第12卷第1期 控制理论与应用 Vol. 12, No. 1
1995年2月 CONTROL THEORY AND APPLICATIONS Feb., 1995

Qualitative Analysis for Distributed Iterative Stochastic Large-Scale Systems*

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Abstract: This paper gives a qualitative analysis for distributed iterative stochastic large-scale systems. Criteria are established for connective mean-square convergence of time-invariant linear, time-varying linear and nonlinear iterative stochastic large-scale systems. In case of all isolated subsystems to be convergent, the mean-square convergence of the iterative stochastic large-scale systems can be always guaranteed for any interconnection and under structural perturbations, only by choosing the appropriate multipliers in the subsystems.

Key words: large-scale system; connective mean-square convergence; iterative stochastic system

1 Introduction

Convergence and stability of distributed iterative stochastic large-scale systems had been recently studied by [1 ~ 4]. However, structural perturbations have not been considered in the results obtained in [1 ~ 4]. A distributed iterative stochastic large-scale systems consisting of many subsystems can hardly be expected to stay"in one piece"over long periods of operation. Quite commonly, either by design or by fault, the subsystems are disconnected and reconnected during the functioning of the system structure which may destroy convergence and cause the system to diverge. To prevent the divergence, systems should be built to have the desirable convergence properties invariant under structural perturbations, that is, to be connectively convergent.

In order to formulate precisely what we mean by connective convergence, we need a detailed description of the structural perturbations of distributed iterative stochastic systems via interconnection matrices. A description for distributed iterative stochastic systems under structural perturbations is given in Section 2.

In this paper, the definition of connective mean-square convergence is given for iterative stochastic large-scale systems. Criteria are established for connective mean-square convergence of time-invariant linear, time-varying linear and nonlinear iterative stochastic large-scale systems. By only choosing the appropriate multipliers in the stochastic large-scale systems, the connective mean-square convergence of the iterative stochastic large-scale systems can always be guaranteed for any interconnection under structural perturbations, provided that all the isolated subsystems

* This work was supported by the Science Foundation of Guangdong Province.
Manuscript received Jan. 8, 1993, revised Jul. 25, 1994.

are mean-square convergent.

2 System Description and Definitions

Consider the following linear distributed iterative stochastic large-scale system

$$x_i(k+1) = x_i(k) + a_i[A_i x_i(k) + \sum_{j=1}^{N} e_{ij} H_{ij} x_j(k) w(k)], \quad i = 1, \cdots, N \tag{1}$$

where $x_i \in R^{n_i} (n_i \leqslant n)$ is the state of the ith subsystem at the step $k \in T$, where T denotes the set of nonnegative integers, a_i with $0 < a_i < 1$ represents multiplier which is used to scale the ith subsystem and which can be chosen appropriately; H_{ij} is an $n_i \times n_i$ constant matrix; e_{ij} is the element of $N \times N$ interconnection matrix; $E = (e_{ij})$ which is generated by an $N \times N$ fundamental interconnection matrix $\bar{E} = (\bar{e}_{ij})$ (denoted by $E \in \bar{E}$), i. e. , $\bar{e}_{ij} = 0$ implies e_{ij}, where e_{ij} is either 0 or 1; $\{w(k), k \in T\}$ is a scalar stochastic process defined on the complete probability space $(\Omega, \widetilde{F}, P)$ satisfying

$$E\{w(k) | \widetilde{F}_k\} = 0, \quad E\{w^2(k) | \widetilde{F}_k\} = 1$$

where $\widetilde{F}_k$ denotes an increasing family of a sub-σ-field of $\widetilde{F}$ such that $\{w(j), 0 \leqslant j \leqslant k-1\}$ and the initial state $x_{i0} = x_{i0}(0)$ is $\widetilde{F}_k$ -measurable.

More generally, we consider the following time-varying linear distributed iterative stochastic large-scale system

$$x_i(k+1) = x_i(k) + a_i[A_{ii} x_i(k) + \sum_{j=1}^{N} e_{ij}(k) H_{ij}(k) x_{ij}(k) w(k)], \quad i = 1, \cdots, N \tag{2}$$

where e_{ij} with $0 \leqslant e_{ij}(k) \leqslant 1$ is the function of $k \in T$, which is also generated by the $N \times N$ fundamental interconnection matrix $\bar{E} = (\bar{e}_{ij})$, $H_{ij}(k)$ $(i \neq j)$ is an $n_i \times n_j$ time-varying matrix.

Furthermore, we consider the following nonlinear distributed iterative stochastic large-scale system

$$x_i(k+1) = x_i(k) + a_i[A_{ii} x_i(k) + \sum_{j=1}^{N} f_{ij}(k, x_j(k), e_{ij}(k)) w(k)], \quad i = 1, \cdots, N \tag{3}$$

where $f_{ij}(i \neq j)$ is a function of k satisfying

$$|f_{ij}(k, x_j(k), e_{ij}(k))| \leqslant \bar{e}_{ij} a_{ij} |x_j(k)|, \quad f_{ij}(0, e_{ij}(k)) \equiv 0.$$

It is obvious that both system (1) and (2) are the special case of system (3).

Let I_i denote the $n_i \times n_i$ identical matrix. And let

$$n = \sum_{i=1}^{N} n_i, \quad x = (x_1^T, \cdots, x_N^T)^T,$$

$$\Lambda = \text{block-diag } \{a_1 I_1 \cdots, a_N I_N\}, A = \text{block-diag } \{A_{11}, \cdots, A_{NN}\},$$

$$f_j(k, e_{1j}, \cdots, e_{Nj}, x_j(k)) = (f_{1j}^T(k, x_j(k), e_{1j}(k)), \cdots, f_{Nj}^T(k, x_j(k), e_{Nj}(k)))^T.$$

H and $H(k)$ are the matrices consisting of the block matrices, in which the block matrix at the ith and jth column are $e_{ij} H_{ij}$ and $e_{ij}(k) H_{ij}(k)$ respectively. It should be noted that Λ is a multiplier diagonal-block matrix which can be chosen appropriately.

By using the above notations, the distributed iterative stochastic large-scale system (1), (2) and (3) can be rewritten in the following equivalent forms respectively.

$$x(k+1) = x(k) + \Lambda[Ax(k) + Hx(k)w(k)] \tag{4}$$

$$x(k+1) = x(k) + \Delta[Ax(k) + H(k)x(k)w(k)] \tag{5}$$

and

$$x(k+1) = x(k) + \Delta[Ax(k) + \sum_{j=1}^{N} f_j(k, e_{1j}, \cdots, e_{Nj}, x_j(k))w(k)]. \tag{6}$$

The iterative stochastic large-scale system (6) (or (4), (5)) can be considered as an interconnection of the following N isolated subsystems

$$x_i(k+1) = x_i(k) + \alpha_i A_{ii} x_i(k), \quad i = 1, \cdots, N. \tag{7}$$

For the definition of mean-square convergene for the iterative stochastic system (4), (5), or (6), refer to Chapter 6 of [5].

Definition 1 System (6) is called to be connectively mean-square convergent, if it is mean-square convergent for all $E \in \bar{E}$.

Lemma 1 Let L be the difference operator defined by system (6) (or (4), or (5)) (refer to (6.4) in Chapter 6 of [5]). For the iterative stochastic system (6) (or (4), or (5)), if there exist a symmetric and positive definite Lyapunov function $v(x)$ which is continuously differentiable and a positive number γ with $0 < \gamma < 1$, such that

$$Lv(x) \leqslant -\gamma v(x).$$

Then system (6) is mean-square convergent.

Proof The proof follows from the results in [5].

Lemma 2 For any y, $z \in \mathrm{R}^n$, $A = (a_{ij})_{n\times n}$, $c > 0$, the following inequality holds

$$-c|y|^2 + y^{\mathrm{T}} A z \leqslant -\frac{c}{2}|y|^2 + \frac{1}{2c}|A|^2|z|^2 \tag{8}$$

where $|\cdot|$ denotes the norm of a vector or a matrix which is induced by the Euclidean norm.

Proof Refer to [4] for the proof.

In this paper, $x(k)$ and $x_i(k)$ is abbreviated by x and x_i respectively, if there is no confusion. $\lambda_{\mathrm{m}}(\cdot)$ and $\lambda_{\mathrm{M}}(\cdot)$ denote the minimal the maximal eigenvalues respectively.

3 Results for Connective Mean-Square Convergence

Theorem 1 The distributed iterative stochastic system (4) is connectively mean-square convergent, if there exist $n_i \times n_i$ symmetric and positive definitive matrices P_i and Q_i as well as an appropriate constant α_i with $0 < \alpha_i < 1$, such that the following conditions are satisfied:

i) $A_{ii}^{\mathrm{T}} P_i + P_i A_{ii} + A_{ii}^{\mathrm{T}} P_i A_{ii} + Q_i = 0, \quad i = 1, \cdots, N$;

ii) $0 < \beta_1 < 1$,

where β_1 is defined by

$$\beta_1 = \min_{1 \leqslant i \leqslant N} \left\{ \frac{1}{\lambda_{\mathrm{m}}(P_i)} \left[\frac{1}{2}\alpha_i \lambda_{\mathrm{m}}(Q_i) - \sum_{k=1}^{N} \frac{N}{2\alpha_k \lambda_{\mathrm{m}}(Q_k)} \left| \sum_{j=1}^{N} \alpha_j^2 H_{jk} P_j H_{ji} \right| \right] \right\}.$$

Proof For the ith subsystem, we choose the following Lyapunov function

$$v_i(x_i) = x_i^{\mathrm{T}} P_i x_i \tag{9}$$

where P_i is given by condition (i). In fact, it is easy to get that

$$\lambda_{\mathrm{m}}(P_i)|x_i|^2 \leqslant v_i(x_i) \leqslant \lambda_{\mathrm{M}}(P_i)|x_i|^2. \tag{10}$$

Let L_i be the difference operator determined by the ith subsystem (refer to (6.4) in Chapter

6 of [5]). By condition (i), we have

$$\begin{aligned} L_i v_i &= E\{v_i(x_i(k+1))\,|\,\widetilde{F}_k\} - v_i(x_k) \\ &= -\alpha_i^{\mathrm{T}} x_i^{\mathrm{T}}(k)[A_{ii}^{\mathrm{T}} P_i + P_i A_{ii} + A_{ii}^{\mathrm{T}} P_i A_{ii}] x_i(k) \\ &= -\alpha_i x_i^{\mathrm{T}}(k) Q_i x_i(k) \\ &\leqslant -\alpha_i \lambda_{\mathrm{m}}(Q_i)\,|x_i(k)|^2. \end{aligned} \tag{11}$$

And it is obvious that one can choose $0<\alpha_i<1$ such that $0<\alpha_i\lambda_{\mathrm{m}}(Q_i)\lambda_M^{-1}(P_i)<1$. By (11) we have

$$L_i v_i \leqslant -\alpha_i \lambda_{\mathrm{m}}(Q_i)\lambda_{\mathrm{M}}^{-1}(P_i) v_i, \tag{12}$$

Hence, by the results obtained in Chapter 6 of [5], each isolated subsystem (7) is mean-square convergent.

For the entire distributed iterative stochastic large-scale system (4) (or (1)), we choose the following Lyapunov function

$$v(x) = \sum_{i=1}^{N} v_i(x_i). \tag{13}$$

Let L be the operator determined by (4). We have

$$\begin{aligned} Lv(x) &= \sum_{i=1}^{N} [L_i v_i(x_i) + \alpha_i^2 \sum_{j=1}^{N}\sum_{k=1}^{N} x_j^{\mathrm{T}} H_{ij}^{\mathrm{T}} P_i H_{ik} x_k] \\ &= \sum_{i=1}^{N} (-\alpha_i x_i^{\mathrm{T}} Q_i x_i) + \sum_{i=1}^{N}\sum_{k=1}^{N} x_j^{\mathrm{T}} (\sum_{j=1}^{N} \alpha_j^2 e_{ij} e_{jk} H_{ji}^{\mathrm{T}} P_j H_{jk}) x_k \\ &\leqslant \sum_{i=1}^{N}\sum_{k=1}^{N}\Big[-\frac{1}{N}\alpha_i\lambda_{\mathrm{m}}(Q_i)\,|x_i|^2 + x_i^{\mathrm{T}}\Big(\sum_{j=1}^{N} e_{ij}e_{jk}\alpha_j^2 H_{ji}^{\mathrm{T}} P_j H_{jk}\Big) x_k\Big]. \end{aligned} \tag{14}$$

By applying Lemma 2 to (14), we have

$$\begin{aligned} Lv(x) &\leqslant \sum_{i=1}^{N}\sum_{k=1}^{N}\Big[-\frac{1}{2N}\alpha_i\lambda_{\mathrm{m}}(Q_i)\,|x_i|^2 + \frac{N\bar{e}_{ij}\bar{e}_{jk}}{2\alpha_i\lambda_{\mathrm{m}}(Q_i)}\Big|\sum_{j=1}^{N}\alpha_j^2 H_{ji}^{\mathrm{T}} P_j H_{jk}\Big|^2 |x_k|^2\Big] \\ &\leqslant \sum_{i=1}^{N}\Big[-\frac{1}{2}\alpha_i\lambda_{\mathrm{m}}(Q_i)\Big]|x_i|^2 + \sum_{i=1}^{N}\sum_{k=1}^{N}\Big[\frac{N}{2\alpha_i\lambda_{\mathrm{m}}(Q_i)}\Big|\sum_{j=1}^{N}\alpha_j^2 H_{ji}^{\mathrm{T}} P_j H_{jk}\Big|^2 |x_k|^2\Big] \\ &\leqslant \sum_{i=1}^{N}\Big[-\frac{1}{2}\alpha_i\lambda_{\mathrm{m}}(Q_i)\Big]|x_i|^2 + \sum_{i=1}^{N}\Big[\sum_{k=1}^{N}\frac{N}{2\alpha_k\lambda_{\mathrm{m}}(Q_k)}\Big|\sum_{j=1}^{N}\alpha_j^2 H_{jk}^{\mathrm{T}} P_j H_{ji}\Big|^2 |x_i|^2\Big] \\ &\leqslant -\sum_{i=1}^{N}\Big[\frac{1}{2}\alpha_i\lambda_{\mathrm{m}}(Q_i) - \sum_{k=1}^{N}\frac{N}{2\alpha_k\lambda_{\mathrm{m}}(Q_k)}\Big|\sum_{j=1}^{N}\alpha_j^2 H_{jk}^{\mathrm{T}} P_j H_{ji}\Big|^2\Big]|x_i|^2 \\ &\leqslant -\beta_1 v(x) \end{aligned}$$

where condition ii) is applied. By Lemma 1 we get that the distributed iterative stochastic large-scale system (4) (or (1)) is connectively mean-square convergent. This completes the proof.

Theorem 2 The distributed iterative stochastic system (5) is connectively mean-square convergent, if there exist $n_i \times n_i$ symmetric and positive definite matrices P_i and Q_i as well as an appropriate constant α_i with $0<\alpha_i<1$, such that the following conditions are satisfied:

i) $A_{ii}^{\mathrm{T}} P_i + P_i A_{ii} + A_{ii}^{\mathrm{T}} P_i A_{ii} + Q_i = 0, \quad i=1,\cdots,N;$

ii) $0<\beta_2<1$,

where β_2 is given by

$$\beta_2 = \min_{1\leq i\leq N} \inf_{k\in T}\left\{\frac{1}{\lambda_m(P_i)}\left[\frac{1}{2}\alpha_i\lambda_m(Q_i) - \sum_{k=1}^{N}\frac{N}{2\alpha_k\lambda_m(Q_k)}\left|\sum_{j=1}^{N}\alpha_j^2 H_{jk}(k)P_jH_{ji}(k)\right|^2\right]\right\}.$$

Proof The proof is analogous to that of Theorem 1 and therefore is omitted here.

Theorem 3 The distributed iterative stochastic system (6) is connectively mean-square convergent, if there exist $n_i \times n_i$ symmetric and positive definitive matrices P_i and Q_i as well as an appropriate constant α_i with $0 < \alpha_i < 1$, such that the following conditions are satisfied:

i) $A_{ii}^T P_i + P_i A_{ii} + A_{ii}^T P_i A_{ii} + Q_i = 0, \quad i = 1,\cdots,N;$

ii) $0 < \beta_3 < 1$,

where β_3 is defined by

$$\beta_3 = \min_{1\leq i\leq N}\left\{\frac{1}{\lambda_m(P_i)}\left[\frac{1}{2}\alpha_i\lambda_m(Q_i) - \sum_{k=1}^{N}\frac{N}{2\alpha_k\lambda_m(Q_k)}\sum_{j=1}^{N}\alpha_j^2 a_{jk}a_{ji}|P_j|\right]\right\}.$$

Proof Analogously to the proof of Theorem 1, for the entire distributed iterative stochastic large-scale system (6), we choose the following Lyapunov function

$$v(x) = \sum_{i=1}^{N} v_i(x_i) \tag{16}$$

where $v_i(x)$ is defined by (9).

Let L be the operator determined by (6). We have

$$\begin{aligned} Lv(x) &= \sum_{i=1}^{N}\left\{L_i v_i(x_i) + \alpha_i^2\sum_{j=1}^{N}\sum_{k=1}^{N} f_{ij}^T(k,\ x_j(k),\ e_{ij}(k))P_i f_{ik}(k, x_j(k),\ e_{ij}(k))\right\} \\ &\leqslant \sum_{i=1}^{N}(-\alpha_i\lambda_m(Q_i))|x_i|^2 + \sum_{i=1}^{N}\sum_{k=1}^{N}|x_i|\left(\sum_{j=1}^{N}\alpha_j^2\bar{e}_{ij}\bar{e}_{jk}a_{ji}a_{jk}|P_j|\right)|x_k| \\ &\leqslant \sum_{i=1}^{N}\sum_{k=1}^{N}\left[-\frac{1}{N}\alpha_i\lambda_m(Q_i)|x_i|^2 + |x_i|\left(\sum_{j=1}^{N}\alpha_j^2 a_{ji}a_{jk}|P_j|\right)|x_k|\right]. \end{aligned} \tag{17}$$

By applying Lemma 2 to (17), we have

$$\begin{aligned} Lv(x) &\leqslant \sum_{i=1}^{N}\sum_{k=1}^{N}\left[-\frac{1}{2N}\alpha_i\lambda_m(Q_i)|x_i|^2 + \frac{N}{2\alpha_i\lambda_m(Q_i)}\left(\sum_{i=1}^{N}\alpha_j^2 a_{ji}a_{jk}|P_j|\right)^2|x_k|^2\right] \\ &\leqslant \sum_{i=1}^{N}\left[-\frac{1}{2}\alpha_i\lambda_m(Q_i)\right]|x_i|^2 + \sum_{i=1}^{N}\sum_{k=1}^{N}\left[\frac{N}{2\alpha_i\lambda_m(Q_i)}\left(\sum_{j=1}^{N}\alpha_j^2 a_{ji}a_{jk}|P_j|\right)^2|x_k|^2\right] \\ &\leqslant -\sum_{i=1}^{N}\left[\frac{1}{2}\alpha_i\lambda_m(Q_i) - \sum_{k=1}^{N}\frac{N}{2\alpha_k\lambda_m(Q_k)}\left(\sum_{j=1}^{N}\alpha_j^2 a_{ji}a_{jk}|P_j|\right)^2\right]|x_i|^2 \\ &\leqslant -\beta_3 v(x) \end{aligned} \tag{18}$$

where conditon ii) is applied. By Lemma 1, we get that the distributed iterative stochastic large-scale system (6) is connectively mean-square convergent. The proof is complete.

4 Conclusion

The distributed iterative stochastic large-scale systems are considered. Definition and criteria are given for connective mean-square convergence of such systems. Provided that all isolated subsystems are mean-square convergent, the connective mean-square convergence of the distributed iterative stochastic large-scale systems can always be guaranteed for any interconnection and under structural perturbations, only by choosing the appropriate multipliers in the subsystems.

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分布式迭代随机大系统的定性分析

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摘要: 本文对分布式迭代随机大系统给出了定性分析. 对时不变线性、时变线性和非线性迭代随机大系统建立了关联均方收敛性判据. 在所有孤立子系统都收敛的情形,只需选择子系统中适当的乘子,迭代随机大系统就能对任意互连项和在结构扰动下保持均方收敛性.

关键词: 大系统; 关连均方收敛性; 迭代随机系统

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