控制理论与应用
CONTROL THEORY AND APPLICATIONS

Vibrational Stabilization of Linear Systems Under Coefficient Perturbation

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Abstract: Vibrational control is a method for modification of dynamic properties of linear and nonlinear systems by introduction of fast, zero—average oscillations in a system parameters. This paper is concerned with the issue of stabilization of unstable linear systems under coefficient perturbation by using vibrational control. Illustrative example is also considered.

Key words: Linear System; Robust Stability; Vibrational Control

1. Introduction

The problem of determining the stabilizability of perturbed systems has attracted much recent attention. This, of course, is due to the fundamental importance of the issue. A control system is usually unacceptable if small system perturbation would result in catastrophe.

In [1], the problem of stabilization of unstable linear interval dynamical systems by using state feedback has been considered. The application of the state feedback requires on—line measurement of states. However, in many cases it is not possible to measure all states, thus not possible to apply full state feedback. Recently, a new technique has been proposed in [2]—[6], which consists of the introduction of zero mean oscillations in the system parameters so that its behavior is modified in a desired manner. Unlike state feedback, this technique, termed vibrational control, does not require on—line measurement and therefore can be used in situations where state feedback methods are not applicable. The theory of vibrational control for linear and nonlinear systems has been developed in [2]—[6], where the general concepts of stabilizability and controllability by parametric oscillations have been introduced and a number of applications have been considered. This technique is based on reducing the system equations to a standard form and a subsequent application of Bogoliubov averaging principle [7].

The aim of this work is to present that stabilization can be achieved even for unstable linear dynamical systems under coefficient perturbation by using vibrational control.

2. Main Result

An nth order LTI dynamical system can be modeled by

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0, \quad x^{(i)} = d^i x/dt^i, \quad 1 \leqslant i \leqslant n. \tag{2.1}$$

where each coefficient a_i is subject to uncertain but bounded perturbation, i. e., $a_i \in [a_i^-, a_i^+]$ (as-

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sume $a_i^+ \geqslant a_i^-$). Clearly, the characteristic polynomial of such a system is

$$P(s,a) = s^n + a_1 s^{n-1} + \cdots + a_n. \tag{2.2}$$

$$\text{Let} \quad a = [a_1 \quad a_2 \quad \cdots \quad a_n]', \tag{2.3}$$

be the coefficient vector $a \in R_a$ where R_a is the perturbation region defined as

$$R_a = \{a \in R^n | a_i^- \leqslant a_i \leqslant a_i^+, \quad 1 \leqslant i \leqslant n\}. \tag{2.4a}$$

Furthermore, let

$$a_i^0 = 1/2(a_i^+ + a_i^-), \quad 1 \leqslant i \leqslant n. \tag{2.4b}$$

denote the nominal value of the coefficients a_i and

$$\delta_i = 1/2(a_i^+ - a_i^-) = \omega_i \mu, \quad 1 \leqslant i \leqslant n. \tag{2.4c}$$

the tolerance of a_i, where μ is a positive perturbation and ω_i is a set of nonnegative weights defined in the vector form as

$$\omega = [\omega_1 \quad \omega_2 \quad \omega_3 \quad \cdots \quad \omega_n]'. \tag{2.4d}$$

The theorem below introduces the capability for stabilizing the unstable linear dynamical systems under coefficient perturbation by using the vibrational control.

Theorem 2.1: Let the system (2.1) be unstable over R_a. Then there exist α_i $2 \leqslant i \leqslant n$, such that

$$x^{(n)} + a_1 x^{(n-1)} + [a_2 + (\alpha_2/\varepsilon) f(t/\varepsilon) x^{(n-2)} + \cdots + [a_n + (\alpha_n/\varepsilon) f(t/\varepsilon)]x = 0, \tag{2.5}$$

is as stable over R_a iff $a_1^- > 0$, where $f(t/\varepsilon)$ is a periodic or almost periodic zero mean function and $0 < \varepsilon \ll 1$.

Proof: Necessarity of $a_1^- > 0$ is quite obvious. Since a_1 can not be adjusted from the vibrational control theory (theorem 1 in [2]), it is necessary that a_1^- has to be positive for system (2.5) to be stabilized by introducing the zero mean oscillations in the system' s parameters as in (2.5).

Proof of sufficient conceptually means to find a certain set of α_i $2 \leqslant i \leqslant n$, such that system (2.5) is stable under the condition of $a_i^- > 0$.

For this end, rewrite (2.5) in the state equation form.

$$\dot{z}_i = z_{i+1}, \qquad 1 \leqslant i \leqslant n-1$$

$$\dot{z}_n = -a_1 z_n - [a_2 + (\alpha_2/\varepsilon) f(t/\varepsilon)] z_{n-1} - \cdots - [a_n + (\alpha_n/\omega) f(t/\varepsilon)] z_1, \tag{2.6}$$

$$z_1 = x, z_2 = \dot{x}, z_3 = \ddot{x}, \cdots, z_n = x^{(n-1)}. \tag{2.7}$$

To reduce this system to the standard form, introduce a substitution in (2.6):

$$z_i = y_i, \qquad 1 \leqslant i \leqslant n-1$$

$$z_n = -\alpha_2 u y_{n-1} - \alpha_3 u y_{n-2} - \cdots - \alpha_n u y_1 + y_n. \tag{2.8}$$

where $u = \int f(\tau) d\tau$ is a periodic zero mean value function;

$\tau = t/\varepsilon$ is fast time.

Substituting (2.8) into (2.6), this yields:

$$dy_i/d\tau = \varepsilon y_{i+1}, \qquad 1 \leqslant i \leqslant n-2$$

$$dy_{n-1}/d\tau = \varepsilon(-\alpha_2 u y_{n-1} - \alpha_3 u y_{n-2} \cdots - \alpha_n u y_1 + y_n),$$

$$dy_n/d\tau = \varepsilon[\alpha_2 u(-\alpha_2 u y_{n-1} - \alpha_3 u y_{n-2} \cdots - \alpha_n u y_1 + y_n)$$

$$+ \alpha_3 u y_{n-1} + \cdots + \alpha_n u y_2] - \varepsilon[a_1(-\alpha_2 u y_{n-1} - \alpha_3 u y_{n-2} \cdots$$

$$-\alpha_n u y_1 + y_n) + a_2 y_{n-1} + \cdots + a_n y_1]. \tag{2.9}$$

In time $\tau = t/\varepsilon$ this is an equation in the standard form[7]. Therefore, applying the averaging principle [7], the following asymptotic approximation can be obtained

$$\begin{aligned} d\bar{y}_i/d\tau &= \varepsilon \bar{y}_{i+1}, \quad 1 \leqslant i \leqslant n-1 \\ d\bar{y}_n/d\tau &= \varepsilon[-a_1\bar{y}_n - (a_2 + \alpha_2^2 \overline{f(\tau)^2})\bar{y}_{n-1} - (a_3 + \alpha_2\alpha_3 \overline{f(\tau)^2})\bar{y}_{n-2} \\ &\quad - \cdots - (a_n + \alpha_2\alpha_n \overline{f(\tau)^2}\bar{y}_1]. \end{aligned} \tag{2.10}$$

where
$$\lim_{T\to\infty}\left(\int_0^T u(\tau)d\tau\right)/T \to 0, \quad \overline{f(\tau)^2} = \lim_{T\to\infty}\left(\int_0^T u^2(\tau)d\tau\right)/T. \tag{2.11}$$

According to the main theorem of the averaging theory, this yields

$$\| y(\tau) - \bar{y}(\tau) \| \leqslant \varepsilon, \quad \tau \in [0, 1/\varepsilon].$$

where
$$y = [y_1 \quad y_2 \quad \cdots \quad y_n]', \quad \bar{y} = [y_1 \quad y_2 \quad \cdots \quad y_n]'.$$

It follows from the averaging theorem that (2.9) has an asymptotically stable periodic solution if (2.10) has such a solution. Averaging equation in the original time t is as follows:

$$\begin{aligned} d\bar{y}_i/dt &= \bar{y}_{i+1}, \quad 1 \leqslant i \leqslant n-1 \\ d\bar{y}_n/dt &= -a_1\bar{y}_n - [a_2 + \alpha_2^2 \overline{f(t/\varepsilon)^2}]\bar{y}_{n-1} - [a_3 + \alpha_2\alpha_3 \overline{f(t/\varepsilon)^2}]\bar{y}_{n-2} \\ &\quad - \cdots - [a_n + \alpha_2\alpha_n \overline{f(t/\varepsilon)^2}]\bar{y}_1. \end{aligned} \tag{2.12}$$

Taking into account (2.1) and (2.7), (2.12) is described equivalently by $\bar{x}, \dot{\bar{x}}, \cdots, \bar{x}^{(n-1)}, \bar{x}^{(n)}$,

$$\bar{x}^{(n)} + a_1\bar{x}^{(n-1)} + (a_2 + \beta_2)\bar{x}^{(n-2)} + \cdots + (a_n + \beta_n)\bar{x} = 0. \tag{2.13}$$

where
$$\beta_i = \alpha_2\alpha_i \overline{f(t/\varepsilon)^2}, \quad 2 \leqslant i \leqslant n. \tag{2.14}$$

Let $\beta = [0 \; \beta_2 \quad \beta_3 \cdots \beta_n]'$ (2.15)

denote the vector of vibrational control parameters and

$$q = [q_1 \quad q_2 \quad \cdots \quad q_n]' \tag{2.16}$$

the coefficient vector of averaging equation (2.13), so that $q = a + \beta$.

From above, the lower and upper bound vectors become:

$$q^+ = a^+ + \beta, q^- = a^- + \beta. \tag{2.17}$$

where
$$a^+ = [a_1^+ a_2^+ a_3^+ \cdots a_n^+]',$$
$$a^- = [a_1^- a_2^- a_3^- \cdots a_n^-]'. \tag{2.18}$$

Furthermore, define

$$Q = \{q \in R^n | q_i^- \leqslant q_i \leqslant q_i^+, \quad 1 \leqslant i \leqslant n\} \tag{2.19}$$

as the perturbation region of q, i.e., $q \in Q$.

The characteristic polynomial of (2.13) is as follows:

$$P(s, q) = s^n + q_1 s^{n-1} + q_2 s^{n-2} + \cdots q_n. \tag{2.20}$$

Let

$$q^* = \{q^* \in R^n | q_i^* = 1/2(q_i^+ + q_i^-), \quad 1 \leqslant i \leqslant n\} \tag{2.21}$$

denote the nominal value of q and

$$\delta = \{\delta \in R^n | \delta_i = 1/2(q_i^+ - q_i^-) = \omega_i\mu, \quad 1 \leqslant i \leqslant n\} \tag{2.22}$$

the tolerance of q, so that

$$q^+ = q^* + \delta, \quad q^- = q^* - \delta. \tag{2.23}$$

According to the result in [1], for the given $\delta_i \geqslant 0$ and $\mu > 0$, there exists q^* with $P(s,q^*)$ to be strictly Hurwitz such that

$$\mu_{\max}(q^*) > \mu, \text{or } \omega_i \mu_{\max}(q^*) > \delta_i, \qquad 1 \leqslant i \leqslant n. \tag{2.24}$$

Therefore, using the Barmish theorem [1], the characteristic polynomial (2.20) of the averaging equation (2.13) remains strictly Hurwitz over Q.

It is clear that if the characteristic polymonial (2.20) is strictly Hurwitz over Q, the asymptotical stability of the averaging system (2.13) can be guaranteed over Q. Asymptotical stability of (2.13) implies the asymptotical stability of (2.12). As it follows from the averaging theory [7] that if (2.12) is asymptotically stable, then equation (2.9) is also asymptotically stable, Due to (2.6), (2.7) and (2.8), the asymptotical stability of (2.9) implies also the asymptotical stability of (2.5).

From (2.17) and (2.21), we have

$$\begin{aligned} q_i^* &= 1/2(a_i^+ + a_i^-) + \beta_i \\ &= a_i^\circ + \beta_i, \qquad 1 \leqslant i \leqslant n. \end{aligned} \tag{2.25}$$

Thus, the vibrational control parameters are as follows:

$$\beta_i = q_i^* - 1/2(a_i^* + a_i^-), \qquad 2 \leqslant i \leqslant n. \tag{2.26}$$

From (2.14), this yields

$$\begin{aligned} \alpha_2 &= \sqrt{\beta_2 / \overline{f(t/\varepsilon)^2}}, , \\ \alpha_i &= \beta_i / (\alpha_2 \overline{f(t/\varepsilon)^2}), \quad 3 \leqslant i \leqslant n. \end{aligned} \tag{2.27}$$

3. Example

Consider a fourth order system

$$x^{(4)} + a_1 x^{(3)} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0,$$

with coefficients a_i uncertain but bounded in the interval $a_i \in [a_i^-, a_i^+]$ for $i = 1,2,3,4$, where

$$a^+ = [7 \quad 2 \quad -6 \quad 1]', \text{and } a^- = [3 \quad -4 \quad -14 \quad -1]'.$$

From (2.4b), the nominal value of a is $a^\circ = [5 \quad -1 \quad -10 \quad 0]$ and from (2.4c), the tolerance of a is $\delta = [2\ 3\ 4\ 1]$. Let $\mu = 1$, we have $\omega = [2\ 3\ 4\ 1]'$. Obviously, the nominal value a° is not stritly Hurwitz. It means that the given system is unstable over the interval $[a^-, a^+]$.

Since $a_1^- = 3 > 0$, it is possible to introduce vibrational control in order to stailize the entire family of systems over the given perturbed interval.

$$\begin{aligned} x^{(4)} &+ a_1 x^{(3)} + [a_2 + (\alpha_2/\varepsilon)\sin(t/\varepsilon)]\ddot{x} + [a_3 + (\alpha_3/\varepsilon)\sin(t/\varepsilon)]\dot{x} \\ &+ [a_4 + (\alpha_4/\varepsilon)\sin(t/\varepsilon)]x = 0, \end{aligned}$$

where $0 < \varepsilon \ll < 1$.

Introducing a substitution to reduce the equation above into the standard form, and then applying averaging principle, we obtain the following asymptotical approximation.

$$\bar{x}^{(4)} + a_1 \bar{x}^{(3)} + (a_2 + \beta_2)\ddot{\bar{x}} + (a_3 + \beta_3)\dot{\bar{x}} + (a_4 + \beta_4)\bar{x} = 0,$$

where $\quad \beta_2 = \alpha_2^2/2, \qquad \beta_3 = \alpha_2 \alpha_3 / 2, \qquad \beta_4 = \alpha_2 \alpha_4 / 2.$

The corresponding characteristic polynomial is as follows:

$$P(s,q)=s^4+q_1s^3+q_2s^2+q_3s+q_4.$$

where
$$q^-=[3\quad -4+\beta_2\quad -14+\beta_3\quad -1+\beta_4],$$
$$q^+=[7\quad 2+\beta_2\quad -6+\beta_3\quad 1+\beta_4].$$

and the following four Hurwitz testing matrices[1]

$$H_1(\mu)=H(q_1^*-2\mu,q_2^*-3\mu,q_3^*+4\mu,q_4^*+\mu),$$
$$H_2(\mu)=H(q_1^*+2\mu,q_2^*+3\mu,q_3^*-4\mu,q_4^*-\mu),$$
$$H_3(\mu)=H(q_1^*-2\mu,q_2^*+3\mu,q_3^*+4\mu,q_4^*-\mu),$$
$$H_4(\mu)=H(q_1^*+2\mu,q_2^*-3\mu,q_3^*-4\mu,q_4^*+\mu).$$

After several iterations, we obtain that

$$q^*=[5\quad 14\quad 6\quad 2]'$$

is strictly Hurwitz and its corresponding maximal admissible perturbation is

$$\mu_{max}(q^*)=1.0071>\mu=1.$$

Clearly, since $\omega_i\mu_{max}(q^*)>\delta_i$, $1\leqslant i\leqslant 4$, the vibrational control obtained guarantees stability of the given system under coefficient perturbation.

From (2.26) and (2.27), we have the vibrational control parameters introduced in the given system's parameters as follows:

$$\beta=[0\quad 15\quad 16\quad 2]',\ \alpha=[0\quad 5.4772\quad 5.8424\quad 0.7303]'.$$

In the mean sense, vibrational control moves the given system from the unstable interval $[a^-,a^+]$ into the stable interval $[a^-+\beta,a^++\beta]$.

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系数扰动的线性系统的振动控制稳定性

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摘要 振动控制是在线性或非线性系统的参数中引入快速的、零均值的小幅振动来改善系统的动态特性. 本文研究利用振动控制方法来稳定系数扰动的非稳定线性系统问题,并给出计算实例.

关键词:线性系统;鲁棒稳定性;振动控制