Study on Stability of Nonlinear Control System Based on Generalized Frequency Response Functions

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Abstract: Based on the representation of generalized frequency rpsponse functions (GFRF), a stability criterion for a class of nonlinear control systems is proposed in this paper. The open-loop case is considered here, and this criterion is demonstrated by numerous simulation examples.

Key words: stability; nonlinear control system; generalized frequeney response function

1 Introduction

The nonlinear transfer function or generalized frequency response function which is based on the Volterra series, provides a new train of thought to study the nonlinear system. The nonlinear frequency analysis is an extension of the classical linear frequency analysis method and its most importance is experimental verifiablity. Some authors have studied the nonlinear frequency analysis methed[1,2], and they have made significant progress in the nonlinear system simulation, nonlinear system identification , GFRF's computational method and its applications in industrial control systems. An analytical relationship between nonlinear integrodifferential equations and the generalized frequency response functions has been obtained[3], and a new recursive algorithm of generalized frequency response funcions for a class of differential equations is proposed[4]. However, few papers on the stability of nonlinear control system based on the GFRF method have been published.

In this paper, based on the representation of generalized frequency response functions, an open-loop stability criterion for a class of nonlinear control systems is proposed. In the second section, the nonlinear control system is described. The input-output stability criterion is given in the third section. In the fourth section, the further stability conditons for the four special cases are discussed. Finally, some simulation examples are used for illustrating the efficiency of the stability criterion.

2 Description of the Nonlinear Control System

The polynomial class of nonlinear control systems is considered as

$$\sum_{n=1}^{N}\{\sum_{p_1=1}^{M}\cdots\sum_{p_n=1}^{M}[a_{n,p_1,\cdots,p_n}\prod_{i=1}^{n}D^{p_i}y(t)+\sum_{p_{n+1}=0}^{M}\cdots\sum_{p_{2n}=0}^{M}b_{n,p_1\cdots,p_{2n}}\prod_{i=1}^{n}\prod_{k=n+1}^{2n}D^{p_i}y(t)\cdot D^{p_k}u(t)$$

Manuscript received Apl. 4, 1995, revised Feb. 26, 1996.

$$+ c_{n,p_1,\cdots,p_n}\prod_{i=1}^{n}[D^{p_i}u(t)]\} = 0, \tag{2.1}$$

where D is the differential operator , M is the maximum of differential orders and N the maximum of multicate degrees, a's, b's and c's are coefficients. u and y are the input and output of the system respectively.

Assume that the nonlinear control system (2.1) possesses the Volterra series solution, and the time-domain and frequency-domain representations of the solution can be respectively described as

$$y(t) = \sum_{n=1}^{\infty} y_n(t),$$

$$y_n(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)\mathrm{d}\tau_i, \quad n \in \mathrm{N}, \tag{2.2}$$

where N is the natural integer set, and $\{h_n(\tau_1,\cdots,\tau_n)\}_{n=1,2,\cdots}$ are the Volterra kernels or the generalized pulse response functions of the system. Furthermore, by using the Fourier transform, the Volterra series can be represented as [4]

$$\hat{y}(\omega) = \mathscr{F}(y(t)) = \sum_{n=1}^{\infty}\mathscr{F}(y_n(t)) = \sum_{n=1}^{\infty}\hat{y}_n(\omega),$$

$$\hat{y}(\omega) = (2\pi)^{-(n-1)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{h}_n(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)\cdot\hat{u}(\omega-\omega_2-\cdots-\omega_n)$$
$$\cdot\,\hat{u}(\omega_2)\cdots\hat{u}(\omega_n)\mathrm{d}\omega_2,\cdots\mathrm{d}\omega_n, \quad n \in \mathrm{N}, \tag{2.3}$$

where $\hat{y}$, $\hat{y}_n$, $\hat{u}$ are Fourier transforms of y, y_n and u respectively; ω's are the frequency variables, $\hat{h}_n$ is the multi-imensional Fourier transform of h_n, i.e.,

$$\hat{h}_n(\omega_1,\cdots,\omega_n) = \mathscr{F}(h_n(\tau_1,\cdots,\tau_n)) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}e^{-j\omega_i\tau_i}\mathrm{d}\tau_i, \quad n \in \mathrm{N}. \tag{2.4}$$

Let
$$A_n(j\omega_1,\cdots,j\omega_n) = \sum_{p_1=0}^{M}\cdots\sum_{p_n=0}^{M}a_{n,p_1,\cdots,p_2}\prod_{i=1}^{n}(j\omega_i)^{p_i},$$

$$B_n(j\omega_1,\cdots,j\omega_{2n}) = \sum_{p_1=0}^{M}\cdots\sum_{p_n=0}^{M}b_{n,p_1,\cdots,p_{2n}}\prod_{i=1}^{n}(j\omega_i)^{p_i}\prod_{k=n+1}^{2n}(j\omega_k)^{p_i},$$

$$C_n(j\omega_1,\cdots,j\omega_n) = \sum_{p_1=0}^{M}\cdots\sum_{p_n=0}^{M}c_{n,p_1,\cdots,p_2}\prod_{i=1}^{n}(j\omega_i)^{p_i}, \quad n \in \mathrm{N}. \tag{2.5}$$

It can be proven[8] that

$$\hat{h}_1(\omega_1) = -A_1^{-1}(j\omega_1)\cdot C_1(j\omega_1),$$

$$\hat{h}_2(\omega_1,\omega_2) = -A_1^{-1}(j(\omega_1+\omega_2))[C_2(j\omega_1,j\omega_2) + A_2(j\omega_1,j\omega_2)\hat{h}_1(\omega_1)\hat{h}_2(\omega_2)$$
$$+ B_1(j\omega_1,j\omega_2)(\hat{h}_1(\omega_1)+\hat{h}_2(\omega_2))/2],$$

$$\cdots$$

$$\hat{h}_n(\omega_1,\cdots,\omega_n) = -A_1^{-1}(j(\omega_1+\cdots+\omega_n))\{C_n(j\omega_1,\cdots,j\omega_n)$$

$$
\begin{aligned}
&+\sum_{s=2}^{\alpha_1}\sum_{\text{all permutation}}\sum_{k_1+k_2+\cdots+k_s=n}[A_s(j(\omega_1+\cdots+\omega_{k_1}),\\
&j(\omega_{k_1+1}+\cdots+\omega_{k_1+k_2}),\cdots,j(\omega_{n-k'_s+1}+\cdots+\omega_n)),\\
&\hat{h}_{k_1}(\omega_1,\cdots,\omega_{k_1'})\cdots\hat{h}_{k_s'}(\omega_{n-k_s+1},\cdots,\omega_n)]/\text{number of permutation}\\
&+\sum_{s=1}^{\alpha_2}\sum_{\text{all permutation}}\sum_{k_1'+k_2'+\cdots+k_s'=n-s}[B_s(j(\omega_1+\cdots+\omega_{k_1'}),\\
&j(\omega_{k_1'}+\cdots+\omega_{k_1'+k_2'}),\cdots,j(\omega_{ns-k_s'+1}+\cdots+\omega_{n-s}),\\
&j\omega_{n-s+1},\cdots,j\omega_n)\cdot\hat{h}_{k_1'}(\omega_1,\cdots,\omega_{k_1'})\cdot\hat{h}_{k_2}(\omega_{k_1'+1},\cdots,\omega_{k_1'+k_2'})\cdots\\
&\hat{h}_{k_s'}(\omega_{n-s-k_s'+1},\cdots,\omega_{n-s})]/\text{number of permutation},\quad n\in\mathrm{N},
\end{aligned}\tag{2.6}
$$

where the "number of permutation" is for the repeat times of terms in different $k_i', k_i' \in \mathrm{N}$, which are of different permutation, and

$$
\alpha_1=\begin{cases} n, & \text{if } n\leqslant N,\\ N, & \text{if } n>N;\end{cases}
$$

$$
\alpha_2=\begin{cases} n/2, & \text{if } n\leqslant 2N \text{ and } n \text{ is even},\\ (n-1)/2, & \text{if } n\leqslant 2N \text{ and } n \text{ is odd},\\ N, & \text{if } n>2N.\end{cases}\tag{2.7}
$$

3 Open-Loop Stability Based on GFRF

Consider that x: R→R is a time-domain signal with L_p-norm defined as

$$
\| x\|_p=[\int_{-\infty}^{\infty}|x(t)|^p\mathrm{d}t]^{1/p}<\infty,\quad 1\leqslant p<\infty,\tag{3.1}
$$

or L_∞-norm defined as

$$
\| x\|_\infty=\operatorname*{ess\,sup}_{t\in\mathrm{R}}|x(t)|<\infty.\tag{3.2}
$$

The Fourier transform $\hat{x}$ of x with H_p-norm or H_∞-norm in frequency-domain defined respectively as

$$
\|\hat{x}\|_p=[\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{x}(\omega)|^p\mathrm{d}\omega]^{1/p}<\infty,\quad 1\leqslant p<\infty;\tag{3.3}
$$

$$
\|\hat{x}\|_\infty=\sup_{\omega\in\mathrm{R}}|\hat{x}(\omega)|<\infty.
$$

Defintion 3.1 The nonlinear control system (2.1) is said to be open-loop stable in L_p-norm (or L_p-stable), if

$$
u\in L_p(-\infty,\infty)\Rightarrow y\in L_p(-\infty,\infty),\tag{3.4}
$$

i.e., if $\| u\|_p<L(L>0)$, then $\exists\ K>0$, such that

$$
\| y\|_p\leqslant K.\tag{3.5}
$$

Lemma 3.1 Assume that $\hat{h}_n\in H_\infty^n(-\infty,\infty)$, i.e., $\sup_{\omega_1,\cdots,\omega_n\in\mathrm{R}}|\hat{h}_n(\omega_1,\cdots,\omega_n)|<\infty$, and $\hat{u}\in H_1\cap H_2$, then

$$
\|\hat{y}_n\|_2\leqslant(2\pi^{(n-1)/2}\|\hat{h}_n\|_\infty\cdot\|\hat{u}\|_1^{n-1}\|\hat{u}\|_2.\tag{3.6}
$$

Proof It can be proven by

$$\|\hat{y}_n\|_2 = [(2\pi)^{-n}\int_{-\infty}^{\infty} |\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \hat{h}_n(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)$$

$$\cdot \hat{u}(\omega-\omega_2-\cdots-\omega_n)\cdot\hat{u}(\omega_2)\cdots\hat{u}(\omega_n)\mathrm{d}\omega_2\cdots\mathrm{d}\omega_n|^2\mathrm{d}\omega]^{1/2}$$

$$\leqslant[(2\pi)^{-n}\int_{-\infty}^{\infty}\sup_{\omega\in R}\{|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{h}_n(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)$$

$$\cdot\hat{u}(\omega_2)\cdots\hat{u}(\omega_n)\mathrm{d}\omega_2\cdots\mathrm{d}\omega_n|^2\}|\hat{u}(\omega-\omega_2-\cdots-\omega_n)|^2\mathrm{d}\omega]^{1/2}$$

$$=(2\pi)^{-(n-1)/2}\sup_{\omega\in R}\{|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{h}_n(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)$$

$$\cdot\hat{u}(\omega_2)\cdots\hat{u}(\omega_n)\mathrm{d}\omega_2\cdots\mathrm{d}\omega_n|\}\|\hat{u}\|_2$$

$$\leqslant(2\pi)^{-(n-1)/2}[\int_{-\infty}^{\infty}\sup_{\omega,\omega_2\in R}\{|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{h}_n(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)$$

$$\cdot\hat{u}(\omega_3)\cdots\hat{u}(\omega_n)\mathrm{d}\omega_3\cdots\mathrm{d}\omega_n|\}|\hat{u}(\omega_2)|\mathrm{d}\omega_2]\|\hat{u}\|_2$$

$$=(2\pi)^{-(n-1)/2}\sup_{\omega,\omega_2\in R}\{|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{h}_n(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)$$

$$\cdot\hat{u}(\omega_3)\cdots\hat{u}(\omega_n)\mathrm{d}\omega_3\cdots\mathrm{d}\omega_n|\}\|\hat{u}\|_1\|\hat{u}\|_2$$

$$\cdots$$

$$=(2\pi)^{-(n-1)/2}\sup_{\omega,\omega_2,\cdots,\omega_n\in R}|\hat{h}_n(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)|\cdot\|\hat{u}\|_1^{n-1}\|\hat{u}\|_2$$

$$=(2\pi)^{-(n-1)/2}\|\hat{h}_n\|_\infty\|\hat{u}\|_1^{n-1}\|\hat{u}\|_2.$$

Theorem 3.1 Assume that the nonlinear control system (2.1) possesses the Volterra series solution (2.2) or (2.3), the control input $u\in L_2(-\infty,\infty)$, and its Fourier transform $\hat{u}\in H_1\cap H_2$, i.e., $\exists\ L>0$, such that

$$\|\hat{u}\|_2,\ \|\hat{u}\|_1<L; \tag{3.7}$$

and assume that GFRF's of the system are attenuate in $n!$, i.e., $\exists\ K>0$, such that

$$\|\hat{h}_n\|_\infty\leqslant\frac{1}{n!}(2\pi)^{-(n-1)/2}e^{-L}K,\quad\forall\ n\in N, \tag{3.8}$$

then the system is L_2-stable.

Proof From Lemma 3.1 and Parseval Theorem[9], we have

$$\|y\|_2=(2\pi)^{-1/2}\|\hat{y}\|_2=(2\pi)^{-1/2}\sum_{n=1}^{\infty}\|\hat{y}_n\|_2\leqslant(2\pi)^{-1/2}\sum_{n=1}^{\infty}\frac{L^n}{n!}e^{-L}K=(2\pi)^{-1/2}K.$$

Remark 1 The condition (3.8) is equivalent to that $\{n!\,\|\hat{h}_n\|_\infty\}$, $n\in N$ is an uniformly bounded sequence. For the finite Volterra series, it is equivalent to that every $\|\hat{h}_n\|_\infty$ is bounded.

Remark 2 The stability conditions in Theorem 3.1 are sufficient but may not be necessary.

Remark 3 For nonlinear control system, it is difficult to determine the system stability by using the system gain defined as

$$\mathrm{gain}(S)=\sup\|y\|_2/\|u\|_2,$$

because the gain is dependent on u. The GFRF's however, are independent on input u, so that the stability criterion based on GFRF is only dependent on the structure and parameters of the system.

4 Further Study on Open-Loop Stability

Now we focus our attention on the special cases of system (2.1). At first, we consider the pure input nonlinear system

$$\sum_{p_1=0}^{M} a_{1,p_1} D^{p_1} y(t) + \sum_{n=0}^{M} \sum_{p_1=0}^{M} \cdots \sum_{p_k=0}^{M} c_{n,p_1,\cdots,p_n} \sum_{i=1}^{n} D^{p_1} u(t) = 0. \tag{4.1}$$

The GFRF's are[4]

$$\hat{h}_n(\omega_1,\cdots,\omega_n) = -A_1^{-1}(j(\omega_1+\cdots+\omega_n))C_n(j\omega,\cdots,j\omega_n), \quad n=1,2,\cdots,N,$$
$$\hat{h}_n = 0, \quad \forall\, n > N. \tag{4.2}$$

Theorem 4.1 For the nonlinear control system (4.1), if

i) for any $n \in \{1,2,\cdots,N\}$, $\hat{h}_n(\omega_1,\cdots,\omega_n)$ is a proper rational fraction, i.e., the degree of C_n is less than that of A_1 for all n;

ii) $A_1(s) = \sum_{p_1=0}^{M} a_{1,p_1} s^{p_1}$ is a Hurwitz polynomial, i.e., all zeros of A_1 are in the open left half s-plane;

iii) the input spectrum satisfying $\hat{u} \in H_1 \cap H_2$,

then the system is open-loop L_2-stable.

Proof From i), we have $\lim_{\omega_i \to \pm\infty} |\hat{h}_n(\omega_1,\cdots,\omega_n)| = 0, i=1,2,\cdots,n$, i.e., there exists $\omega_0 > 0$, such that

$$\|\hat{h}_n\|_\infty = \sup_{|\omega_1| \leqslant \omega_0} |\hat{h}_n(\omega_1,\cdots,\omega_n)|;$$

Let $D = \{(s_1,\cdots,s_n); s_i = \sigma_i + j\omega_i, \sigma_i > 0, i=1,2,\cdots,n\}$, then from ii), it is known that $\hat{h}_n$ is a bounded analytic function in D, i.e., $\|\hat{h}_n\|_\infty = K_n, n = 1,2,\cdots,N$, Let $K = \max_{n \in \{1,2,\cdots,N\}} \{n!K_n\}$; therefore, $n!\|\hat{h}_n\| \leqslant K, \forall\, n \in N$, from Theorem 3.1, it is known that the system is open-loop L_2-stable.

Next, a more complex case, the pure output nonlinear system is considered

$$\sum_{n=0}^{M} \sum_{p_1=0}^{M} \cdots \sum_{p_n=0}^{M} a_{n,p_1,\cdots,p_n} \prod_{i=1}^{n} D^{p_i} y(t) + \sum_{p_i=0}^{M} c_{1,p_1} D^{p_1} u(t) = 0, \tag{4.3}$$

the GFRF's are[4]

$$\hat{h}_1(\omega_1) = -A_1^{-1}(j\omega_1)C_1(j\omega_1),$$
$$\hat{h}_2(\omega_1,\omega_2) = -A_1^{-1}(j(\omega_1+\omega_2))A_2(j\omega_1,j\omega_2)\hat{h}_1(\omega_1)\hat{h}_1(\omega_2),$$
$$\cdots$$
$$\hat{h}_n(\omega_1,\cdots,\omega_2) = -A_1^{-1}(j(\omega_1+\cdots+\omega_n)) \sum_{s=2}^{n} \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n} A_s(j(\omega_1+\cdots+\omega_{k_1})),$$
$$j(\omega_{k_1+1}+\cdots+\omega_{k_1+k_2}),\cdots,j(\omega_{n-k_s+1}+\cdots+\omega_n))\hat{h}_{k_1}(\omega_1,\cdots,\omega_{k_1})$$
$$\cdot \hat{h}_{k_2}(\omega_{k_1+1},\cdots,\omega_{k_1+k_2}),\cdots,\hat{h}_{k_s}(\omega_{n-k_s+1},\cdots,\omega_n)/\text{number of permutation}, \tag{4.4}$$

where $k_1 \in N$, and $\alpha = \begin{cases} n; & \text{if } 2 \leqslant n \leqslant N, \\ N, & \text{if } n > N. \end{cases}$

Theorem 4.2 For the nonlinear control system (4.3), if

i) for any $n \in \mathrm{N}, \hat{h}_n(\omega_1, \cdots, \omega_n)$ is a proper rational fraction i.e., the degrees of C_1 are less than degree of A_1 ;

ii) $A_1(s) = \sum_{p_1}^{M} a_{1,p_1} s^{p_1}$ is a stable polynomial;

iii) Let $\rho_1 = \sup_{\omega_1} |A_1^{-1}(j\omega_1)C_1(j\omega_1)|$,

$$\rho_n = \sup_{\omega_1,\cdots,\omega_n} |A_1^{-1}(j(\omega_1 + \cdots + \omega_n))A_n(j\omega_1, \cdots, j\omega_n)|, \quad n = 2, \cdots, N,$$

and $\theta_1 = 1, \theta_2 = \rho_2, \cdots, \theta_n = \sum_{\text{all permutation}} \sum_{s=2}^{\alpha} \rho_2 \sum_{k_1+\cdots+k_s=n} \theta_{k_1} \cdots \theta_{k_s}$/ number of permutation;

where $\alpha = \begin{cases} n, & \text{if } 2 \leqslant n \leqslant N, \\ N, & \text{if } n > N \end{cases}$ and the series $\sum_{n=1}^{\infty} \theta_n \rho_1^n$ is convergent;

iv) $\hat{u} \in H_1 \cap H_2$, i.e., the input is absolutely integrable and squarely integrable , then the system is open-loop L_2 -stable.

Proof From i) we have $\lim_{\omega \to \pm\infty} |A^{-1}(j\omega)C_1(j\omega)| = 0$,

$$\lim_{\omega \to \pm\infty} |A_1^{-1}(j(\omega_1 + \cdots + \omega_n))A_n(j\omega_1, \cdots, j\omega_n)| = 0, \quad i = 1, 2, \cdots, n; \quad n = 2, 3, \cdots, N.$$

so there exists $\omega_0 > 0$, such that

$$\rho_1 = \| \hat{h}_1 \|_\infty = \sup_{|\omega_1| \leqslant \omega_0} |A_1^{-1}(j\omega_1)C_1(j\omega_1)|,$$

$$\rho_n = \sup_{|\omega_i| \leqslant \omega_0} |A_1^{-1}(j(\omega_1 + \cdots + \omega_n))A_n(j\omega_1, \cdots, j\omega_n)|, \quad n = 2, 3, \cdots, N.$$

Similar to Theorem 4.1, from ii), $A_1^{-1}(j(\omega_1 + \cdots + \omega_n))A_n(j\omega_1, \cdots, j\omega_n)$ is a bounded analytic function in $D, n = 1, 2, \cdots, N$, so that $\{\rho_1, \rho_2, \cdots, \rho_N\}$ is an uniformly bounded sequence. Because

$$\| \hat{h}_1 \|_\infty = \rho_1, \quad \| \hat{h}_2 \|_\infty \leqslant \rho_2 \| \hat{h}_1 \|_\infty^2 = \theta_2 \rho_1^2, \cdots,$$

$$\| \hat{h}_n \|_\infty \leqslant \sum_{s=2}^{\alpha} \rho_s \sum_{\text{all permutation}} \sum_{k_1+k_2+\cdots+k_s=n} \theta_{k_1} \rho_1^{k_1} \cdots \theta_{k_s} \rho_1^{k_s} / \text{number of permutation};$$

$$= [\sum_{s=2}^{\alpha} \rho_s \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n} \theta_{k_1} \cdots \theta_{k_2} / \text{number of permutation}] \rho_1^n = \theta_n \rho_1^n,$$

and from iii), the series $\sum_{n=2}^{\infty} \theta_n \rho_1^n$ is convergent. By using convergency, it can be proven that

$$\sum_{n=1}^{\infty} \| \hat{h}_n \|_\infty \leqslant \sum_{n=1}^{\infty} \theta_n \rho_1^n \leqslant K_0 = e^K = \sum_{n=1}^{\infty} \frac{K^n}{n!},$$

and $n! \| \hat{h}_n \|_\infty \leqslant K_n$, for some $n_0 \in N, \forall\, n \geqslant n_0$, where $K_n = K^n$. From iv), there exists $L > 0$. such that $\| \hat{u} \|_1, \| \hat{u} \|_2 \leqslant L$, and , therefore, we have

$$\| \hat{y}_n \|_2 \leqslant \frac{K_n}{n!} L^n = \frac{1}{n!}(KL)^n, \quad \forall\, n \geqslant n_0,$$

$$\| y \|_2 = (2\pi)^{-1/2} \cdot \| \hat{y} \|_2 \leqslant (2\pi)^{-1/2} \cdot \sum_{n=1}^{\infty} \| \hat{y}_n \|_2 = (2\pi)^{-1/2} (\sum_{n=1}^{n_0-1} \| \hat{y}_n \|_2 + \sum_{n=n_0}^{\infty} \| \hat{y}_n \|_2)$$

$$\leqslant (2\pi)^{-1/2} \cdot (\sum_{n=1}^{n_0-1} \| \hat{y}_n \|_2 + \sum_{n=n_0}^{\infty} \frac{(KL)^n}{n!}) = (2\pi)^{-1/2} \cdot (M + e^{KL}) \doteq M_0.$$

That is ,the system is L_2 -stable.

Similarly, the pure cross-product nonlinear system can be described as

$$\sum_{p_1=0}^{M} a_{1,p_1} D^{p_1} y(t) + \sum_{n=0}^{N} \sum_{p_1=0}^{M} \cdots \sum_{P_{2n}=0}^{M} b_{n,p_1,\cdots,p_{2n}} \prod_{i=1}^{n} D^{p_i} y(t) \cdot \prod_{k=n+1}^{2n} D^{p_k} u(t) + \sum_{p_1=0}^{M} c_{1,p_1} D^{p_1} u(t) = 0, \tag{4.5}$$

and its GFRF's are follows[4].

$$\hat{h}_1(\omega_1) = - A_1^{-1}(j\omega_1) C_1(j\omega_1),$$

$$\hat{h}_2(\omega_1, \omega_2) = - A_1^{-1}(j(\omega_1 + \omega_2)) B_1(j\omega_1, j\omega_2) \hat{h}_1(\omega_1),$$

$$\cdots$$

$$\begin{aligned} \hat{h}_n(\omega_1, \cdots, \omega_n) = & - A_1^{-1}(j(\omega_1 + \omega_2 + \cdots + \omega_n)) \\ & \cdot \sum_{s=2}^{\alpha} \rho_s \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n-s} B_s(j(\omega_1 + \omega_2 + \cdots + \omega_{k_1}), \\ & j(\omega_{k_1+1} + \cdots + \omega_{k_1+k_2}), \cdots, j(\omega_{n-s-k_s+1} + \cdots + \omega_{n-s}), \\ & j\omega_{n-s+1}, \cdots, j\omega_n) \hat{h}_{k_1}(\omega_1, \cdots, \omega_{k1}) \hat{h}_{k_2}(\omega_{k_1+1}, \cdots, \omega_{k_1+k_2}) \cdots \\ & \hat{h}_{k_s}(\omega_{n-s-k_s+1}, \cdots, \omega_{n-s}) / \text{number of permutation} \end{aligned} \tag{4.6}$$

where $k_i \in N$, and $\alpha = \begin{cases} n/2, & \text{if } n \leqslant 2N \text{ and } n \text{ is even}, \\ (n-1)/2, & \text{if } n \leqslant 2N \text{ and } n \text{ is odd}, \\ N, & \text{if } n \geqslant 2N. \end{cases}$

Theorem 4.3 For the nonlinear control system (4.5), if

i) for any $n \in N$, $\hat{h}_n(\omega_1, \cdots, \omega_n)$ is a proper rational fraction of $\omega_i, i = 1, 2, \cdots, N$;

ii) $A_1(s) = \sum_{p_1=0}^{M} \alpha_{1,p_1} s^{p_1}$ is a stable polynomial;

iii) Let $\rho_0 = \sup_{\omega_1} | A_1^{-1}(j\omega_1) C_1(j\omega_1) | \neq 0$,

$\rho_n = \sup_{\omega_1, \cdots, \omega_{2n}} | A_1^{-1}(j(\omega_1 + \cdots + \omega_{2n})) B_n(j\omega_1, \cdots, j\omega_{2n}) |, n = 1, 2, \cdots, N$ and $\rho_1 \neq 0$,

$\theta_1 = \theta_2 = \theta_3 = \rho_0, \quad \theta_4 = (\rho_1^{-3} \rho_2^2 + 1) \rho_0,$

$\theta_n = \theta_{n-1} + \sum_{s=2}^{\alpha} \rho_s \rho_1^{1-2s} \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n-s} \theta_{k_1} \cdots \theta_{k_2} / \text{number of permutation}, n \geqslant 3,$

where $\alpha = \begin{cases} n/2, & \text{if } n \leqslant 2N \text{ and } n \text{ is even}, \\ (n-1)/2, & \text{if } n \leqslant 2N \text{ and } n \text{ is odd}, \\ N, & \text{if } n \geqslant 2N. \end{cases}$ the series $\sum_{n=1}^{\infty} \theta_n \rho_1^{n-1}$ is convergent;

iv) $\hat{u} \in H_1 \cap H_2$,

then the system is open-loop L_2- stable.

Proof Because

$$\|\hat{h}_1\|_\infty = \rho_0,\quad \|\hat{h}_2\|_\infty \le \rho_1\|\hat{h}_1\|_\infty = \rho_0\rho_1 = \theta_2\rho_1,$$

$$\|\hat{h}_3\|_\infty = \rho_1\|\hat{h}_2\|_\infty = \rho_0\rho_1^2 = \theta_3\rho_1^2,$$

$$\|\hat{h}_4\|_\infty \le \rho_1\|\hat{h}_3\|_\infty + \rho_2\|\hat{h}_1\|_\infty^2 \le \rho_0\rho_1^3 + \rho_2\rho_0^2 = (\rho_0 + \rho_1^{-3}\rho_2\rho_0^2)\rho_1^3 = \theta_4\rho_1^3,$$

$$\cdots$$

$$\|\hat{h}_n\|_\infty \le \sum_{s=1}^{\alpha}\rho_s \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n-s} \|\hat{h}_{k_1}\|_\infty \cdots \|\hat{h}_{k_s}\|_\infty/\text{number of permutation}$$

$$\le \sum_{s=1}^{\alpha}\rho_s \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n-s} \theta_{k_1}\rho_1^{k_1-1}\cdots\theta_{k_s}\rho_1^{k_s-1}/\text{number of permutation}$$

$$= \sum_{s=1}^{\alpha}\rho_s\rho_1^{n-2s} \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n-s} \theta_{k_1}\cdots\theta_{k_s}/\text{number of permutation}$$

$$= \rho_1^{n-1}\theta_{n-1} + \sum_{s=2}^{\alpha}\rho_s\rho_1^{n-2s} \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n-s} \theta_{k_1}\cdots\theta_{k_s}/\text{number of permutation}$$

$$= \rho_1^{n-1}[\theta_{n-1} + \sum_{s=2}^{\alpha}\rho_s\rho^{1-2s} \sum_{\text{all permutation}} \sum_{k_1+\cdots+k_s=n-s} \theta_{k_1}\cdots\theta_{k_s}/\text{number of permutation}]$$

$$= \theta_n\rho_1^{n-1},$$

by using the method in Theorem 4.2, the system is L_2- stable.

Finally, we discuss the general case, its GFRF's are those in (2.6).

Theorem 4.4 For the nonlinear system (2.1), if

i) for any $n \in N$, $\hat{h}_n(j\omega_1,\cdots,j\omega_n)$ is a proper rational fraction;

ii) $A_1(s) = \sum_{p_1=0}^{M}\alpha_{1,p_1}s_1^{p_1}$ is a stable polynomial;

iii) $\{\sum_{i=1}^{n}\|\hat{h}_i\|_\infty\}$, $n \in N$ is convergent;

iv) $\hat{u} \in H_1 \cap H_2$,

then the system is open-loop L_2-stable.

Proof This is a straightforward result of Theorem 4.1～4.3.

5 Simulation Examples

Example 5.1 The pure output nonlinear system is considered as

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y - u + 0.1y^2 = 0,$$

the GFRF's $\hat{h}_1(\omega)$, $\hat{h}_2(\omega_1,\omega_2)$, $\hat{h}_3(\omega_1,\omega_2,\omega_3)$ can be calculated as

$$\hat{h}_1(\omega) = 1/[-j\omega^3 - 6\omega^2 + j11\omega + 6],$$

$$\hat{h}_2(\omega_1,\omega_2) = -0.1/\{[-j\omega_1^3 - 6\omega_1^2 + j11\omega_1 + 6]\cdot[-j\omega_2^3 - 6\omega_2^2 + j11\omega_2 + 6]$$
$$\cdot[-j(\omega_1+\omega_2)^3 - 6(\omega_1+\omega_2) + j11(\omega_1+\omega_2) + 6]\}$$

$$\hat{h}_3(\omega_1,\omega_2,\omega_3) = -0.1[\hat{h}_2(\omega_1,\omega_2)\hat{h}_1(\omega_3) + \hat{h}_2(\omega_1,\omega_3)\hat{h}_1(\omega_2)$$
$$+ \hat{h}_2(\omega_2,\omega_3)\hat{h}_1(\omega_1)]/\{3[-j(\omega_1+\omega_2+\omega_3)^3$$

$$-6(\omega_1+\omega_2+\omega_3)^2+j11(\omega_1+\omega_2+\omega_3)+6]\}.$$

It can be checked by Theorem 4.2 that i) $A_1(s)=s^3+6s^2+11s+6$ is stable; ii) $\rho_1=0.166667, \rho_2=0.019468, \theta_1=1, \theta_2=\rho_2=0.019468; \theta_3=2\rho_2^2=0.000758, \cdots$, by using the recusive computing test, the series $\sum_{n=1}^{\infty}\theta_n\rho_1^n$ is convergent; the system is open-loop L_2-stable. The response diagram of the system is showed in Fig. 1.

Example 5.2 The following system is also pure output nonlinear

$$\dddot{y}+3\ddot{y}+0.8\dot{y}+2.2y-u+0.65y\dot{y}=0,$$

the GFRF's $\hat{h}_1(\omega), \hat{h}_2(\omega_1,\omega_2), \hat{h}_3(\omega_1,\omega_2,\omega_3)$ can be calculated as

$$\hat{h}_1(\omega)=1/[-j\omega^3-3\omega^2+j0.8\omega+2.2],$$

$$\hat{h}_2(\omega_1,\omega_2)=j0.65\omega_1\omega_2[\omega_1+\omega_2]\hat{h}_1(\omega_1)\hat{h}_1(\omega_2)/\{2[-j(\omega_1+\omega_2)^3-3(\omega_1+\omega_2)^2+j0.8(\omega_1+\omega_2)+2.2]\},$$

$$\begin{aligned}\hat{h}_3(\omega_1,\omega_2,\omega_3)=j0.65\{&[(\omega_1+\omega_2)\omega_3^2+(\omega_1+\omega_2)^2\omega_3]\hat{h}_2(\omega_1,\omega_2)\hat{h}_1(\omega_3)\\&+[(\omega_1+\omega_3)\omega_2^2+(\omega_1+\omega_3)^2\omega_2]\cdot\hat{h}_2(\omega_1,\omega_3)\hat{h}_1(\omega_2)\\&+[(\omega_2+\omega_3)\omega_1^2+(\omega_2+\omega_3)^2\omega_1]\hat{h}_2(\omega_2,\omega_3)\hat{h}_1(\omega_1)\}\\&/\{3[-j(\omega_1+\omega_2+\omega_3)^3\\&-3(\omega_1+\omega_2+\omega_3)^2+j0.8(\omega_1+\omega_2+\omega_3)+2.2]\}.\end{aligned}$$

$A_1(s)$ is stable, and $\rho_1=18.1591$, $\rho_2=0.0963$, $\theta_1=1$, $\theta_2=\rho_2=0.0963$, $\theta_3=2\rho_2^2=2\times0.0963^2, \cdots$, by using the recusive computing test, the series $\sum_{n=1}^{\infty}\theta_n\rho_1^n$ is divergent. The time respones diagram of the system is shown is Fig. 2.

6 Conclusion

The stability criteria for the polynomial class of nonlinear control systems based on GFRF's are similar to ones of linear control system, and the zeros of the linear polynomial $A_1(s)$ play main role. The convergence of GFRF's norm, however, is a special desire for nonlinear control systems, and the absolute integrability of the input is a further condition of the nonlinear stability. From the open-loop stability criterion, some closed-loop stability criterion can be obtained.

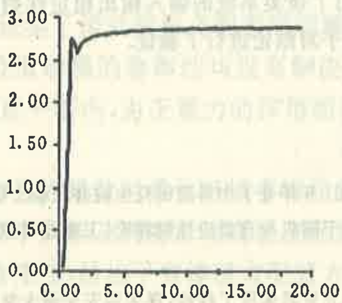

Fig. 1 Time response diagram

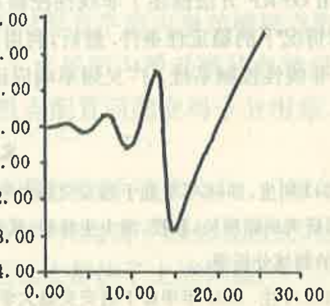

Fig. 2 Time response diagram

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基于广义频率响应函数的非线性控制系统稳定性研究

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摘要：基于 Volterra 级数的广义频率响应函数(GFRF)或称非线性传递函数，为研究非线性系统提供了新的思路，它是对线性频率分析方法的推广，其最重要的价值是实验可验证性. 一些学者已经对非线性传递函数理论进行了研究，在非线性控制系统仿真、非线性系统辨识、GFRF 计算方法，以及非线性传递函数理论的工业应用等方面取得了一些有意义的成果. 但到现在为止，对具有无限项 GFRF 的非线性系统的稳定性研究还没有涉及. 本文基于广义频率响应函数，提出了一类非线性控制系统的开环稳定性判据条件. 作者首先用 GFRF 方法描述了非线性控制系统，然后给出了该类系统的输入输出稳定性判据，进而讨论了四种特殊情况下的稳定性条件. 最后，利用大量的仿真例子对结论进行了验证.

关键词：非线性控制系统；广义频率响应函数；稳定性

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