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Circular Pole and Variance-Constrained Design for Linear Discrete Systems*

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Abstract: The problem of controller design for linear discrete systems with circular pole and variance constraints is considered in this paper. The goal of this problem is to design the controller such that the closed-loop system satisfies the prespecified circular pole constraints and the prespecified individual variance constraints, simultaneously. An algebraic, modified Riccati equation approach is developed to solve the above problem.

Key words: Discrete stochastic systems; constrained variance design; circular pole placement

1 Introduction

The performance requirements of many engineering control problems are naturally described in terms of the acceptable variance values of the system states. LQG controllers minimize a linear quadratic performance index which lacks guaranteed variance constraints with respect to individual system states. Recently, the covariance control theory[2~4] was developed to provided a more direct methodology for achieving the individual variance constraint than the LQG control theory. However, much of the covariance control literature focuses on the steady-state behavior and the transient properties are seldom considered. To this end, this paper will introduce an approach which deals with the circular pole assignment technique[5,6] and individual variance constraint for linear discrete systems, simultaneously.

Most previous work about pole assignment focuses on the problem of exact pole assignment. It is often the case in practice, however, that exact closed-loop pole locations are not required. Rather, because the problem considered in this paper is a multiobjective design task, the exact locations of assignable poles might be difficult to attain. Owing to the above important reasons, an attempt will be made in this work to design controllers which can achieve a specified state covariance upper bound, such that the variance of each state meets the specified constraints and such that the closed-loop poles lie within a specified circular region. An effective, algebraic, modified Riccati equation approach is developed to solve the addressed problem.

2 Problem Formulation and Preliminaries

Consider the stationary vector process x generated by

$$x(k+1) = Ax(k) + Bu(k) + Dw(k). \tag{2.1}$$

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Here $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, w \in \mathbb{R}^{n_w}$ and A,B,D are constant matrices with appropriate dimensions. w is a zero mean white noise process with covariance $W > 0$, and $w(k)$ and $x(0)$ are uncorrelated. The notation "$[\cdot] > 0$" and "$[\cdot] \geqslant 0$", respectively, denote positive definite and positive semidefinite. The pair (A,B) and (A,D) are, respectively, assumed to be stabilizable and controllable.

When a state feedback control law

$$u(k) = Gx(k) \tag{2.2}$$

is applied to system (2.1), a closed-loop system is obtained as

$$x(k+1) = A_c x(k) + Dw(k), \quad A_c = A + BG. \tag{2.3}$$

The steady-state covariance X of the closed-loop system defined as

$$X = \lim_{k\to\infty} E[x(k)x^{\mathrm{T}}(k)] \tag{2.4}$$

is the solution to the following discrete Lyapunov equation

$$X = A_c X A_c^{\mathrm{T}} + DWD^{\mathrm{T}}. \tag{2.5}$$

We consider a circular region $D(q,r)$ with the center at $q + j0 (q \geqslant 0)$ and the radius r which may be shown in Fig. 1 for the discrete system.

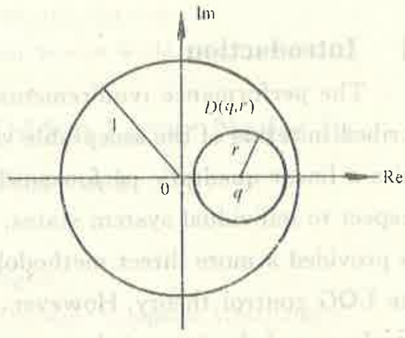

Fig. 1 Circular region $D(q,r)$ in the unit disk

Now, we may conclude that circular pole and variance-constrained design (CPVCD) problem considered in this paper is to determine the controller G such that the following performance criteria are satisfied.

a) The individual state variance constraints are satisfied, i. e.

$$[X]_{ii} \leqslant \sigma_i^2, \quad i = 1, 2, \cdots, n_x \tag{2.6a}$$

where $[X]_{ii}$ is the ith diagonal element of X, and $\sigma_i (i = 1, 2, \cdots, n_x)$ denotes the root-mean-squared value constraints for the variance of system state.

b) The closed-loop poles are constrained to lie within the circular region $D(q,r)$, i. e.,

$$\sigma(A_c) \subset D(q,r). \tag{2.6b}$$

Theorem 2.1 Given a circular region $D(q,r)$. Then the requirement b) is satisfied if the following matrix equation

$$A_c Q A_c^{\mathrm{T}} + (q^2 - r^2)Q + DWD^{\mathrm{T}} - q(A_c Q + QA_c^{\mathrm{T}}) = 0 \tag{2.7}$$

has a positive definite solution Q. Furthermore, in this case, if the positive definite solution Q meets

$$(A_c - \frac{q^2 - r^2 + 1}{2q} I)Q + Q(A_c - \frac{q^2 - r^2 + 1}{2q} I)^{\mathrm{T}} < 0 \tag{2.8}$$

then the steady-state covariance X given by (2.4) exists and satisfies

$$X < Q. \tag{2.9}$$

Proof See the Appendix.

By using the above theorem, we can assign a desired value to the positive definite matrix Q, such that this matrix Q meets

$$[Q]_{ii} \leqslant \sigma_i^2, \quad i = 1, 2, \cdots, n_x \tag{2.10}$$

and seek the set of the feedback controller G which satisfies (2. 7), (2. 8) for the specified Q. If such a controller exists and can be obtained, then from Theorem 2. 1, we will have $[X]_{ii} < [Q]_{ii} \leqslant \sigma_i^2 (i=1,2,\cdots,n_x)$ and $\sigma(A_c) \subset D(q,r)$, and therefore the design task will be accomplished. The above problem is referred to as the "Q-matrix assignment" problem and then the problem CPVCD can be converted to such an auxiliary "Q-matrix assignment" problem.

3 Main Results and Derivation

Lemma 3. 1[4] Let $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{m\times p} (m \leqslant p)$. There exists a matrix V which satisfies simultaneously

$$N = MV, \quad VV^{\mathrm{T}} = I \tag{3. 1}$$

if and only if

$$MM^{\mathrm{T}} = NN^{\mathrm{T}}. \tag{3. 2}$$

In this case, a general solution for V can be expressed as

$$V = V_M \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_N^{\mathrm{T}}, \quad U \in \mathbb{R}^{(n-r_M)\times(p-r_M)}, \quad UU^{\mathrm{T}} = I \tag{3. 3}$$

where V_M and V_N come from the singular value decompositions (SVD) of M and N, respectively,

$$M = U_M \begin{bmatrix} Z_M & 0 \\ 0 & 0 \end{bmatrix} V_M^{\mathrm{T}} = [U_{M1} \quad U_{M2}] \begin{bmatrix} Z_M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{M1}^{\mathrm{T}} \\ V_{M2}^{\mathrm{T}} \end{bmatrix},$$

$$N = U_N \begin{bmatrix} Z_N & 0 \\ 0 & 0 \end{bmatrix} V_N^{\mathrm{T}} = [U_{N1} \quad U_{N2}] \begin{bmatrix} Z_N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{N1}^{\mathrm{T}} \\ V_{N2}^{\mathrm{T}} \end{bmatrix},$$

and $\quad r_M = \mathrm{rank}(M), \quad U_M = U_N, \quad Z_M = Z_N.$

To make the problem more tractable, we give the following definition.

Definition 3. 1 Given a desired circular region $D(q,r)$. Let Q be a prespecified positive definite matrix which meets (2. 10). Then Q is called a D-assignable matrix if there exists a set of controllers G such that the equation (2. 7) has the positive definite solution Q and this Q meets the inequality (2. 8).

Now, let Q be some prespecified positive definite matrix and $Q^{1/2}$ be the unique positive definite square root of Q. To obtain the conditions for the existence of desired controllers, G, for the circular pole and variance-constrained design problem, we can rearrange (2. 7) as follows:

$$(A_c Q^{1/2} - qQ^{1/2})(A_c Q^{1/2} - qQ^{1/2})^{\mathrm{T}} = r^2 Q - DWD^{\mathrm{T}}. \tag{3. 4}$$

Consider (3. 4), since its left-hand side is positive semidefinite, Q is required to meet

$$Q \geqslant (1/r^2) DWD^{\mathrm{T}} \tag{3. 5}$$

which gives the lower bound of Q.

Now we first define $P = r^2 Q - DWD^{\mathrm{T}}$, and take the square root of P

$$P = TT^{\mathrm{T}}, \quad T \in \mathbb{R}^{n_x \times n_x}. \tag{3. 6}$$

From Lemma 3. 1 and (3. 6), (3. 4) is equivalent to

$$A_c Q^{1/2} - qQ^{1/2} = TV \tag{3. 7a}$$

or

$$BG = TVQ^{-1/2} + qI - A \tag{3.7b}$$

where $V \in \mathbb{R}^{n_x \times n_x}$ is some orthogonal matrix.

Hence, we obtain the following result.

Lemma 3.2 Suppose that (3.5) is satisfied. Equation (2.7) has a solution for G if and only if (3.7a) or (3.7b) has a solution for G.

Lemma 3.3[1] There exists an orthogonal matrix V such that (3.7b) has a solution for G, if and only if there exists an orthogonal matrix V such that

$$(I - BB^+)(TVQ^{-1/2} + qI - A) = 0 \tag{3.8}$$

where B^+ denotes the Moore-Penrose inverse of B.

Now, prior to considering the inequality (2.8), we first take the following SVD:

$$M = (I - BB^+)T = U_M\begin{bmatrix} Z_M & 0 \\ 0 & 0 \end{bmatrix}V_M^{\mathrm{T}}, \tag{3.9}$$

$$N = (I - BB^+)(A - qI)Q^{1/2} = U_N\begin{bmatrix} Z_N & 0 \\ 0 & 0 \end{bmatrix}V_N^{\mathrm{T}}. \tag{3.10}$$

Lemma 3.4 The inequality (2.8) is satisfied for the positive definite solution Q of (2.7) if and only if, there exists an orthogonal matrix $U \in \mathbb{R}^{(n_x - r_M)\times(n_x - r_M)}$ satisfying

$$TVQ^{1/2} + (TVQ^{1/2})^{\mathrm{T}} < \frac{1-(q^2+r^2)}{q}Q \tag{3.11}$$

where $V = V_M\begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}V_N^{\mathrm{T}}$, $r_M = \mathrm{rank}M$ and M, N, V_M, V_N are defined as in (3.9), (3.10). T is the square root of $r^2Q - DWD^{\mathrm{T}}$.

Proof From (3.4), (3.7) and (3.8), we know that the equation (2.7) has a positive definite solution Q if and only if

$$A_c = TVQ^{-1/2} + qI \tag{3.12}$$

where $V \in \mathbb{R}^{n_x \times n_x}$ is an orthogonal matrix and this matrix V satisfies

$$(I - BB^+)TV = (I - BB^+)(A - qI)Q^{1/2} \tag{3.13}$$

or

$$N = MV. \tag{3.14}$$

By using Lemma 3.1, the orthogonal matrix V satisfying (3.14) can be expressed as

$$V = V_M\begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}V_N^{\mathrm{T}}, \quad U \in \mathbb{R}^{(n_x - r_M)\times(n_x - r_M)} \tag{3.15}$$

where matrix U is arbitrary orthogonal. Substituting (3.12) and (3.15) into (2.8) yields (3.11). This proves the lemma.

Theorem 3.1 A specified positive definite matrix Q satisfying (2.10), (3.5) is D-assignable, if and only if

1) $(I - BB^+)[r^2Q - DWD^{\mathrm{T}} - (A - qI)Q(A - qI)^{\mathrm{T}}](I - BB^+) = 0.$ (3.16)

2) There exists an orthogonal matrix V such that

$$TVQ^{1/2} + (TVQ^{1/2})^{\mathrm{T}} < \frac{1-(q^2+r^2)}{q}Q, \tag{3.17}$$

where the orthogonal matrix V is defined in (3.15) and T is the sqaure root of $r^2Q - DWD^{\mathrm{T}}$.

Proof It is clear from Lemma 3.2, Lemma 3.3 and Lemma 3.4 that the given $Q > 0$ satisfying (2.10), (3.5) is D-assignable if and only if, there exists an orthogonal matrix V satisfying (3.8) and (3.17). We can rewrite (3.8) as the following

$$(I - BB^+)TV = (I - BB^+)(A - qI)Q^{1/2}. \tag{3.18}$$

Using Lemma 3.1, (3.18) has an orthogonal solution V if and only if

$$\begin{aligned}&[(I - BB^+)T][(I - BB^+)T]^{\mathrm{T}} \\ &= [(I - BB^+)(A - qI)Q^{1/2}][(I - BB^+)(A - qI)Q^{1/2}]^{\mathrm{T}}\end{aligned} \tag{3.19}$$

or equivalently

$$\begin{aligned}&(I - BB^+)(r^2Q - DWD^{\mathrm{T}})(I - BB^+) \\ &= (I - BB^+)[(A - qI)Q(A - qI)^{\mathrm{T}}](I - BB^+)\end{aligned} \tag{3.20}$$

and the orthogonal solution V of (3.19) can be expressed as (3.15). The equation (3.21) is equivalent to (3.17). This proves the theorem.

In what follows, the solutions of the circular pole and variance-constrained design problem is introduced.

Theorem 3.2 Assume that the given positive definite matrix Q satisfying the condition (2.10), (3.5) is D-assignable, then the set of all controllers that assign this Q is parameterized as

$$G = B^+\left(TV_M\begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}V_N^{\mathrm{T}}Q^{-1/2} + qI - A\right) + Z - B^+BZ \tag{3.21}$$

where $TT^{\mathrm{T}} = r^2Q - DWD^{\mathrm{T}}$, $Z \in \mathbb{R}^{n_u \times n_x}$ is arbitrary, $U \in \mathbb{R}^{(n_x - r_M)\times(n_x - r_M)}$ is arbitrary orthogonal, $r_M = \operatorname{rank} M$, and M, N, V_M, V_N are defined as in (3.9), (3.10).

Proof From the proof of Theorem 3.1, we know that the given Q is D-assignable if and only if there exists a solution G to (3.7b) for some orthogonal matrix V satisfying (3.18). Hence, a general solution of (3.7b) is

$$G = B^+(TVQ^{-1/2} + qI - A) + Z - B^+BZ \tag{3.22}$$

where V defined in (3.15) is the orthogonal matrix satisfying the inequality (3.18). This proves the theorem.

Theorem 3.2 provides the set of all feedback controllers G which can achieve a specified circular region for closed-loop poles and a specified state covariance upper bound Q for (2.7). By appropriately assigning Q with $[Q]_{ii} \leqslant \sigma_i^2$, where σ_i have been defined in (2.6a), then from (2.9), we have $[X]_{ii} < [Q]_{ii} \leqslant \sigma_i^2$. Hence, the variance constraints (2.6a) and the circular pole constraints (2.6b) will be achieved by using the feedback controllers G which can be obtained from (3.22), and the following result is easily accessible.

Theorem 3.3 Given the desired circular region $D(q,r)$ and the individual state variance-constraints $\sigma_i^2 (i = 1, 2, \cdots, n_x)$. Assume that a specified positive definite matrix Q satisfying (2.10), (3.5) is D-assignable, i.e., this Q meets (3.16), (3.17). Then the solution of the circular pole and variance-constrained design problem can be obtained from (3.21).

4 A Numerical Example

Consider the linear stochastic discrete system (2.1) where the parameters are given as

$$A=\begin{bmatrix}0.6 & 0 & 0\\ 0 & 1.5 & 1\\ 0 & 0 & 1.5\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix},\quad D=\begin{bmatrix}0.2\\ 0\\ 0\end{bmatrix},\quad W=I. \tag{4.1}$$

We assume that the constraints for variance of system states are

$$[X]_{11}\leqslant 1.41,\quad [X]_{22}\leqslant 2.23,\quad [X]_{33}\leqslant 1.15 \tag{4.2}$$

and the condition for closed-loop poles is

$$D(q,r)=D(0.5,0.2). \tag{4.3}$$

Subject to the conditions (3.5), (3.16) and the constraints (4.2), (4.3), we can choose an appropriate Q-matrix as

$$Q=\begin{bmatrix}4/3 & 0.01 & 0\\ 0.01 & 2 & 0.1\\ 0 & 0.1 & 1\end{bmatrix}.$$

Using the results provided in the previous section, we can obtain

$$T=\begin{bmatrix}0.11546 & 0.00101 & -0.00003\\ 0.00101 & 0.28272 & 0.00829\\ -0.00003 & 0.00829 & 0.19983\end{bmatrix},\quad V_M=V_N=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

It is easy to test that the condition (3.17) is satisfied for $U=I_2$. Therefore, substituting $A,B,T,V_M,V_N,Q^{-1/2},q$ and $U=I_2, Z=0$ into (3.21) yields a desired feedback controller

$$G=\begin{bmatrix}0.0002 & -0.80001 & -1\\ -0.00001 & 0 & -0.80001\end{bmatrix}.$$

Finally, by simulating the responses of this example, we can obtain the variance of states, i. e., $\mathrm{var}(x_1(k))=1.21236$, $\mathrm{var}(x_2(k))=1.83411$, $\mathrm{var}(x_3(k))=0.90024$. Moreover, the poles of closed-loop system are 0.6, 0.69999, 0.69999. Clearly, these results satisfy the constraints (4.2) and (4.3).

5 Conclusions

This paper has introduced a theory for designing feedback controllers such that the closed-loop system meets the prespecified variance and circular pole constraints. A simple, effective, generalized Lyapunov equation approach has been developed to solve the above problem. It is shown that the above problem can be converted to "Q-matrix assignment" problem and this Q-matrix assignment problem has been solved completely. The existence conditions of the desired controllers and the set of solutions have been introduced in Section 3 of the present paper.

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Appendix

Proof of Theorem 2.1.

The proof of (2.6b) can be made by a simple modification of the proof of Lemma 1 in [6]. Furthermore, it is well known that the steady-state covariance X of the system(2.3) satisfies (2.5) where the matrix A_c is stable, i.e., the poles of A_c are located within the unit disk with the center at the origin.

To prove (2.9), subtract (2.5) from (2.7) to obtain

$$A_c(Q-X)A_c^{\mathrm{T}}-(Q-X)=q(A_cQ+QA_c^{\mathrm{T}})-(q^2-r^2+1)Q.$$

From the Lyapunov stability theory, we know that if

$$\Pi=q(A_cQ+QA_c^{\mathrm{T}})-(q^2-r^2+1)Q<0, \tag{A1}$$

then $Q-X>0$ by virtue of the stability of A_c. Clearly (A1) holds if $q=0$. Therefore, it is assumed that $q>0$. We can rewrite(A1) as the following:

$$\Pi=q\left[\left(A_c-\frac{q^2-r^2+1}{2q}I\right)Q+Q\left(A_c-\frac{q^2-r^2+1}{2q}I\right)^{\mathrm{T}}\right]<0. \tag{A2}$$

Let $Y=A_c-\frac{q^2-r^2+1}{2q}I$. To show that the positive definite matrix Q satisfying (A2) exists, from the Lyapunov stability theory, we now prove that $\sigma(Y)\subset C_g$, where $C_g=\{s\in \Xi \mid \mathrm{Re}(s)<0, \Xi \text{ is a complex plane}\}$. Noting that $\sigma(A_c)\subset D(q,r)$ and $q+r\leqslant 1$, we have $\mathrm{Re}(s)<q+r$, where s is an eigenvalue of A_c, and

$$\mathrm{Re}(s)-\frac{q^2-r^2+1}{2q}<q+r-\frac{q^2-r^2+1}{2q}=\frac{1}{2q}[(q+r)^2-1]\leqslant 0$$

which yields $\sigma(Y)\subset C_g$. This proves Theorem 2.1.

圆形区域极点及方差约束下线性离散系统的控制设计

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摘要：本文考虑线性离散随机系统在圆形区域极点及方差约束下的控制器设计问题.即设计状态反馈控制器,使闭环系统同时满足预先给定的圆形区域极点约束以及预先给定的各状态分量的方差约束.文中利用代数黎卡提方程方法解决了上述问题.

关键词：离散随机系统；约束方差设计；圆形区域极点配置.

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