第 13 卷第 6 期
1996 年 12 月

控制理论与应用
CONTROL THEORY AND APPLICATIONS

Vol. 13, No. 6
Dec., 1996

Stability Analysis of Multirate Sampled-Data Systems

FANG Yuguang
(Department of Systems Engineering, Case Western Reserve University · USA)
CHU Xuedao and WU Yuqiang
(Institute of Automation, Qufu Normal University · Qufu, 273165, PRC)

Abstract: In this paper, new sufficient conditions for stability of asynchronous sampled-data systems are presented. A new robust stability problem is proposed and preliminary results on this problem is given.

Key words: stability; sampled-data; asynchronous; robust

1 Introduction

Multirate digital systems have been a topic of intensive interest since the early 1950's due to the wide applications of computer technologies in industries. Walton[1] and Glasson[2] present a very comprehensive survey on the privious multirate methods in this area, Franklin et al[3] provides a very excellent book on the systematic treatment for analysis and design of dynamical systems and controls. As we can see, the analysis and design of dynamical systems for synchronous sampling cases can not be easy generalize to the cases where the asynchronous samplers are used.

In this paper, we also study the stability of generally formulated sampled-data systems which have a great potentiality of applications in digital control systems with multirate sampling. Consider the following linear control system:.

$$\dot{x}(t) = Ax(t) + Bu(t),$$

and we consider the digital feedback control problem. Let the time sequence t_k denotes the instants that the controller values are executed, that is, $u(t) = u(t_k)$ on the interval $t_k \leqslant t < t_{k+1}$. Let $\sigma_k = t_{k+1} - t_k$, the difference sequence σ_k may not be uniform due to the time delay resulting from communication medium. As in linear time-invariant control theory, we can use the feedback control $Kx(t)$ to stabilize the system. Suppose σ_k is bounded, if the sampled system is stable in discrete-time case, then the closed-loop of the continuous-time linear system with the sampled-data control is also stable in continuous-time sense. Thus, we reduce the digital control problem to the stability analysis of the following time-varying systems:

$$x_{k+1} = \Phi(t_{k+1}, t_k)x_k. \tag{1}$$

where $\{t_k\}$ is a given sequence and $\Phi(.,.)$ is a given mapping. Ritchey and Franklin[4] and Halevi and Ray[5] came up with the similar problem. We will generalize the result in [4] in

Manuscript received Oct. 24, 1994, revised Nov. 27, 1995

this paper.

2 Sufficient Conditions of Sampled-Data Systems

Now we return to the study of the stability of the system (1). We have the following general sufficient condition.

Theorem 2.1 Suppose $\Phi\ (t_{k+1}, t_k)$ is bounded, for any matrix norm $\|\ .\ \|$, group the time interval $\{(t_k,\ t_{k+1}) \mid k = 0,1,\cdots\} \stackrel{\text{def}}{=} G$ into the following classes $G_1, \cdots, G_m$, and $G = \bigcup_{i=1}^{m} G_i$, where

$$G_r = \{(t_i, t_{i+1}) \in G \mid \sigma_{r-1} < \|\Phi\ (t_{i+1}, t_i)\| \leqslant \sigma_r\},\ r = 1,2,\cdots,m$$

and $\sigma_1, \sigma_2, \cdots, \sigma_m$ are certain positive numbers. Let p_r denotes the frequency (or probability) that intervals in G happen to be in G_r for $r = 1,2,\cdots,m$. Then (1) is exponentially stable if

$$\sigma_1^{p_1}\sigma_2^{p_2}\cdots\sigma_m^{p_m} < 1. \tag{2}$$

Proof Let $n(k,r)$ denotes the numbers of intervals which happen to be in G_r among $\{(t_i, t_{i+1}) \mid i = 1,2,\cdots,k\}$. Using the multiplicative property of matrix norm, we obtain

$$\begin{aligned} \|\ x_{k+1}\ \| &= \|\Phi\ (t_{k+1},\ t_k)\cdots\Phi\ (t_1, t_0)x_0\ \| \\ &\leqslant \|\ \Phi\ (t_{k+1}, t_k)\ \| \cdots \|\ \Phi\ (t_1, t_0)\ \|\ \|\ x_0\ \| \\ &\leqslant \sigma_1^{n(k,1)}\sigma_2^{n(k,2)}\cdots\sigma_m^{n(k,m)}\ \|\ x_0\ \| \\ &= [\sigma_1^{n(k,1)/k+1}\sigma_2^{n(k,2)/k+1}\cdots\sigma_m^{n(k,m)/k+1}]^{k+1}\ \|\ x_0\ \|\ . \end{aligned} \tag{3}$$

Since, for any $r \in \{1,2,\cdots,m\}$, we have

$$\lim_{k\to\infty}\frac{n(k,r)}{k+1} = p_r,$$

hence, we obtain

$$\lim_{k\to\infty}\sigma_1^{n(k,1)/k+1}\sigma_2^{n(k,2)/k+1}\cdots\sigma_m^{n(k,m)/k+1} = \sigma_1^{p_1}\sigma_2^{p_2}\cdots\sigma_m^{p_m}.$$

Thus, if $\sigma_1^{p_1}\sigma_2^{p_2}\cdots\sigma_m^{p_m} < 1$, from (3), we can conclude that the system (1) is exponentially stable.

Remark To effectively use the above sufficient condition, we have to appropriately choose a matrix norm so that $\sigma_1^{p_1}\sigma_2^{p_2}\cdots\sigma_m^{p_m}$ is as small as possible. In [4], Ritchey and Franklin assume that $\Phi\ (t_{k+1}, t_k)$ is diagonalizable and they use the matrix 2-norm to obtain a sufficient condition for asymptotic stability of (1). In a similar manner, we can use any matrix norm to obtain a more general sufficient condition. Let $\Phi\ (t_{k+1}, t_k) = S_k\Gamma_k S_k^{-1}$ where Γ_k is a diagonal matrix. Given a sequence of positive numbers $\{\mu_k\}$, dedfine

$$\Delta_k = \sqrt{\frac{\Gamma_k}{\mu_k}}S_k^{-1}S_{k-1}\sqrt{\frac{\Gamma_{k-1}}{\mu_{k-1}}}.$$

using this notation, we can give the following result.

Theorem 2.2 For any matrix norm $\|\ .\ \|$ and any given sequence of positive numbers $\{\mu_k\}$, group the Δ's and μ's into numbered classes and define upper bounds ξ_j and σ_i such that $\|\Delta\| \leqslant \sigma_i$ for all the Δ's in class "i" and $\mu \leqslant \xi_j$ for all μ's in class "j". Let p_i and q_j denote the probability of occurence of sampling intervals with Δ in class i and μ in class

j, respectively. Then the system (1) is exponentially stable if

$$\prod_i \sigma_i^{p_i} \prod_j \eta_j^{q_j} < 1.$$

Proof This can be proved in a similar manner as in [4].

Suppose that $\Phi\ (t_{k+1}, t_k)$ is only dependent on the difference $t_{k+1} - t_k$ as we discussed in section, we can reduce to the stability analysis of the following linear system:

$$x_{k+1} = A_k(\xi_k) x_k \tag{4}$$

where $\xi_k = t_{k+1} - t_k$. This problem is also studied in [6], where $A_k(\xi_k) = \exp(A_k \xi_k)$ and the sampling period ξ_k is assumed to be independently identically distributed and the moment stability is studied. The sufficient conditions for exponentially stability in Theorem 2. 1 and 2. 2 can be used to study almost sure stability of (4). In fact, we can further simplify the sufficient condition for the system studied in [6] using the concept of matrix measure. Given a matrix norm $\| \cdot \|$, a matrix measure of a matrix A induced by the given matrix norm is defined as

$$\mu\ (A) = \lim_{\theta \to 0} \frac{\| I + \theta A \| - 1}{\theta},$$

where I is the identity matrix with the same dimension as A. For detailed properties and its applications, the reader may refer to [7]. Using the property of matrix measure, we obtain

Theorem 2. 3 Consider the system (4) with $A_k(\xi_k) = \exp(A_k \xi_k)$, we have the following: (if the samplimg sequence $\{\xi_k\}$ is a random process, then the stability means almost sure stability)

a) The system (4) is exponentially stable if there exists a matrix measure such that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \mu\ (A_i) \xi_k < 0.$$

b) Choose a matrix measure $\mu\ (\cdot)$, group $G \stackrel{\text{def}}{=\!=} \{\mu\ (A_k)\}$ into a finite number of classes. say, $G_1, \cdots, G_m$, where,

$$G_i = \{\mu\ (A_k) \mid \mu\ (A_k) \in G, \mu_{i-1} < \mu\ (A_k) \leqslant \mu_i\}, i = 1, 2, \cdots, m,$$

where $\mu_0 = -\infty$ and $\mu_1, \cdots, \mu_m$ are some apporopriately chosen numbers. Let π_i denotes the probability of occurence of $\mu\ (A_k)$ in the class G_i, $i = 1, 2\cdots$, then (4) is exponentially stable if $\pi_1 \mu_1 + \pi_2 \mu_2 + \cdots + \pi_m \mu_m < 0$.

c) If A_k only assumes a finite number of matrices, say, $A_k \in \{F_1, F_2, \cdots, F_m\}$, let π_i denotes the probability of occurence of A_k assuming F_i in a long run, then (4) is exponentially stable if there exists a matrix measure $\mu(\cdot)$ such that $\pi_1 \mu\ (F_1) + \pi_2 \mu\ (F_2) + \cdots + \pi_m \mu\ (F_m) < 0$.

Proof b) and c) are just direct consequences of a). We only need to prove a). Let $\| \cdot \|$ be the matrix norm which induces the matrix measure. Then, applying the property of matrix measure $\| \exp(At) \| \leqslant \exp(\mu\ (A)t)$, we have

$$\begin{aligned}\| x_{k+1} \| &\leqslant \| e^{A_k \xi_k} \| \cdots \| e^{A_0 \xi_0} \| \| x_0 \| \\ &\leqslant e^{\mu(A_k)\xi_k} \cdots e^{\mu(A_0)\xi_0} \| x_0 \| \\ &= (e^{\sum_{i=0}^{k} \mu(A_i)\xi_i / k})^k \| x_0 \| .\end{aligned}$$

From this, the proof can be easily completed.

In many applications, the sampling period sequence $\{\xi_k\}$ may only assume a finite number of values and $A_k(\xi_k)$ only depends on ξ_k[5]. For simplicity of notations, we may assume that ξ_k is an integer-valued function, say, $\{1,2,\cdots,N\}$ and $A_k(\xi_k) = A(\xi_k)$, $\xi_k = i$ means that $A(i)$ is the state transition matrix with the i-th period. For such case, we have the following result for stability.

Theorem 2.4 Let π_i denotes the probability of the occurence of state transition matrix $A(i)$ in a long run, then (4) is exponentially stable if there exists a matrix norm $\| \cdot \|$ such that

$$\| A(1) \|^{\pi_1} \| A(2) \|^{\pi_2} \cdots \| A(N) \|^{\pi_N} < 1.$$

Proof This can be easily derived from Theorem 2.1. Using a similar technique, we obtain the following result.

Theorem 2.5 Let π_i denotes the probability of the occurence of state transition matrix $A(i)$ in a long run. Suppose that $A(1),\cdots,A(N)$ can be transformed using one single similarity transformation into upper triangular forms

$$\begin{bmatrix} \lambda_1(i) & * & * & * \\ & \lambda_2(i) & * & * \\ & & \ddots & \vdots \\ & & & \lambda_n(i) \end{bmatrix}, \quad i = 1,2,\cdots,n \tag{5}$$

then (4) is exponentially stable if and only if

$$\lambda_i(1)^{\pi_1} \lambda_i(2)^{\pi_2} \cdots \lambda_i(N)^{\pi_N} < 1, \quad i = 1,2,\cdots,n; \tag{6}$$

If $A(1),\cdots,A(N)$ pairwise commute, then they can be transform using a single similarity transformation into the forms (5), hence (4) is exponentially stable if and only if (6) holds.

In [5], Halevi and Ray studied the stability of feedback control systems with communication delays. It is shown that when a control system shares a single communication medium through the feedback loop, time delay will occur in the feedback loop, the closed-loop system can be modelled as a system with communication delay. When the time delay is periodic, Halevi and Ray provided a sufficient condition for the stability of the closed-loop system. Using our notations in the above, we restate their result in the following: If the samping sequence $\{\xi_k\}$ has a period $M>0$, i.e., $\xi_{k+M}=\xi_k$, then (4) is exponentially stable if the matrix $A(\xi_{M-1})A(\xi_{M-2})\cdots A(\xi_0)$ is Schur stable, i.e., its eigenvalues are inside the unit circle. This motivates the following problem: Given a finite number of matrices

$A(1), A(2), \cdots, A(N)$ and a finite number of integers $i_1, i_2, \cdots, i_p \in \{1,2,\cdots,N\}$, what is the condition for the matrix $A(i_1)A(i_2)\cdots A(i_p)$ to be (Schur) stable? For this problem, we have the following result.

Theorem 2.6 a) The matrix $A(i_1)\cdots A(i_p)$ is stable if and only if there exists a matrix norm $\|\cdot\|$ such that $\|A(i_1)\cdots A(i_p)\|<1$; The matrix is also stable if there exists a matrix norm $\|\cdot\|$ such that $\|A(i_r)\|<1$ for any $r=1,2,\cdots,p$.

b) The matrix $A(i_1)\cdots A(i_p)$ is stable if there exists a positive definite matrix P such that $A^T(i_r)PA(i_r)-P$ is a negative definite matrix for any $r=1,2,\cdots,p$.

Proof a) This can be proved by the fact that a matrix A is stable if its spectral radius $\rho(A)<1$.

b) From Lyapunov stability theory, A is stable if and only if there exists a positive definite matrix P such that A^TPA-P is a negative definite. We use $P>Q$ to denote that $P-Q$ is a positive definite matrix for any positive definite matrices P and Q. If, for any $r=1,2,\cdots,p$, positive matrix P satisfies $Q(i_r)\overset{\text{def}}{=}A^T(i_r)PA(i_r)-P<0$, then

$$
\begin{aligned}
&[A(i_1)\cdots A(i_p)]^TP[A(i_1)\cdots A(i_p)]-P \\
&=A^T(i_p)\cdots A^T(i_2)PA(i_2)\cdots A(i_p)-A^T(i_p)\cdots A^T(i_2)Q(i_1)A(i_2)\cdots A(i_p)-P \\
&\leqslant A^T(i_p)\cdots A^T(i_2)PA(i_2)\cdots A(i_p)-P\leqslant A^T(i_p)PA(i_p)-P<0.
\end{aligned}
$$

From this, the matrix $A(i_1)\cdots A(i_p)$ is stable.

3 Conclusion

In this paper, we study the stability of sampled-data systems. Some very general sufficient conditions are obtained using the multiplicative property of matrix norm. As we observed that different choice of matrix norm gives different sufficient condition for (almost sure) stability.

References

[1] Walton, V. M.. State Space Stability of Multirate-Multiloop Sampled Data System. Proc. AAS/ALAA Astrodynam. Specialist Conf. 1981, Aug., 81—201

[2] Glasson, D. P.. Development and Application of Multirate Digital Control. Control Systems Magazine, 1983, (1): 2—8

[3] Franklin, G. F., Powell J. D. and Workman, M. L.. Digital Control of Dynamic Systems. 2nd ED., Addison-Wesley Publishing Company, New York, 1990

[4] Ritchey, V. S. and Franklin, G. F.. A Stability Criterion for Asynchronous Multirate Linear Systems. IEEE Trans. Automat. Contr., 1989, AC-34(5): 529—535

[5] Halevi, Y. and Ray, A.. Integrated Communication and Control Systems: Part I-Analysis. ASME J. Dynam. Syst. and Contr., 1989, 110: 367—373

[6] Bergen, A. R.. Stability of Systems with Randomly Time-Varying Parameters. IEEE Trans. Automat. Contr., 1960, AC-5(2): 265—269

[7] Vidyasagar, M.. Nonlinear Systems Analysis. 2nd., Prentice-Hall, Inc., 1993

多速率采样数据系统的稳定性分析

方玉光 初学导 武玉强
（美国凯斯西部保留地大学系统工程系） （曲阜师范大学自动化研究所·山东曲阜，273165）

摘 要：本文对异步采样数据系统的稳定性给出了新的充分条件. 一种新的鲁棒稳定性问题被提出，并给出了这一问题的初步结果.

关键词：稳定性；采样数据；异步的；鲁棒

本文作者简介

方玉光 1963 年生. 1987 年曲阜师范大学硕士研究生毕业，获理学硕士学位，1987 年留曲阜师范大学自动化研究所工作，1988 年在美国 Case Western Reserve University 读博士研究生，1994 年获博士学位，现为曲阜师范大学自动化研究所教授，美国波士顿大学博士后. 主要研究方向有：神经网络控制，随机系统分析等. 目前已在 IEEE AC，Inter. J. Contr. 等国际学术刊物发表论文 20 余篇.

初学导 1939 年生，曲阜师范大学自动化研究所所长，教授. 发表论文 40 余篇，其中 12 篇发表于国内外主要学术刊物，主要研究方向为智能控制与自适应控制.

武玉强 1962 年生. 1988 年曲阜师范大学硕士研究生毕业，获硕士学位，1988 年留曲阜师范大学自动化研究所工作，1994 年东南大学自动控制理论及应用专业博士研究生毕业，获工学博士学位. 现为曲阜师范大学自动化研究所教授. 主要研究方向有：自适应控制与变结构控制.

（上接第 822 页）