Simultaneous Stabilization: Inverse LQ Approach *

Cao Yongyan and Sun Youxian

(National Laboratory of Industrial Control Technology, Zhejiang University · Hangzhou, 310027, P.R. China)

(Institute of Industrial Process Control, Zhejiang University · Hangzhou, 310027, P.R. China)

Abstract: In this paper, the simultaneous stabilization problem is addressed using the inverse LQ approach. A necessary and sufficient condition for simultaneous stabilizability of r strictly proper MIMO plants via static output feedback is obtained, and a solution of the simultaneous stabilization via state feedback is presented. The condition is described by some coupled algebraic Riccati inequalities. It is shown that any such stabilizing feedback gain is the solution of some coupled linear quadratic control problems where every cost functional has a suitable cross term.

Key words: simultaneous stabilization; algebraic Riccati inequality (ARI); static output feedback; state feedback; linear systems

同时镇定:逆 LQ 方法

曹永岩　孙优贤

(浙江大学工业控制技术国家重点实验室·杭州, 310027)

(浙江大学工业控制技术研究所·杭州, 310027)

摘要: 本文使用逆 LQ 方法研究了同时镇定问题,首先得到 r 个正则 MIMO 对象可静态输出反馈同时镇定的充要条件,然后给出了状态反馈同时镇定的一种解.这些条件均以一组耦合 Riccati 方程和 Riccati 不等式的形式给出.本文证明,任一同时镇定反馈增益均可描述为一组具有合适交叉项的耦合 LQ 控制问题的解.

关键词: 同时镇定; 代数 Riccati 不等式; 静态输出反馈; 状态反馈; 线性系统

1 Introduction

The purpose of this paper is to find a necessary and sufficient condition for the existence of a static output feedback controller and linear state feedback controller which simultaneously stabilizes a set of given linear time-invariant multiple-input multiple-output plants. The main contribution of this paper can be summarized as follows. There exists a simultaneously stabilizing controller via static output feedback or linear state feedback if and only if there exists a stabilizing solution for a set of coupled algebraic Riccati inequalities or a set of coupled Linear Quadratic (LQ) control problems in which every cost functional has suitable cross term satisfying an inequality constraint. Our design strategy consists of finding suitable weighting matrices such that the solution of these coupled LQ control problems corresponds to a simultaneously stabilizing output feedback controller.

Simultaneous stabilization is an important problem in the area of robust control. It is the problem of determining a single controller which will simultaneously stabilize a finite set of plants. The result may apply to linear plants characterized by different modes of operation (for instance, failure modes) or to the stabilization of non-linear plants linearized at several different equilibria. Also, some research results clearly show the relevance of simultaneous stabilization to system robustness issues (see for example[1]).

2 Preliminaries

First, let us consider the linear time-invariant plant G described by the equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector and A, B are constant matrices of appropriate dimensions. The following lemma is well known[2,3].

Lemma 1 Consider linear time-invariant system (1). The following statements are equivalent:

1) The system (1) is stabilizable via state feedback.

2) There exist matrices $Q > 0$ and $R > 0$ such that

* This work was partially supported by the National Natural Science Foundation of China under grant 69604007 and Chinese post-doctoral foundation.
Manuscript received Jan. 14, 1997, revised Mar. 5, 1999.

the following ARE

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}} + Q = 0$$

has a unique solution $P > 0$.

3) There exist matrices $P > 0$ and $R > 0$ such that the following ARI

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P < 0.$$

Remark 1 In fact, if (A, B) is stabilizable, then for any $Q > 0$ and $R > 0$ the above ARE must exist a unique solution $P > 0$, and for any $R > 0$ the above ARI has a feasible solution.

The LQ control problem with cross term, which we take as the basis for our development, involves minimizing the cost

$$v(x_0) = \int_0^\infty (x^{\mathrm{T}}Qx + 2u^{\mathrm{T}}Sx + u^{\mathrm{T}}Ru)\mathrm{d}t, \quad (2)$$

where $Q > 0, R > 0$, and S are constant weighting matrices satisfying

$$Q - S^{\mathrm{T}}R^{-1}S > 0. \quad (3)$$

The solution of the LQ control problem associated with (1), (2) and (3) is (see[2])

$$u = Kx, \quad K = -R^{-1}(B^{\mathrm{T}}P + S),$$

$$PA + A^{\mathrm{T}}P - (B^{\mathrm{T}}P + S)^{\mathrm{T}}R^{-1}(B^{\mathrm{T}}P + S) + Q = 0. \quad (4)$$

Lemma 2 For a given system (1), the following statements are equivalent:

1) It is stabilizable via state feedback.

2) There exist matrices $P > 0, Q > 0, R > 0$ and S of compatible dimensions such that (3) and (4) hold.

3) There exist matrices $P > 0, R > 0$ and S of compatible dimensions such that the following modified ARI with cross term holds

$$PA + A^{\mathrm{T}}P - (B^{\mathrm{T}}P + S)^{\mathrm{T}}R^{-1}(B^{\mathrm{T}}P + S) + S^{\mathrm{T}}R^{-1}S < 0. \quad (5)$$

Remark 2 Lemma 2 was established in [5]. Based on its proof in [5] and Lemma 1, it can be shown that for any $R > 0$, ARI (5) has a feasible solution (P, S) if and only if (A, B) is stabilizable.

3 Main results

Now we consider the simultaneous stabilization of r plants G_i

$$\begin{cases} \dot{x}_i(t) = A_ix_i(t) + B_iu_i(t), \\ y_i(t) = C_ix_i(t), \end{cases} \quad i = 1, \cdots, r, \quad (6)$$

where the state $x_i(t) \in \mathbb{R}^n$, the input $u_i(t) \in \mathbb{R}^m$, the output $y_i(t) \in \mathbb{R}^p$ and n is the order of G_i. We assume

$$C_i = C, \quad i = 1, \cdots, r \quad (7)$$

and C has full rank. For example, this condition is satisfied for single-output plants of the same order since it is always possible to realize them in minimal observable form. In addition, for multiple model control of r models $G_i(s)$ of an uncertain plant, uncertainty often exists only in the system matrix and (or) input matrix. In these situations, condition (7) is satisfied. Let E be the right inverse of C, i.e., $CE = I_p$. Since C has full rank, E can be constructed from

$$E = C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}. \quad (8)$$

So $E_\perp = EC$ is the orthogonal projection matrix on Im (C^T) and $x_\perp = E_\perp x = Ey$.

Theorem 1 For every $i, i = 1, \cdots, r$, let G_i be given by (6). Then the following statements are equivalent:

1) They are simultaneously stabilizable via static output feedback.

2) There exist matrices $P_i > 0, R > 0, Q_i > 0$ and M satisfying the following coupled AREs

$$P_iA_i + A_i^{\mathrm{T}}P_i - (S_i + B_i^{\mathrm{T}}P_i)^{\mathrm{T}}R^{-1}(S_i + B_i^{\mathrm{T}}P_i) + Q_i = 0, \quad i = 1, \cdots, r, \quad (9)$$

$$Q_i - S_i^{\mathrm{T}}R^{-1}S_i > 0, \quad (10)$$

$$S_i = ME_\perp - B_i^{\mathrm{T}}P_i. \quad (11)$$

If such matrices can be found, the simultaneously stabilizing static output feedback controller can be constructed as

$$K = -R^{-1}MC^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}. \quad (12)$$

3) There exist matrices $P_i > 0, R > 0$ and M satisfying the following coupled ARIs

$$P_iA_i + A_i^{\mathrm{T}}P_i - (S_i + B_i^{\mathrm{T}}P_i)^{\mathrm{T}}R^{-1}(S_i + B_i^{\mathrm{T}}P_i) + S_i^{\mathrm{T}}R^{-1}S_i < 0, \quad i = 1, \cdots, r, \quad (13)$$

$$S_i = ME_\perp - B_i^{\mathrm{T}}P_i.$$

Proof 1)⇒2). Suppose the plants are simultaneously stabilizable by static output feedback, then there exists a state feedback gain $F = KC$ and r matrices $P_i > 0$ such that

$$x^{\mathrm{T}}[(A_i + B_iF)^{\mathrm{T}}P_i + P_i(A_i + B_iF)]x < 0, i = 1, \cdots, r.$$

Without loss of generality, we may suppose that the feedback gain matrix has the form of $F = R^{-1}ME_\perp$ where M is a suitable matrix [4]. So the last inequality is equivalent to

$$x^{\mathrm{T}}[(A_i^{\mathrm{T}}P_iP_iA_i)x < 0,$$

$$x \in \ker(F), \quad x \neq 0, \quad i = 1,\cdots,r.$$

Define

$$\alpha^* = \max_{i=1,\cdots,r}(\alpha_i^*),$$

$$\alpha_i^* = \max_x \frac{x^{\mathrm{T}}(P_iA_i + A_i^{\mathrm{T}}P_i)x}{x^{\mathrm{T}}(E_\perp M^{\mathrm{T}}R^{-1}R^{-1}ME_\perp)x},$$

$$x \notin \ker(ME_\perp),$$

$$x \neq 0, i = 1,\cdots,r.$$

From [5], we know that $\alpha_i^* < \infty$, $i = 1,\cdots,r$, So $\alpha^* < \infty$, i.e., α^* is upper bounded. Since $F = R^{-1}ME_\perp$, for any $\alpha = \max(0, \alpha^*)$,

$$x^{\mathrm{T}}(A_i^{\mathrm{T}}P_i + P_iA_i)x < \alpha x^{\mathrm{T}}(E_\perp M^{\mathrm{T}}R^{-1}R^{-1}ME_\perp)x,$$

$$x \neq 0 \quad i = 1,\cdots,r.$$

Thus, for any symmetric matrix R satisfying $R \geqslant \alpha I$, we have

$$x^{\mathrm{T}}(A_i^{\mathrm{T}}P_i + P_iA_i)x < x^{\mathrm{T}}(E_\perp M^{\mathrm{T}}R^{-1}RR^{-1}ME_\perp)x,$$

$$x \neq 0, \quad i = 1,\cdots,r,$$

i.e.,

$$A_i^{T}P_i + P_iA_i - E_\perp M^{\mathrm{T}}R^{-1}ME_\perp < 0, \quad i = 1,\cdots,r,$$

which in turn implies the existence of a matrix $Q_i > 0$ satisfying the matrix equation (9). In fact, the last two inequalities are the ARIs given by (13). From Lemma 2, the inequality (10) is also a necessary condition. In fact, the foregoing proof implies 1)$\Rightarrow$3).

2)$\Rightarrow$1). Suppose the coupled AREs (9) have a feasible solution P_i, Q_i, R and M. From Lemma 2, the state feedback

$$F = -R^{-1}(S_i + B_i^{\mathrm{T}}P_i) = -R^{-1}ME_\perp = KC,$$

minimizes the performance index

$$J_i = \int_0^\infty (x_i^{\mathrm{T}}Q_ix_i + 2u_i^{\mathrm{T}}S_ix_i + u_i^{\mathrm{T}}Ru_i)\mathrm{d}t,$$

$$i = 1,\cdots,r$$

and stabilizes the plants. Hence all $A_i + B_iKC$ are asymptotically stable. On the other hand, it is clear that 3)$\Rightarrow$2). Therefore the proof is completed.

This theorem implies that simultaneous stabilization via static output feedback can be viewed as the solution of r coupled LQ control problems with suitable weighting matrices Q_i, R and S_i in the functional (2) satisfying the coupled constraint (11).

Corollary 1 Given the plant (A, B, C), it is stabilizable via static output feedback if and only if there exist matrices $Q > 0$, $R > 0$ and M such that the following constrained ARE

$$PA + A^{\mathrm{T}}P - (S + B^{\mathrm{T}}P)^{\mathrm{T}}R^{-1}(S + B^{\mathrm{T}}P) + Q = 0, \tag{14}$$

$$Q - S^{\mathrm{T}}R^{-1}S > 0, \tag{15}$$

$$S = ME_\perp - B^{\mathrm{T}}P, \tag{16}$$

has a solution $P > 0$, or equivalently, there exist matrices $P > 0$, $R > 0$ and M satisfying the following constrained ARI with suitable cross term

$$PA + A^{\mathrm{T}}P - (S + B^{\mathrm{T}}P)^{\mathrm{T}}R^{-1}(S + B^{\mathrm{T}}P) + S^{\mathrm{T}}R^{-1}S < 0, \tag{17}$$

Remark 2 Comparing Theorem 1 and Corollary 1 with Lemma 2, the range of the cross matrix S has been constrained. In the case of static output feedback, constraint (16) appears, and so $M \in \mathbb{R}^{m\times p}$ becomes the free variable instead of $S \in \mathbb{R}^{m\times m}$. S is related to the output matrix C such that $S \in \mathrm{Range}(C)$. For the static output feedback simultaneous stabilization of r plants, M is the coupling variable.

Corollary 2 For r given plants G_i in (6), the following statements are equivalent

1) They are simultaneously stabilizable via state feedback.

2) There exist matrices $P_i > 0$, $R > 0$, $Q_i > 0$ and M satisfying the following coupled AREs

$$P_iA_i + A_i^{\mathrm{T}}P_i - (S_i + B_i^{\mathrm{T}}P_i)^{\mathrm{T}}R^{-1}(S_i + B_i^{\mathrm{T}}P_i) + Q_i = 0, \quad i = 1,\cdots,r, \tag{18}$$

$$Q_i - S_i^{\mathrm{T}}R^{-1}S_i > 0,$$

$$S_i = M - B_i^{\mathrm{T}}P_i, \tag{19}$$

and the simultaneously stabilizing state feedback control law can be constructed as

$$K = -R^{-1}M. \tag{20}$$

3) There exist matrices $P_i > 0$, $R > 0$ and M satisfying the following coupled ARIs

$$P_iA_i + A_i^{\mathrm{T}}P_i - (S_i + B_i^{\mathrm{T}}P_i)^{\mathrm{T}}R^{-1}(S_i + B_i^{\mathrm{T}}P_i) + S_i^{\mathrm{T}}R^{-1}S_i < 0, \quad i = 1,\cdots,r, \tag{21}$$

$$S_i = M - B_i^{\mathrm{T}}P_i.$$

Note that this is the case that $C = I$ in Theorem 1, this result can be easily proven.

4 Examples

Example 1 Let us consider the system (6) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = [1 \quad 0]^{\mathrm{T}}, \quad C_1 = [1 \quad -\alpha],$$

which is not stable since its two eigenvalues are 1 and -1. It is not difficult to find this system unable to be stabilized via static output feedback when $\alpha > 0$, while it can be when $\alpha < 0$, Let $\alpha = -1$ and $R = 1$, it is easy to find that when

$$M = [2 \quad 2], \quad S_1 = [0 \quad 1.5],$$

$$P_1 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(9) ~ (13) hold. So $F = 2.0$ is a stabilizing output feedback gain. In fact, both eigenvalues of the closed loop system are -1.0. If we want to find an output feedback gain to simultaneously stabilize the last system and the following system

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = [1 \quad 1].$$

We can find the following selected matrices

$$R = 1, \quad M = [2 \quad 3], \quad S_2 = [1.8182 \quad 1.2727],$$

$$P_2 = \begin{bmatrix} 0.5455 & 0.1818 \\ 0.1818 & 0.7273 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.3306 & 0.2314 \\ 0.2314 & 0.5620 \end{bmatrix}$$

satisfy (9) ~ (13). So static output feedback gain $F = 2.0$ stabilizes these two systems.

Example 2 Let us consider the following 3 systems

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.75 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$R = 1, \quad M = [7.0490 \quad 10.8616 \quad 5.6523],$$

$$P_1 = \begin{bmatrix} 16.3325 & 16.4135 & 4.1315 \\ 16.4135 & 20.5165 & 6.1651 \\ 4.1315 & 6.1651 & 3.0531 \end{bmatrix},$$

$$S_1 = [2.9174 \quad 4.6965 \quad 2.5992],$$

$$P_2 = \begin{bmatrix} 18.1623 & 12.6059 & 3.4580 \\ 12.6059 & 22.5605 & 9.5924 \\ 3.4580 & 9.5924 & 4.6590 \end{bmatrix},$$

$$S_2 = [5.3200 \quad 6.0654 \quad 3.3228],$$

$$P_3 = \begin{bmatrix} 4.0162 & 4.5318 & 1.8994 \\ 4.5318 & 13.8963 & 7.8264 \\ 1.8994 & 7.8264 & 4.7233 \end{bmatrix},$$

$$S_3 = [4.0368 \quad 7.4627 \quad 4.2277].$$

These matrices satisfy the inequality (21). So the state feedback gain

$$K = [-7.0490 \quad -10.8616 \quad -5.6523]$$

stabilizes simultaneously these 3 plants. In fact, the eigenvalues of the closed-loop systems are $\{-0.4677 \pm 0.9589i, -3.4666\}$, $\{-0.7171 \pm 1.3759i, -3.2669\}$ and $\{-7.7011, -0.7978 \pm 0.4628i\}$, respectively.

5 Concluding remarks

This paper has proposed a method to solve the problem of simultaneous stabilization via static output feedback. A necessary and sufficient condition for simultaneous stabilizability of a set of MIMO plants via static output feedback is given using a collection of coupled AREs and ARIs. We notice that the dynamic case may be straightforwardly transformed into a static output feedback design problem by simply constructing an augmented system. Also, by using the method proposed in this paper, we may solve simultaneous stabilization problem since we can obtain their same order stabilizable and detectable realization (6) of the observable form for a set of linear MIMO plants with different order. So we may say that we have recast the simultaneous stabilization problem as a computational procedure.

References

1 Djaferis T E. To stabilze a k real parameter affine family of plants it suffices to simultaneously stabilize 4^k polynomials. Syst. Contr. Lett., 1991, 16(3): 187 – 193

2 Anderson B D O and Moore J B. Optimal Control: Linear Quadratic Methods. Englewood Cliffs, NJ: Prentice-Hall, 1990

3 Boyd S, El Ghaoui L, Feron E and Balakrishnan V. Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia, 1994

4 Geromel J C, Peres P L D and Souza S R. Convex analysis of output feedback control problems: robust stability and performance. IEEE Trans. Automat. Contr., 1996, 41(7): 997 – 1002

5 Cao Y Y, Sun Y X and Mao W J. A new necessary and sufficient condition for static output feedback stabilitzability and comments on "Stabilization via static output feedback". IEEE Trans. Automat. Contr., 1998, 43(8): 1110 – 1111

本文作者简介

曹永岩 见本刊 1999 年第 1 期第 112 页.

孙优贤 见本刊 1999 年第 1 期第 112 页.