

New Method for Optimization of Hierarchical Control

HUANG Xunan, SHAO Huihe and HAN Zhengzhi

(Department of Automatic Control, Shanghai Jiaotong University • Shanghai, 200030, PRC)

Abstract: This paper presents a new method of hierarchical control for large-scale linear systems via the shifted Chebyshev series (SCS). The main idea is that convert the differential equations to algebraic solutions. The algorithm is simple in form and computationally advantageous. An example is given to demonstrate the applicability of the proposed method.

Key words: optimization; hierarchical control; Chebyshev series

1 Introduction

The polynomial series analysis of linear systems has been developed well recently by many authors^[1,2]. For a class of large-scale interconnected systems, one analysed the interaction prediction approach with quadratic performance via polynomial series approximation^[3~5]. According to these method, the series approach is only employed to solve the two-point boundary value problem.

In this paper, we study the algorithm of the Goal Coordination Method (GCM)^[6] and suggest a new method of obtaining an analytic solution for large-scale systems. By using particular properties of SCS, we will show that for a large-scale systems, each of whose variables is a linear combination of SCS, the hierarchical control can be obtained by solving a set of linear algebraic equations. The proposed algorithm can easily be adapted for computer programming. An numerical example is given to illustrate the effectiveness of proposed method.

2 Description of Hierarchical Control Problem

The problem is to minimize J where

$$J = \frac{1}{2} \sum_{i=1}^n \int_0^t (x_i^T Q_i x_i + u_i^T R_i u_i) dt \quad (1)$$

where Q_i are positive semi-definite and R_i are positive definite matrices. Notation n is the number of interconnected subsystems which comprise the overall system, x_i is the n_i -dimensional state vector of the i th subsystem and u_i is the corresponding m_i -dimensional control vector. $\sum_{i=1}^n n_i = N$ and

$\sum_{i=1}^n m_i = M$. J in (1) is to be minimized subject to subsystem dynamics

$$\dot{x}_i = A_i x_i + B_i u_i + C_i Z_i, \quad i = 1, 2, \dots, n \quad (2)$$

where

$$z_i = \sum_{j=1}^n L_{ij} x_j, \quad i = 1, 2, \dots, n. \quad (3)$$

Here z_i is the q_i -dimensional input vector which comes in from the other subsystems. Such a problem can be viewed in the integrated or composite sense as a large multi-variable regulator but with a special structure. In order to utilize this structure to reduce the computer storage, define the dual function $\pi(\lambda)$ where

$$\pi(\lambda) = \min \{L(x_i, u_i, z_i)\}, \quad i = 1, 2, \dots, n. \quad (4)$$

where
$$\{L(x_i, u_i, \lambda, z_i)\} = \frac{1}{2} \sum_{i=1}^n \int_0^{t_f} [(x_i^T Q_i x_i + u_i^T R_i u_i) + 2\lambda_i^T (z_i - \sum_j L_{ij} x_j)] dt \quad (5)$$

where $\lambda(t)$ is the $\sum_{i=1}^n q_i = q$ -dimensional vector of Lagrange multipliers.

Then by the theorem of strong Lagrange duality

$$\max_{\lambda} \pi(\lambda) = \min_{u_i} J, \quad i = 1, 2, \dots, n. \quad (6)$$

Thus an alternative way of optimizing J is to maximize $\pi(\lambda)$ in (6).

Now from eq. (5), for a given $\lambda(t)$, the Lagrangian L is separable into n independent minimization problems, the i th of which is given by

$$\min L_i = \frac{1}{2} \int_0^{t_f} [(x_i^T Q_i x_i + u_i^T R_i u_i) + 2\lambda_i^T (z_i - \sum_j L_{ij} x_j)] dt \quad (7)$$

subject to (2) and (3).

This leads to two-level structure shown in Fig. 1 where on the first level, for given λ , L_i in eq. (7) is minimized subject to eq. (2), (3) and on level 2, the $\lambda(t)$ trajectory is improved using, say, the steepest ascent method, i. e. from iteration j to $j+1$:

$$\lambda(t)^{j+1} = \lambda(t)^j + \alpha^j d^j \quad (8)$$

here $\alpha^j > 0$ is the step length and d^j the steepest ascent search direction. At the optimum $d^j \rightarrow 0$ and the appropriate Lagrange multiplier trajectory is the optimal one.

3 Some properties of SCS

The Chebyshev polynomial $T_i(\bar{z})$ is defined as [7]:

$$T_i(\bar{z}) = \cos(i \cos^{-1} \bar{z}), \quad -1 \leq \bar{z} \leq 1 \quad (9)$$

where the independent variable is defined between -1 and 1 . In order to solve the optimal control problem of a linear large-scale system, the domain may be transformed into values between 0 and β , where

$$z = \beta(1 - \bar{z})/2. \quad (10)$$

The shifted Chebyshev polynomials are then obtained as follows:

$$\begin{cases} T_0(z) = 1, \\ T_1(z) = 1 - 2z/\beta, \\ T_2(z) = 8(z/\beta)^2 - 8(z/\beta) + 1, \\ \vdots \\ T_{i+1}(z) = (2 - 4z/\beta)T_i(z) - T_{i-1}(z). \end{cases} \quad (11)$$

Furthermore an arbitrary function, $f(z)$, can be approximated by Chebyshev series; it is

$$f(z) = \sum_{i=1}^{m-1} f_i T_i(z) = f^T T(z) \quad (12)$$

where the superscript T denotes transpose, f is the Chebyshev coefficient vector and $T(z)$ is the Chebyshev vector. These two vector are defined as

$$f = [f_0, f_1, \dots, f_{m-1}]^T \quad (13)$$

and

$$T(z) = [T_0(z), T_1(z), T_2(z), \dots, T_{m-1}(z)]^T. \quad (14)$$

When m is large enough, it is well known that the approximation error of a finite SCS is less than a finite value which decreases as m increases.

The operational matrix of integration for the shifted Chebyshev series be approximated as

$$\int_0^z T(z) dz = HT(z) \quad (15)$$

where H is the operational matrix of integration:

$$H = \beta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{2m(m-2)} & 0 & 0 & 0 & \frac{1}{4(m-2)} & 0 & \dots \end{bmatrix}. \quad (16)$$

Since

$$T_i(z)T_j(z) = [T_{i+j}(z) + T_{i-j}(z)]/2$$

an important result can be obtained as follows^[1]

$$W = \int_0^\beta T(z)T^T(z)dz$$

$$= \frac{\beta}{2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \dots & \frac{(-1)^{m-1}-1}{2(m-2)(m-3)} & \frac{(-1)^{m-1}}{2m(m-2)} \\ 0 & -\frac{1}{3} & 0 & \dots & \frac{(-1)^m-1}{2m(m-2)} & \frac{(-1)^{m+1}-1}{2(m+1)(m-1)} \\ -\frac{1}{3} & 0 & -\frac{1}{15} & \dots & \frac{(-1)^{m+1}-1}{2(m+1)(m-1)} & \frac{(-1)^{m+2}-1}{2(m+2)m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{(-1)^m-1}{2m(m-2)} & \frac{(-1)^{m+1}-1}{2(m+1)(m-1)} & \frac{(-1)^{m+2}-1}{2(m+2)m} & \dots & 0 & -\frac{1}{(2m-1)(2m-3)} \end{bmatrix}$$

$$+ \frac{\beta}{2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \cdots & \frac{(-1)^{m-1}-1}{2(m-2)(m-3)} & \frac{(-1)^{m-1}}{2m(m-2)} \\ 0 & 1 & 0 & \cdots & \frac{(-1)^{m-2}-1}{2(m-2)(m-4)} & \frac{(-1)^{m-1}-1}{2(m-1)(m-3)} \\ -\frac{1}{3} & 0 & 1 & \cdots & \frac{(-1)^{m-3}-1}{2(m-3)(m-5)} & \frac{(-1)^{m-2}-1}{2(m-2)(m-4)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{(-1)^m-1}{2m(m-2)} & \frac{(-1)^{m-1}-1}{2(m-1)(m-3)} & \frac{(-1)^{m-2}-1}{2(m-1)(m-4)} & \cdots & 0 & 1 \end{bmatrix}. \quad (17)$$

4 Hierarchical Control via SCS

Now, we propose a new algorithms of hierarchical feedback control for linear systems via SCS.

1) Consider the lower level problems. For given λ , L_i in eq. (7) is minimized subject to eqs. (2) and (3). The subscript i in (2), (3) and (7) for i th subsystem is omitted for the sake of simplicity, except for some special situations. In order to solve (2) we approximate $x_i(t)$, $u_i(t)$ and $z_i(t)$ by a finite term of SCS as follows:

$$x(t) = [x_0, x_1, \dots, x_{m-1}]T(t) = XT(t), \quad (18)$$

$$u(t) = [u_0, u_1, \dots, u_{m-1}]T(t) = UT(t), \quad (19)$$

$$z(t) = [z_0, z_1, \dots, z_{m-1}]T(t) = ZT(t) \quad (20)$$

where x , u and z are $(n_i \times 1)$, $(m_i \times 1)$ and $(n_i \times 1)$ vectors.

In order to find the optimal control variables $u(t)$ via SCS, one first substitutes eqs. (18)~(20) into eq. (2) and integrates from it to get

$$XT(t) - X(0)T(t) = AXHT(t) + BUHT(t) + ZHT(t) \quad (21)$$

where H is the operational matrix of integration defined in eq. (16), and

$$X(0) = [x(0), 0, \dots, 0] \quad (22)$$

where the initial value $x(0)$ is specified.

Since $T_i(t)$ are mutually independent, (21) reduces to

$$X = AXH + BUH + ZH + X(0) \quad (23)$$

which is a $(n_i \times m)$ matrix algebraic equation.

Equation (23) can also be expressed as

$$\begin{aligned} \bar{X} &= (I_{mni} - H^T \otimes A)^{-1} [(H^T \otimes B)\bar{U} + (H^T \otimes C)\bar{Z} + \bar{X}(0)] \\ &= B_1\bar{U} + C_1\bar{Z} + F_1 \end{aligned} \quad (24)$$

where I_{mni} is an $(mni \times mni)$ identity matrix, \otimes denotes the Kronecker product, I_{ni} is an $(n_i \times n_i)$ identity matrix and

$$\begin{aligned} \bar{X} &= [x_0^T, x_1^T, \dots, x_{m-1}^T]^T, \quad \bar{U} = [u_0^T, u_1^T, \dots, u_{m-1}^T]^T, \quad \bar{Z} = [z_0^T, z_1^T, \dots, z_{m-1}^T]^T, \\ \bar{X}(0) &= [x^T(0), 0^T, \dots, 0^T]^T, \end{aligned} \quad (25)$$

$$B_1 = (I_{mni} - H^T \otimes A)^{-1} [(H^T \otimes B)], \quad (26)$$

$$C_1 = (I_{mni} - H^T \otimes A)^{-1} [(H^T \otimes C)],$$

$$F_1 = (I_{mi} - H^T \otimes A)^{-1} X(0). \quad (27)$$

The expansion of $x^T Q x$ and $u^T R u$ in terms of SCS is as follows [1]:

$$x^T(t) Q x(t) = \bar{X}^T \psi_{ni}(t) Q \psi_{ni}^T(t) \bar{X}, \quad (28)$$

$$u^T(t) R u(t) = \bar{U}^T \psi_{mi}(t) R \psi_{mi}^T(t) \bar{U}. \quad (29)$$

where $\psi_{ni}(t)$ and $\psi_{mi}(t)$ are $(n_i \times mn_i)$ and $(m_i \times mm_i)$ matrix defined as

$$\psi_{ni}^T(t) = [I_{ni} T_0(t), I_{ni} T_1(t), \dots, I_{ni} T_{m-1}(t)], \quad (30)$$

$$\psi_{mi}^T(t) = [I_{mi} T_0(t), I_{mi} T_1(t), \dots, I_{mi} T_{m-1}(t)]. \quad (31)$$

Let SCS approximation of $\lambda(t)$

$$\lambda(t) = \psi_{ni}^T(t) [\lambda_0^T, \lambda_1^T, \dots, \lambda_{m-1}^T]^T = \psi_{ni}^T(t) \bar{\lambda}.$$

The expansion of $\lambda^T(t) z(t)$ and $\lambda_j^T(t) L_{ji} x(t)$ in term of SCS is as follows:

$$\lambda^T(t) z(t) = \bar{\lambda}^T \psi_{ni}(t) \psi_{ni}^T(t) \bar{Z}, \quad (32)$$

$$\lambda_j^T(t) L_{ji} x(t) = \bar{\lambda}_j^T \psi_{nj}(t) \psi_{nj}^T(t) \bar{L}_{ji} \bar{X}. \quad (33)$$

Now substituting eqs. (28)~(33) into (7)

$$L = \frac{1}{2} [\bar{X}^T \bar{Q}^T \bar{X} + \bar{U}^T \bar{R} \bar{U} + 2(\bar{\lambda}^T \bar{W}_{qi} \bar{Z} - \sum_j \bar{\lambda}_j^T \bar{W}_{qj} \bar{L}_{ji} \bar{X})] \quad (34)$$

where

$$\bar{Q} = W_{ni} (I_m \otimes Q), \quad (35)$$

$$\bar{R} = W_{mi} (I_m \otimes R), \quad (36)$$

$$\bar{L}_{ji} = (I_m \otimes L_{ji}) \quad (37)$$

and

$$\bar{W}_{ni} = \int_0^{t_f} \psi_{ni}(t) \psi_{ni}^T(t) dt = W \otimes I_{ni},$$

$$\bar{W}_{mi} = \int_0^{t_f} \psi_{mi}(t) \psi_{mi}^T(t) dt = W \otimes I_{mi},$$

$$\bar{W}_{qi} = \int_0^{t_f} \psi_{qi}(t) \psi_{qi}^T(t) dt = W \otimes I_{qi},$$

$$\bar{W}_{qj} = \int_0^{t_f} \psi_{qj}(t) \psi_{qj}^T(t) dt = W \otimes I_{qj}$$

where W has been given in eq. (17).

Using eq. (24), eq. (34) becomes

$$L = \frac{1}{2} [(B_1 \bar{U} + C_1 \bar{Z} + F_1)^T \bar{Q} (B_1 \bar{U} + C_1 \bar{Z} + F_1) + \bar{U}^T \bar{R} \bar{U} + 2 \bar{\lambda}^T \bar{W}_{qi} \bar{Z} - 2 \sum_j \bar{\lambda}_j^T \bar{W}_{qj} \bar{L}_{ji} (B_1 \bar{U} + C_1 \bar{Z} + F_1)]. \quad (38)$$

Thus far, the optimal problem is reduced to one of finding the vectors \bar{U} and \bar{Z} to minimize L for λ^* given by the second level. From the necessary condition of optimization, one has

$$\frac{\partial L}{\partial \bar{U}} = 0, \quad \frac{\partial L}{\partial \bar{Z}} = 0. \quad (39)$$

Therefore, one can obtain

$$\bar{U} = C_2 \bar{Z} + F_{21}, \quad (40)$$

$$\bar{Z} = B_2 \bar{U} + F_{22} \quad (41)$$

where

$$C_2 = -(\bar{R} + B_1^T \bar{Q} B_1)^{-1} B_1^T \bar{Q} C_1, \quad (42)$$

$$F_{21} = -(\bar{R} + B_1^T \bar{Q} B_1)^{-1} B_1^T \bar{Q} F_1 - \sum_j \bar{\lambda}_j^*{}^T \bar{W}_{qj} \bar{L}_{ji} B_1, \quad (43)$$

$$B_2 = -(C_1^T \bar{Q} C_1)^{-1} C_1^T \bar{Q} B_1, \quad (44)$$

$$F_{22} = -(C_1^T \bar{Q} C_1)^{-1} (C_1 \bar{Q} F_1 + \bar{\lambda}^*{}^T \bar{W}_q - \sum_j \bar{\lambda}_j^*{}^T \bar{W}_{qj} \bar{L}_{ji} C_1), \quad (45)$$

From (40) and (41), we obtain

$$\bar{U} = (I_{mmi} - C_2 B_2)^{-1} (C_2 F_{22} + F_{21}), \quad (46)$$

$$\bar{Z} = B_2 (I_{mmi} - C_2 B_2)^{-1} (C_2 F_{22} + F_{21}) + F_{22}. \quad (47)$$

Substituting (46) and (47) into (24)

$$\begin{aligned} \bar{X} = & B_1 (I_{mmi} - C_2 B_2)^{-1} (C_2 F_{22} + F_{21}) \\ & + C_1 B_2 (I_{mmi} - C_2 B_2)^{-1} (C_2 F_{22} + F_{21}) + C_1 F_{22} + F_1. \end{aligned} \quad (48)$$

2) The task of the second level is to improve λ_i , such that the global optimum is achieved.

$\lambda(t)$ ($0 \leq t \leq t_f$) trajectory is improved using the equation (8). d_j obtain from the equation (36)

$$\nabla_{\lambda} \pi(\bar{\lambda})|_{\lambda_i = \lambda_i^*} = \frac{\partial L}{\partial \lambda} = W_{qi} (\bar{Z}_i - \sum_j \bar{L}_{ij} \bar{X}_j) = e_i, \quad i = 1, 2, \dots, n \quad (49)$$

where \bar{Z}_i and \bar{X}_j are the values of (47), (48), respectively. From the coordination rules for the second level via SCS, we have

$$\bar{\lambda}^{j+1} = \bar{\lambda}^j + \alpha^j \nabla_{\lambda} \pi(\bar{\lambda}).$$

The interaction error is normalized from

$$\begin{aligned} \text{Error} = & \sum_{i=1}^n \int_0^{t_f} (z_i - \sum_j L_{ij} x_j)^T (z_i - \sum_j L_{ij} x_j) dt \\ = & \sum_{i=1}^n (\bar{Z}_i - \sum_j \bar{L}_{ij} \bar{X}_j)^T \bar{W}_{qi} (\bar{Z}_i - \sum_j \bar{L}_{ij} \bar{X}_j). \end{aligned} \quad (50)$$

The overall optimum is achieved when the total system interaction error is sufficiently closed to zero.

The above approach ensures that the subproblems are not singular, thus eliminating one of the main disadvantages of the standard GCM solution. It is a interesting solutions which we have given by (46)~(48). We next illustrate the approach using an example.

5 Numerical Example

In order to illustrate the proposed method, let us consider the following optimal control problem

$$J = \frac{1}{2} \int_0^1 (x^T Q x + u^T R u) dt$$

$$\text{subject to} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

with the initial conditions $Q = \text{diag}(10, 10)$ and $R = \text{diag}(1, 1)$. We are now in a position to examine the application of the GCM to this problem. It is desired to find optimal control via SCS.

In order to do so, let us decompose our system into two subsystems which are approximated by SCS for $m=15$.

The above formula is programmed on IBM PC 386. Convergence to optimum within a tolerance of 10^{-6} is achieved in 12 second-level iterations, as shown in Fig. 1. Fig. 2~5 show the optimal trajectories of x_1 , x_2 , u_1 and u_2 , respectively.

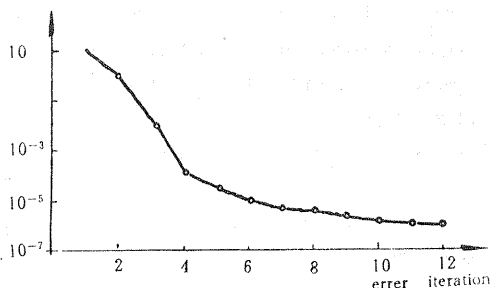
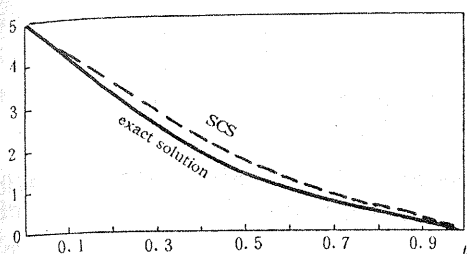
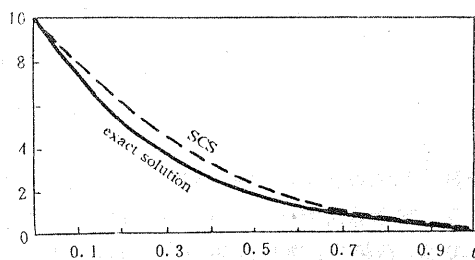
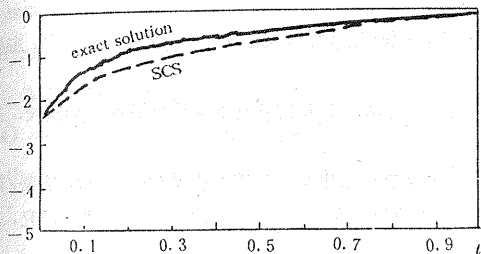
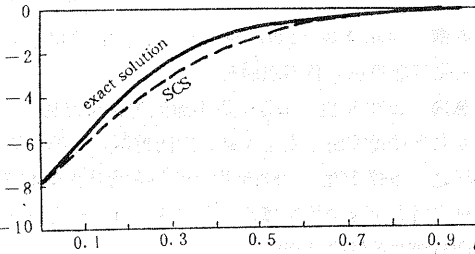


Fig. 1 Convergence of error

Fig. 2 Optimal state x_1 Fig. 3 Optimal state x_2 Fig. 4 Optimal control u_1 Fig. 5 Optimal control u_2

6 Conclusion

The SCS have been applied in analysing linear large-scale systems. It is attractive to solve the hierarchical control using the SCS approximation. The main advantage is that it reduces the large-scale systems problem to the solution of algebraic equations for which the calculation is simple and straightforward.

References

- [1] Horng, I. R., and Chou, J. H.. Application of Shifted Chebyshev Series to the Optimal Control of Linear Distributed-Parameter Systems. *Int. J. Control*, 1985, 42(2):233-243
- [2] Zhu, J. M. and Lu, Y. Z.. Hierarchical Optimal Control for Distributed Parameter Systems via Block Pulse Operator. *Int. J. Control*, 1988, 48:685-703
- [3] Zhu, J. M.. Novel Approach to Hierarchical Control via Single-Term Walsh Series Method. *Int. J. Systems Sci.*, 1988, 19:1859-1870

- [4] Huang, S. N., and Tang, G. Y.. Application of Legendre Series in Hierarchical Control of Large-Scale Systems. J. of Qingdao Institute of Chemical Technology, 1991, 12(2):66—73
- [5] Huang, S. N., and Yu, J. S.. Application of Taylor Series to Large-Scale Systems. Automation and Instrumentation, 1992, (2):15—18
- [6] Pearson, J. D.. Dynamic Decomposition Techniques. Optimization Methods for Large Scale Systems, D. A. Wismer (editor) McGraw-Hill, 1971
- [7] Abramovitz, M. and Stegun, I. A.. Handbook of Mathematical Functions. Washington D. C., National Bureau of Standards, 1967

递阶控制最优化的新方法

黄苏南 邵惠鹤 韩正之

(上海交通大学自动控制系, 200030)

摘要: 本文基于 Chebyshev 级数给出了一种大系统递阶控制新方法. 主要思想是将微分方程转化成代数方程. 整个算法简单、方便. 文中用一个例子说明了该方法的有效性.

关键词: 最优化; 递阶控制; Chebyshev 级数

本文作者简介

黄苏南 1962 年生. 1991 年毕业于华东化工学院自动控制专业, 获硕士学位, 现为上海交通大学博士生. 主要研究方向是智能控制, 鲁棒控制等.

邵惠鹤 1936 年生. 现为上海交通大学自动控制系教授, 博士生导师. 上海交通大学自动化研究所副所长. 主要研究方向是过程控制, 生化控制, 智能控制, 管控一体化等.

韩正之 1947 年生. 1982 年毕业于华东师范大学数学系, 1988 年获华东化工学院工业自动化专业博士学位, 同年进上海交通大学自动控制理论与应用博士后科研流动站工作, 现为教授, 博士生导师. 从事控制系统理论研究, 主要兴趣是非线性系统设计理论.