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# Initial-Value Problem of Singular Distributed Parameter System *

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**Abstract**: This paper defines the concepts of the generalized left inverse, the generalized Fourier transform and the square root of matrix. Meanwhile, it deals with singular distributed parameter systems described by coupled partial differential equations with singualr matrix coefficients. As to the singular distributed parameter systems, the initial-value problem are considered on the basis of the generalized Fourier transform theorem. The solution of the systems is obtained from the method presented here and the possibility of determining acceptable initial-value conditions is also discussed.

**Key words**: singular distributed parameter systems; generalized Fourier transform; left inverse; Drazin inverse; solution

# 广义分布参数系统的初值问题

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**摘要**: 本文定义了广义左逆、广义 Fourier 变换, 矩阵平方根等概念, 论述了由带奇异系数矩阵的耦合偏微分方程描述的广义分布参数系统, 由广义 Fourier 变换定理讨论了广义分布参数系统的初值问题, 得到了该系统的解及其相容的初值条件.

**关键词**: 广义分布参数系统; 广义 Fourier 变换; 广义左逆; Drazin 逆; 解

## 1 Introduction

Coupled systems of second order partial differential equations appear in the study of the temperature distribution in a composite heat conductor[1~3], in signal propagation in a system of electrical cables[2] and in magneto-hydrodynamics[3]. It is also shown that the same problems are of interest in the area of distributed parameter systems[4~6]. Recently initial-boundary-value problems are considered in the light of both the singular 1-D systems theory and the Fourier approach to distributed parameter systems[4], the analysis is based on the separation principle which breaks the problem down into two 1-D problems: an initial problem for a 1-D singular system in time domain and a two-point boundary-value problem for a second-order system in the spatial domain, but the convergence of solution is not solved. It only obtains the formal solution.

Initial homogeneous problems are considered in the light of both the singular 1-D systems and the generalized Fourier transforms to distributed parameter systems.

Trzaska and Marszalek W have discussed initial-boundary-value problems of the singular distributed parameter systems using the separation principle in [4] as follows:

Consider the following system for $y(t,x) \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n\times n}$,

$$\begin{aligned} & A\frac{\partial^2 y}{\partial x^2} = B\frac{\partial y}{\partial t}, \quad 0 \leqslant x \leqslant l, \quad t \geqslant 0, \\ & y(t,0) = 0, \quad y(t,l) = 0, \\ & y(0,x) \equiv F(x) \in \mathbb{R}^n, \\ & \det A = 0, \quad \det B = 0, \quad \det(Bs + A) \neq 0. \end{aligned} \tag{1.1}$$

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Let us assume that due to separation principle, the solution of Eqn. (1.1) can be represented as

$$y(t,x) = G(t)H(x), \qquad (1.2)$$

where

$$G(t) \in \mathbb{R}^{n\times n}, \quad H(x) \in \mathbb{R}^n.$$

The solution of Eqn. (1.1) is the following

$$y(t,x) = \sum_{k=1}^{\infty} \sin(K\frac{\pi}{l}x)\exp(-\lambda_k \bar{B}^D\bar{A}t)\bar{B}\bar{B}^D V_k,$$

$$V_k \in \mathbb{R}^n, \quad k = 1,2,\cdots, \qquad (1.3)$$

$$\sum_{k=1}^{\infty} \sin(K\frac{\pi}{l}x)\bar{B}\bar{B}^D V_k = F(x). \qquad (1.4)$$

Its substantial problem, the convergence of the solution is not solved, but the authors only discuss a class of singular distributed parameter system.

The separation principle (the Fourier approach) is invalid for the initial-value problem and other distinct type singular distributed parameter system. Therefore, we present the generalized Fourier transform approach.

## 2 Preliminaries

It is well known that the singular parameter system for $x(t) \in \mathbb{R}^n$

$$\begin{aligned} E\dot{x} + Bx(t) &= 0, \\ x(t_o) &= C_0, \end{aligned} \qquad (2.1)$$

where $E$ and $B$ are constant matrices of appropriate dimensions, $C_0 \in \mathbb{R}^n$ and det $E = 0$. We have the following unique solution

$$x(t) = \exp(-\bar{E}^D\bar{B}(t-t_0))\bar{E}\bar{E}^D C_0, \qquad (*)$$

where $\bar{E} = (\lambda E + B)^{-1}E, \bar{B} = (\lambda E + B)^{-1}B$ and index $E = k$.

**Definition 2.1**[8] Set $A \in M_n(C)$, index $A = k$, $Y \in M_n(C)$ and

$$YAf_i = f_i, \quad \forall f_i \in \bigcap_{m=0}^{\infty} \mathrm{Im}A^m = \mathrm{Im}A^k$$

is satisfied, then $Y$ is called a generalized left inverse of $A$ for $f_i$.

Where $f_i = (f_{1i}, f_{2i}, \cdots, f_{ni})^{\mathrm{T}}$, $M_n(C)$ is the space of matrix.

**Definition 2.2** Set $A \in M_n(C)$, index $A = k$. $Y \in M_n(C)$ and $YAf = f, f = (f_1, f_2, \cdots, f_n), f_i \in \mathrm{Im}A^k$ is satisfied, then $Y$ is called a generalized left inverse of $A$ for $f$.

**Corollary 2.1** If $A \in M_n(C)$, index $A = k$, then $A^D$ is a generalized left inverse of $A$ for $f$.

**Definition 2.3** If $B^2 = A$, then $B = \sqrt{A}$ is a square root of $A$, where $A \in M_n(C)$, index $A = k$.

**Definition 2.4** If $F(x) \in M_n(L(-\infty, \infty))$, then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\exp(-i\lambda Ix)\mathrm{d}x = \hat{F}(\lambda) \qquad (2.2)$$

is called Fourier transfoms, $\hat{F}(\lambda)$ is called image of Fourier transfoms of $F(x)$.

**Theorem 2.1** (Fourier integral theorem)

If

$$F(x) \in M_n(L(-\infty,\infty) \cap C^1(-\infty,\infty)),$$

then we have

$$(\hat{F}(\lambda))^{\vee} = \lim_{N\to\infty}\int \hat{F}(\lambda)\exp(i\lambda Ix)\mathrm{d}x = F(x). \qquad (2.3)$$

The formula (2.3) is called inverse formula of Fourier transforms.

**The properties of Fourier transform:**

1° Linear property.

If $F_i(x) \in M_n(L(-\infty,\infty)), \alpha_1, \alpha_2 \in C$, then

$$(\alpha_1 F_1 + \alpha_2 F_2)^{\wedge} = \alpha_1\hat{F}_1 + \alpha_2\hat{F}_2. \qquad (2.4)$$

2° Differential property.

If $F(x), F'(x) \in M_n(L(-\infty,\infty)) \cap C(-\infty, \infty))$, then

$$\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^{\wedge} = i\lambda\hat{F}. \qquad (2.5)$$

3° Multiplied polynomial property.

If $F(x), xF(x) \in M_n(L(-\infty,\infty))$, then

$$\left(\frac{\mathrm{d}^m F}{\mathrm{d}x^m}\right)^{\wedge} = (i\lambda)^m\hat{F}(\lambda), \quad m \geqslant 1. \qquad (2.6)$$

**Corollary 2.2** If $F(x), \cdots, F^m(x) \in M_n(L(-\infty,\infty)) \cap C(-\infty,\infty))$, then

$$\left(\frac{\mathrm{d}^m F}{\mathrm{d}x^m}\right)^{\wedge} = (i\lambda)^m\hat{F}(\lambda), \quad m \geqslant 1. \qquad (2.7)$$

4° Translational property.

If $F(x) \in M_n(L(-\infty,\infty))$, then

$$(F(x-\dot{a}))^{\wedge} = \exp(-i\lambda Ia)\hat{F}(\lambda). \qquad (2.8)$$

By Corollary 2.1 we have

5° Matrix property.

If $F(x) \in M_n(L(-\infty,\infty)), A \in M_n(C)$, $AF(x) = F(x)A$, then

$$(F(x))^{\wedge} = A^D\hat{F}(\lambda A^D), \qquad (2.9)$$

where index $A = k$.

6° Symmetry property.

If $F(x) \in M_n(L(-\infty,\infty))$, then

$$(F(x))^{\vee} = \hat{F}(-x). \qquad (2.10)$$

7° Convolution property.

If $F(x), G(x) \in M_n(L(-\infty,\infty))$, then

$$F * G(x) = \int_{-\infty}^{\infty} F(x-t)G(t)\mathrm{d}t \in M_n(L(-\infty,\infty))$$

and

$$(F * G)^{\wedge} = \sqrt{2\pi}\hat{F}\hat{G}. \qquad (2.11)$$

**Lemma 2.1** If $A \in M_n(C)$, $A^D$ is a generalized left inverse of $A$

1) $$\int_{-\infty}^{\infty} \exp(-Ax^2)\mathrm{d}x$$

is integrable, then

$$\int_{-\infty}^{\infty} \exp(-Ax^2)\mathrm{d}x = C_1,$$

where $C_1$ is constant matrix.

2) If $A$ possesses the eigenvalue of positive real part, then

$$\int_{-\infty}^{\infty} \exp(-Ax^2)\mathrm{d}x = \sqrt{A^D\pi},$$

where $A^D$ is the Drazin inverse of $A$.

Proof 1) is obvious.

$$\int_{-\infty}^{\infty} \exp(-Ax^2) = 2\int_0^{\infty} \exp(-Ax^2)\mathrm{d}x = 2K,$$

where $K = \int_0^{\infty}(-Ax^2)\mathrm{d}x$.

Let

$$I(t) = \int_0^{\infty} \frac{\exp(-At(1+x^2))}{1+x^2}\mathrm{d}x,$$

obviously we have

$$I(0) = \frac{\pi}{2}I, \quad \lim_{t\to\infty} I(t) = 0,$$

$$I'(t) = A\int_0^{\infty} \exp(-At(1+x^2))\mathrm{d}x =$$

$$-At^{-\frac{1}{2}}\exp(-At)\int_0^{\infty}\exp(-Au^2)\mathrm{d}u =$$

$$-AKt^{-\frac{1}{2}}\exp(-At).$$

Integrating both sides of equality from $\delta(\delta > 0)$ to $M$, then

$$I(M) - I(\delta) = -AK\int_{\delta}^{M} t^{-\frac{1}{2}}\exp(-At)\mathrm{d}t =$$

$$2AK\int_{\sqrt{M}}^{\sqrt{\delta}} \exp(-Au^2)\mathrm{d}u.$$

Let $M \to \infty$, $\delta \to 0^+$, by Corollary 2.1 we have

$$K = \frac{\sqrt{A^D\pi}}{2},$$

therefore

$$\int_{-\infty}^{\infty} \exp(-Ax^2)\mathrm{d}x = \sqrt{A^D\pi}.$$

**Example 2.1** Suppose

$$F(x) = \frac{\sqrt{A^D}}{\sqrt{2t}}\exp\left(-\frac{A^Dx^2}{4t}\right),$$

extract $\hat{F}(\lambda)$, where $A$ is a constant matrix, $a > 0$ is a constant.

$$\hat{F}(\lambda,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{A^D}}{\sqrt{2t}}\exp\left(-\frac{A^Dx^2}{4t}\right)\cdot$$

$$\exp(-i\lambda Ix)\mathrm{d}x =$$

$$\frac{A^D}{2t}\left[x\frac{\sqrt{A^D}}{\sqrt{2t}}\exp\left(-\frac{A^Dx^2}{4t}\right)\right] =$$

$$-\frac{A^D}{2t\lambda}\frac{\mathrm{d}\hat{F}(\lambda,t)}{\mathrm{d}\lambda},$$

therefore

$$\begin{cases} A^D\dfrac{\mathrm{d}\hat{F}(\lambda,t)}{\mathrm{d}\lambda} = -2t\lambda\hat{F}(\lambda,t), \\ \hat{F}(0,t) = \sqrt{A^DA\pi}. \end{cases}$$

By Corollary 2.1 we get

$$\hat{F}(\lambda,t) = \exp(-A\lambda^2 t).$$

## 3 Second-order singular distributed parameter perturbation systems

First, we consider the following system for $Z(x,t) \in \mathbb{R}^n$, $E, B \in M_n(C)$,

$$\begin{cases} E\dfrac{\partial Z(x,t)}{\partial t} = A\dfrac{\partial^2 Z(x,t)}{\partial x^2} + QZ(x,t), \\ \qquad -\infty < x < \infty, \quad t \geqslant 0, \\ Z(x,0) = \varphi(t), \quad \varphi(x) \in \mathbb{R}^n, \end{cases} \qquad (3.1)$$

where

$$\det E = 0, \quad \det A = 0, \quad \det(\lambda E - A) \neq 0.$$

The above problem has not been analysed. In this paper we propose a method of analysing Eqn. (3.1), our analysis is based on the Fourier transform method.

We shall assume that there exists a $\lambda$ such that $(\lambda E - A)$ is invertible. Then (3.1) becomes

$$\begin{cases} \bar{E}\dfrac{\partial Z(\lambda,t)}{\partial t} = \bar{A}\dfrac{\partial Z(\lambda,t)}{\partial x^2} + \bar{Q}Z(x,t), \\ \qquad -\infty < x < \infty, \\ Z(x,0) = \varphi(x), \quad \varphi(x) \in \mathbb{R}^n, \end{cases} \qquad (3.2)$$

where

$$\bar E = (\lambda E - A)^{-1}E,\quad \bar A = (\lambda E - A)^{-1}A,$$
$$(\bar E^D\bar A)^D = \bar E\bar A^D.$$

Taking Fourier transform of both sides of (3.2) with $x$, we have

$$\begin{cases}\bar E\dfrac{\mathrm{d}\hat Z(\lambda,t)}{dt} = -\lambda^2\bar A\hat Z(\lambda,t) + \bar Q\hat Z(\lambda,t),\\ \hat Z(\lambda,0) = \hat\varphi(\lambda).\end{cases}\tag{3.3}$$

By formula ( * ), we get

$$\hat Z(\lambda,t) = \exp(\bar E^D\bar Q t - \bar E^D\bar A\lambda^2 t)\bar E\bar E^D\hat\varphi(\lambda).$$

Taking inverse Fourier transform of both sides of the equality.

$$Z(x,t) = (\exp(-\bar E^D\bar A\lambda^2 t)\bar E\bar E^D\hat\varphi(\lambda))^{\vee}.$$

From Examples 2.1 and (2.11), we obtain

$$(\exp(-\bar E^D\bar A\lambda^2 t)\bar E\bar E^D\hat\varphi(\lambda))^{\vee} = (\hat g\hat\varphi)^{\vee}$$

where

$$g(x,t) = \frac{\sqrt{\bar E\bar A^D}}{\sqrt{2t}}\exp\left(-\frac{\bar E\bar A^D x^2}{4t}\right)\bar E\bar E^D,$$

therefore

$$Z(x,t) = \int_{-\infty}^{\infty}K(x-\zeta,t)\varphi(\zeta)d\zeta,\tag{3.4}$$

where

$$K(x,t) = \begin{cases}\dfrac{\sqrt{\bar E\bar A^D}}{2\sqrt{\pi t}}\exp\left(\bar E^D\bar Q t - \dfrac{\bar E\bar A^D x^2}{4t}\right)\bar E\bar E^D, & t>0,\\ 0, & t\leqslant 0.\end{cases}\tag{3.5}$$

The formula (3.4) is the solution of initial-problem (3.1), when $\varphi(x)$ satisfies some conditions.

**Theorem 3.1** If $\varphi(x)$ is continuous vector function in $(-\infty,\infty)$, and it is bounded, then the unique solution of (3.1) is given by the formula (3.4) if and only if $\sqrt{\bar A\bar A^D\bar E\bar E^D}\varphi(x) = \varphi(x)$.

Proof The function

$$Z(x,t) = \int_{-\infty}^{\infty}K(x-\zeta,t)\varphi(\zeta)d\zeta,$$

as $t>0$, $Z(x,t)$ is infinite-time which can take arbitrary time derivative with respect to $x$ and $t$ in integral. Therefore as $t>0$, we have

$$\bar E\frac{\partial Z(x,t)}{\partial t} =$$
$$\int_{-\infty}^{\infty}\Big[\frac{-1}{4\sqrt{\pi}t^{3/2}}\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\times$$
$$\bar E\sqrt{\bar E}\ \sqrt{\bar A^D\bar E\bar E^D} + \frac{1}{2\sqrt{\pi t}}\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\times$$
$$\frac{(x-\zeta)^2}{4t^2}\bar E\ \sqrt{\bar E}\ \sqrt{\bar A^D\bar E\bar A^D\bar E\bar E^D}\Big]\varphi(\zeta)d\zeta =$$
$$\int_{-\infty}^{\infty}\Big[\frac{-1}{4\sqrt{\pi}t^{3/2}}\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\times$$
$$\sqrt{\bar A^D}\sqrt{\bar E}\ \bar E^2\bar E^D + \frac{(x-\zeta)^2}{8\sqrt{\pi}t^{5/2}}\cdot$$
$$\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\times$$
$$\bar A^D\sqrt{\bar A^D}\sqrt{\bar E}\ \bar E^3\bar E^D\Big]\varphi(\zeta)d\zeta,\tag{3.6}$$

$$\bar A\frac{\partial^2 Z(x,t)}{\partial x^2} =$$
$$\int_{-\infty}^{\infty}\Big[\frac{1}{2\sqrt{\pi t}}\times\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\cdot$$
$$\frac{(x-\zeta)^2}{4t^2}\bar A(\bar E\bar A^D)^2\sqrt{\bar E}\ \sqrt{\bar A^D\bar E\bar E^D} - \frac{-1}{4\sqrt{\pi}t^{3/2}}\cdot$$
$$\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\times$$
$$\bar A\bar E\bar A^D\sqrt{\bar E}\ \sqrt{\bar A^D\bar E\bar E^D}\Big]\varphi(\zeta)d\zeta =$$
$$\int_{-\infty}^{\infty}\Big[\frac{(x-\zeta)^2}{8\sqrt{\pi}t^{5/2}}\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\times$$
$$\bar A^D\sqrt{\bar A^D}\sqrt{\bar E}\ \bar E^3\bar E^D - \frac{-1}{4\sqrt{\pi}t^{3/2}}\cdot$$
$$\exp\left(\bar E^D\bar Q t - \frac{\bar E\bar A^D(x-\zeta)^2}{4t}\right)\times$$
$$\sqrt{\bar A^D}\sqrt{\bar E}\ \bar E^2\bar E^D\Big]\varphi(\zeta)d\zeta.\tag{3.7}$$

By (3.6) and (3.7), we get

$$\bar E\frac{\partial Z(x,t)}{\partial t} = \bar A\frac{\partial^2 Z(x,t)}{\partial x^2} + \bar Q Z(x,t).$$

Namely

$$E\frac{\partial Z(x,t)}{\partial t} = A\frac{\partial^2 Z(x,t)}{\partial x^2} + Q Z(x,t).$$

In the following, we shall prove

$$\lim_{t\to 0^+}Z(x,t) = \varphi(x).$$

Set $\eta = \dfrac{\zeta - x}{2\sqrt t}$, then

$$Z(x,t) = \frac{\sqrt{\bar E}\ \sqrt{\bar A^D\bar E\bar E^D}}{\sqrt\pi}\cdot$$
$$\int_{-\infty}^{\infty}\exp(\bar E^D\bar Q t - \eta^2\bar E\bar A^D)\varphi(x+2\sqrt{t}\eta)d\eta.$$

By the boundedness of $\varphi(x)$, as $t>0$, the integral is uniform convergence with respect to $t$, therefore

$$\lim_{t \to 0^+} Z(x,t) =$$

$$\lim_{t \to 0^+} \frac{\sqrt{\bar{E}} \quad \sqrt{\bar{A}^D \bar{E}\bar{E}^D}}{\sqrt{\pi}} \cdot$$

$$\int_{-\infty}^{\infty} \exp(\bar{E}^D \bar{Q} t - \eta^2 \bar{E}\bar{A}^D) \varphi(x + 2\sqrt{t}\eta) \mathrm{d}\eta =$$

$$\sqrt{\bar{A}\bar{A}^D} \sqrt{\bar{E}\bar{E}^D} \varphi(x) = \varphi(x).$$

From the above statement, we can see that the solution of Eqn. (3.1) in the form of integral is convergent and unique. The applied range of the generalized Fourier transform approach is wider than the separation principle in singular distributed parameter systems.

In other papers, the approach of using the generalized Fourier transforms can be used to deal with other distinct types of singular distributed parameter systems. The boudary-value problems and initial-boundary-value problems will be discussed.

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