag{A.9}$$

i.e, the perturbed system (S_0) remains asymptotically stable if

$$e(t) \in (e_{\min}, e_{\max}) \tag{A.10}$$

for all $t \in [0, \infty)$.

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线性状态空间模型的稳定鲁棒性改善

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摘　要

本文讨论线性定常系统的稳定鲁棒性问题。文中给出了线性定常系统的稳定鲁棒性范围的新改善界限。在结构扰动情况下，本文给出的稳定鲁棒性界限比引文〔1〕中的相应结果更好。用数值计算方法容易得到这一动扰界限。文中给出的两个例子证实了本文结果的优点。

关键词： 时域；摄动界限；鲁棒性；线性非时变系统