The Solving of Riccati Equations for Large-Scale Systems with Symmetric Circulant Structure*

HUANG Shoudong and ZHANG Siying

(Department of Automatic Control, Northeastern University · Shenyang, 110006, PRC)

Abstract: This paper discusses the solving of the algebraic Riccati equations and the Lyapunov matrix equations for large-scale systems with symmetric circulant structure. It is shown that the solving of the algebraic Riccati equations and the Lyapunov matrix equations for such a system can be simplified by solving $\frac{N}{2}+1$ independent equations of dimension N times smaller than the original equations. As an application, the problems of the linear quadratic optimal control and the robust linear quadratic optimal control for such a system can also be simplified.

Key words: large-scale systems; symmetric circulant structure; Riccati equation; linear quadratic optimal control; robust control

1 Introduction

Many control problems are concerned with the solving of the algebraic Riccati equations and the Lyapunov matrix equations. Generally speaking, the higher the dimensions of the systems are, the more difficult the solving of such equations will be. But, for a special class of large-scale systems, we can use the special structure of the systems to simplify the problems.

In this paper, we study large-scale systems with symmetric circulant structure. Such systems are common in practice and include paper machines, distribution networks, coating processes, and systems consisting of units operating in parallel. Many industrial examples were given in[1]and the references therein. This type of systems also arise in lumped approximations to partial differential equations[2].

Large-scale systems with symmetric circulant structure have been dealt with in several papers. For example, Brockett and Willems[2] studied the controllability, observability and stability of such systems. Hovd and Skogestad[1] studied the H_2 and H_∞ control of such systems.

This paper is organized as follows. In Section 2, the model of large-scale systems with symmetric circulant structure is given. In Section 3, the methods of solving the algebraic Riccati equations and the Lyapunov matrix equations are presented. In Section 4, the problems of the linear quadratic quadratic optimal control and the robust linear quadratic optimal control are studied. Section 5 gives an illustrative example.

2 System Description

Before describing the systems, the definition of block symmetric circulant matrix is need-

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ed.

Definition 2.1 A matrix $C \in \mathbb{R}^{Nm\times Np}$, is called bolck criculant if C has the following structure

$$C = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_N \\ C_N & C_1 & C_2 & \cdots & C_{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ C_2 & C_3 & C_4 & \cdots & C_1 \end{bmatrix}$$

where $C_i \in \mathbb{R}^{m\times p}(i = 1,\cdots,N)$. If $C_i = C_{N-i+2}(i = 2,\cdots,N)$, then the matrix C is called block symmetric circulant, and denoted by scl $[C_1, C_2, \cdots, C_N]$.

Denote $m_j = [1 \quad v_j \quad v_j^2 \quad \cdots \quad v_j^{N-1}]^{\mathrm{T}}, j = 1,2,\cdots,N$, where $v_j = \exp(2\pi(j-1)\cdot\sqrt{-1}/N), j = 1,2,\cdots,N$, i.e. v_j is a root of the equation $v^N = 1$.

Let $R_N = \frac{1}{\sqrt{N}}[r_1 \quad r_2 \quad \cdots \quad r_N]$ with $r_1 = m_1 = [1 \quad 1 \quad \cdots \quad 1]^{\mathrm{T}}, r_{\frac{N}{2}+1} = m_{\frac{N}{2}+1}$ if N is an even number, $r_p = \frac{1}{\sqrt{2}}(m_p + m_{N+2-p}), r_{N+2-p} = \frac{\sqrt{-1}}{\sqrt{2}}(m_p - m_{N+2-p}), (p = 2,3,\cdots,t)$, where $t = \frac{N+1}{2}$ if N is odd and $t = \frac{N}{2}$ if N is even.

Then R_N is a real orthogonal matrix, and the following result holds[1].

Lemma 2.1 Let $C = \mathrm{scl}[C_1\, C_2 \cdots C_N]$ with $C_i \in \mathbb{R}^{m\times p}(i = 1,\cdots,N)$. Then $C_d = (R_N \otimes I_m)^{\mathrm{T}} C (R_N \otimes I_p) = \mathrm{diag}[C_{d1}\, C_{d1} \cdots C_{dN}]$ is a block diagonal matrix, and $C_{di} = C_{d(N+2-i)} (i = 2, 3,\cdots,t)$, where $\otimes$ denotes the Kronecker product, and I_q denotes a $q \times q$ identity matrix.

The relation between $(C_{d1}\, C_{d1} \cdots C_{dN})$ and $(C_1\, C_1 \cdots C_N)$ is:

$$\begin{bmatrix} C_{d1} \\ C_{d2} \\ \vdots \\ C_{dN} \end{bmatrix} = (\sqrt{N} F_N \otimes I_m)^H \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}, \quad \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} = \frac{1}{\sqrt{N}}(F_N \otimes I_m) \begin{bmatrix} C_{d1} \\ C_{d2} \\ \vdots \\ C_{dN} \end{bmatrix}$$

where $F_N^H = \frac{1}{\sqrt{N}}[m_1 \quad m_2 \quad \cdots \quad m_N]$.

Consider a class of large-scale systems composed of N subsystems, each of which is described by

$$\dot{x}_i = Ax_i + Bu_i + \sum_{j=1}^{N} D_{ij} x_j, \quad i = 1,\cdots,N$$

where x_i, u_i are the n-, m-dimensional vectors of the subsystem states, control inputs, respectively.

We further assume that

$$D = (D_{ij}) = \mathrm{scl}[D_1, D_2, \cdots, D_N]. \tag{1}$$

Then ths state-space model of the overall system is

$$\dot{x} = \overline{A}x + \overline{B}u \tag{2}$$

where $x = (x_1', \cdots, x_N')'$, $u = (u_1', \cdots, u_N')'$ and

$$\overline{A} = \mathrm{scl}[A + D_1, D_2, \cdots, D_N] = \mathrm{scl}[A_1, A_2, \cdots, A_N],$$

$$\overline{B} = \mathrm{diag}[B,\cdots,B].$$

We shall hereafter refer to the system (2) as a large-scale system with symmetric circulant structure.

In the following of this paper, we denote $T_i = R_N \otimes I_i$, and denote

$$\begin{cases} A_d = T_n^{-1}\overline{A}T_n = \mathrm{diag}[A_{d1},A_{d2},\cdots,A_{dN}], \\ B_d = T_n^{-1}\overline{B}T_m = \overline{B} = \mathrm{diag}[B,B,\cdots,B]. \end{cases} \tag{3}$$

Let k denote the number of distinct matrices among $A_{d1},A_{d2},\cdots,A_{dN}$. Obviously, $k = \frac{N}{2} + 1$ if N is even; and $k = \frac{N+1}{2}$ if N is odd.

From (3) we can easily get the following fundamental results for the system (2) which were given in[2].

Theorem 2.1 System (2) is completely controllable (c.c.) if and only if the pairs $\{A_{di}, B\}(i = 1,\cdots,k)$ are all c.c.

Theorem 2.2 System (2) is asymptotically stable if and only if $A_{di}(i = 1,\cdots,k)$ are all asymptotically stable. i.e. $\sigma(\overline{A}) \subset C^- \rightleftharpoons \sigma(A_{di}) \subset C^-, i = 1,\cdots,k$ where $\sigma(A)$ denotes the spectrum of matrix A ,and C^- denotes the open left half plane.

3 The Solving of Riccati and Lyapunov Equations

In this section, we will show that we can construct the solutions of the algebraic Riccati equations and the Lyapunov matrix equations for the system (2) from solving corresponding equations for considerably lower order systems. The following results provide the details of these construction.

Theorem 3.1 Suppose the system (2) is c.c. and symmetric and positive definite matrices $P_{di} \in \mathbb{R}^{n\times n}(i = 1,\cdots,k)$ are the solutions of the Riccati equations

$$A_{di}^{\mathrm{T}}P_{di} + P_{di}A_{di} - P_{di}BR_1^{-1}B^{\mathrm{T}}P_{di} + Q_1 = 0, \quad i = 1,\cdots,k, \tag{4}$$

respectively, where $Q_1 \in \mathbb{R}^{n\times n}, R_1 \in \mathbb{R}^{m\times m}$ are arbitrarily selected symmetric and positive definite matrices. Then the unique symmetric and positive definite solution P of the Riccati equation for the system (2), i.e.

$$\overline{A}^{\mathrm{T}}P + P\overline{A} - P\overline{B}R^{-1}\overline{B}^{\mathrm{T}}P + Q = 0. \tag{5}$$

where $Q = \mathrm{diag}[Q_1,\cdots,Q_1]$ and $R = \mathrm{diag}[R_1,\cdots,R_1]$, has the structure

$$P = \mathrm{scl}[P_1,P_2,\cdots,P_N] \tag{6}$$

where

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} = \frac{1}{\sqrt{N}}(F_N \otimes I_n) \begin{bmatrix} P_{d1} \\ P_{d2} \\ \vdots \\ P_{dN} \end{bmatrix}. \tag{7}$$

Proof Denote $P_d = \mathrm{diag}[P_{d1},P_{d2},\cdots,P_{dN}]$, where $P_{di} = P_{d(N+2-i)}$ for $i = k+1,\cdots,N$. From (4) we have

$$A_d^{\mathrm{T}}P_d + P_dA_d - P_dB_dR^{-1}B_d^{\mathrm{T}}P_d + Q = 0. \tag{8}$$

Multiply (8) on the left by T_n and on the right by $T_n^{-1} = T_n^{\mathrm{T}}$ to obtain:

$$T_n A_d^{\mathrm{T}} T_n^{-1} T_n P_d T_n^{-1} + T_n P_d T_n^{-1} T_n A_d T_n^{-1}$$
$$- T_n P_d T_n^{-1} T_n B_d T_m^{-1} T_m R^{-1} T_m^{-1} T_m B_d^{\mathrm{T}} T_n^{-1} T_n P_d T_n^{-1} + T_n Q T_n^{-1} = 0.$$

That is

$$\overline{A}^{\mathrm{T}} P + P\overline{A} - P\overline{B}R^{-1}\overline{B}^{\mathrm{T}} P + Q = 0.$$

Now we have demonstrated that P is a solution of (5). Since system (2) is c. c. ,the equation (5) has unique symmetric and positive definite solution P. The proof is completed.

Similarly, the solution of the Lyapunov matrix equation for large-scale systems with symmetric circulant structure can also be obtained from solving corresponding equations of much lower order, as given in the following result.

Theorem 3. 2 Suppose $\sigma(A_{di}) \subset C^-, i = 1, 2, \cdots, k$ and symmetric and positive definite matrices $P_{di} \in \mathbb{R}^{n\times n} (i = 1, \cdots, k)$ are the solutions of the Lyapunov matrix equations

$$A_{di}^{\mathrm{T}} P_{di} + P_{di} A_{di} + Q_1 = 0, \quad i = 1, \cdots, k,$$

respectively, where $Q_1 \in \mathbb{R}^{n\times n}$ is arbitrarily selected symmetric and positive definite matrix. Then the unique symmetric and positive definite solution P of the Lyapunov equation for the system (2), i. e.

$$\overline{A}^{\mathrm{T}} P + P\overline{A} + Q = 0$$

where $Q = \mathrm{diag}[Q_1, \cdots, Q_1]$, has the structure (6) and (7).

Proof Noting that $\sigma(\overline{A}) = \bigcup_{i=1}^{k} \sigma(A_{di})$, the proof is similar with that of Theorem 3. 1 and is omitted here.

Remark 1 In the special case when $D_2 = D_3 = \cdots = D_N$ in (1). The systems (2) becomes the system considered in [3]. So the Theorem 3. 1 and Theorem 3. 2 in this paper are the generalization of the corresponding results in [3]. Moreover, the proof of Theorem 3. 1 is different from that of Theorem 1 in[3]. In fact, our method could be used in the proof of Theorem 1 in[3] and will shorten the proof.

4 Linear Quadratic Optimal Control

In this section, we consider the linear quadratic optimal control of the system (2).

Suppose the performance index to be minimized is

$$J = \int_0^\infty e^{2\alpha t}[x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} R u(t)]\mathrm{d}t$$

where the weighting matrices $Q = \mathrm{diag}[Q_1, \cdots, Q_1]$, $R = \mathrm{diag}[R_1, \cdots, R_1]$ and $Q_1 \in \mathbb{R}^{n\times n}$ and $R_1 \in \mathbb{R}^{m\times m}$ are assumed to be positive definite, and the real number α is used to prescribe the degree of stability.

From the well-known result of linear quadratic optimal control and Theorem 3. 1, the following conclusion can be easily obtained.

Theorem 4. 1 Suppose the system (2) is c. c. and symmetric and positive definite matrices $P_{di} \in \mathbb{R}^{n\times n} (i = 1, \cdots, k)$ are the solutions of the Riccati equations

$$(A_{di} + \alpha I)^{\mathrm{T}} P_{di} + P_{di}(A_{di} + \alpha I) - P_{di} B R_1^{-1} B^{\mathrm{T}} P_{di} + Q_1 = 0, \quad i = 1, \cdots, k,$$

respectively. Let matrix P be given by (6) and (7), then the state feedback control

$$u(t) = - R^{-1}\overline{B}^{\mathrm{T}} P x(t) \tag{9}$$

minimized J and the closed-loop system $\dot{x} = (\overline{A} - \overline{B}R^{-1}\overline{B}^{T}P)x$ is asymptoically stable.

Now we consider the robustnes of he controller given by (9).

Suppose the system is expressed by differential equation of the form

$$\dot{x} = \overline{A}x + \overline{B}u + f_0(x(t), u(t), t, \theta)$$

where $f_0(x(t), u(t), t, \theta)$ is the nonlinear perturbation, θ is uncertainty vector.

The resulting closed-loop system becomes

$$\dot{x} = (\overline{A} - \overline{B}R^{-1}\overline{B}^{T}P)x + f(x(t), t, \theta) \tag{10}$$

where $f(x(t), t, \theta) = f_0(x(t), -R^{-1}\overline{B}^{T}Px(t), t, \theta)$.

About the stability of the closed-loop system (10), we have the following result.

Theorem 4.2 Denote

$$D_i^* = (A_{di} + \alpha I)^{T} P_{di} + P_{di}(A_{di} + \alpha I) + 2Q_1, \quad i = i, \cdots, k,$$

and

$$\mu = \frac{\min \bigcup_{i=1}^{k} \sigma(D_i^*)}{2\max \bigcup_{i=1}^{k} \sigma(P_{di})} + \alpha \frac{\min \bigcup_{i=1}^{k} \sigma(P_{di})}{\max \bigcup_{i=1}^{k} \sigma(P_{di})}.$$

If $\| f(x, t, \theta) \| \leqslant \mu \| x \|$, then the closed-loop system (10) is robust stable.

The proof of Theorem 4.2 will use the following lemma which was given by Patel et. al.[4].

Lemma 4.1 Denote

$$D^* = (\overline{A} + \alpha I)^{T} P + P(\overline{A} + \alpha I) + 2Q$$

where P is given by (6) and (7), and denote

$$\mu_0 = \frac{\min\sigma(D^*)}{2\max\sigma(P)} + \alpha \frac{\min\sigma(P)}{\max\sigma(P)}.$$

If $\| f(x, t, \theta) \| \leqslant \mu_0 \| x \|$, then the closed-loop system (10) is robust stable.

Proof of Theorem 4.2 It is easy to see that: $T_n^{-1} P T_n = \mathrm{diag}[P_{d1}, P_{d2}, \cdots, P_{dN}]$ and $T_n^{-1} D^* T_n = \mathrm{diag}[D_1^*, D_2^*, \cdots, D_N^*]$ where $D_i^* = D_{(N+2-i)}^*$ for $i = k+1, \cdots, N$, hence $\sigma(P) = \bigcup_{i=1}^{k} \sigma(P_{di}), \sigma(D^*) = \bigcup_{i=1}^{k} \sigma(D_i^*)$, so we have $\mu = \mu_0$. From Lemma 4.1, the proof is completed.

Remark 2 All the results in this paper can be extended to the more general case in which the matrix $\overline{B}$ in (2) is block symmetric circulant. Since the conclusions are similar the details are omitted. Moreover, because the methods used in this paper are somewhat general, the matrices $\overline{A}$ and $\overline{B}$ in (2) needn't be block symmetric circulant. In fact, as long as there exist an orthogonal matrix $R_N \in \mathbb{R}^{N \times N}$ such that $T_n^{-1} \overline{A} T_n$ and $T_n^{-1} \overline{B} T_m$ are all block diagonal where $T_i = R_N \otimes I_i$, all the results in this paper will keep true.

5 Illustrative Example

Consider the system described by (2), where $N = 4, n = 2, m = 1$,

$$A = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = D_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = D_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

By computing directly, we have

$$A_1 = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{d2} = A_{d4} = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}, \quad A_{d3} = \begin{bmatrix} 0 & 5 \\ -4 & 0 \end{bmatrix}.$$

Since the pairs $\{A_{di}, B\}(i = 1,2,3)$ are all c. c. ,from Theorem 2. 1,the pair $\{\overline{A},\overline{B}\}$ is c. c. . Let $R_1 = 1, Q_1 = I_2$, solving the following three 2-dimensional Riccati equations

$$A_{di}^{\mathrm{T}} P_{di} + P_{di} A_{di} - P_{di} B R_1^{-1} B^{\mathrm{T}} P_{di} + Q_1 = 0, \quad i = 1,2,3,$$

we have

$$P_{d1} = \begin{bmatrix} 1.732 & 1 \\ 1 & 1.732 \end{bmatrix}, \quad P_{d2} = \begin{bmatrix} 1.159 & 0.236 \\ 0.236 & 1.554 \end{bmatrix}, \quad P_{d3} = \begin{bmatrix} 1.232 & 0.123 \\ 0.123 & 1.494 \end{bmatrix}.$$

From (7),we have

$$P_1 = \frac{P_{d1} + P_{d3} + 2P_{d2}}{4} = \begin{bmatrix} 1.32 & 0.399 \\ 0.399 & 1.584 \end{bmatrix},$$

$$P_2 = P_4 = \frac{P_{d1} - P_{d3}}{4} = \begin{bmatrix} 0.125 & 0.219 \\ 0.219 & 0.06 \end{bmatrix},$$

$$P_3 = \frac{P_{d1} + P_{d3} - 2P_{d2}}{4} = \begin{bmatrix} 0.162 & 0.163 \\ 0.163 & 0.029 \end{bmatrix}.$$

Thus,from Theorem 3. 1,the solution of the Riccati equation

$$\overline{A}^{\mathrm{T}} P + P\overline{A} - P\overline{B}R^{-1}\overline{B}^{\mathrm{T}} P + Q = 0$$

where $R = I_4, Q = I_8$,is

$$P = \mathrm{scl}[P_1, P_2, P_3, P_2]$$

$$= \begin{bmatrix} 1.32 & 0.399 & 0.125 & 0.219 & 0.162 & 0.163 & 0.125 & 0.219 \\ 0.399 & 1.584 & 0.219 & 0.06 & 0.163 & 0.029 & 0.219 & 0.06 \\ 0.125 & 0.219 & 1.32 & 0.399 & 0.125 & 0.219 & 0.162 & 0.163 \\ 0.219 & 0.06 & 0.399 & 1.584 & 0.219 & 0.06 & 0.163 & 0.029 \\ 0.162 & 0.163 & 0.125 & 0.219 & 1.32 & 0.399 & 0.125 & 0.219 \\ 0.163 & 0.029 & 0.219 & 0.06 & 0.399 & 1.584 & 0.219 & 0.06 \\ 0.125 & 0.219 & 0.162 & 0.163 & 0.125 & 0.219 & 1.32 & 0.399 \\ 0.219 & 0.06 & 0.163 & 0.029 & 0.219 & 0.06 & 0.399 & 1.584 \end{bmatrix}.$$

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具有对称循环结构的大系统 Riccati 方程的求解

黄守东 张嗣瀛

(东北大学自动控制系·沈阳,110006)

摘要:本文研究了具有对称循环结构的大系统的代数 Riccati 方程和 Lyapunov 矩阵方程的求解问题.结果表明,这类系统的代数 Riccati 方程和 Lyapunov 矩阵方程的求解问题可以简化为求解 $\frac{N}{2}+1$ 个独立的低阶方程.做为一个应用,这类系统的二次型最优控制问题和鲁棒二次型最优控制问题也可以简化.

关键词:大系统;对称循环结构;Riccati 方程;二次型最优控制;鲁棒控制

本文作者简介

黄守东 1969 年生.分别于 1987 年和 1990 年在东北工学院数学系获学士和硕士学位,1995 年 9 月起在东北大学自动控制系攻读博士学位.目前主要从事复杂控制系统的结构等方面的研究.

张嗣瀛 1925 年生.1948 年毕业于武汉大学机械系,1957 年至 1959 年赴前苏联莫斯科大学数学力学系进修.现为东北大学自动控制系教授,博士生导师,中国科学院院士,《控制与决策》主编.主要研究方向为对策理论及复杂控制系统结构分析.

《智能控制系统——模糊逻辑·专家系统·神经网络控制》评介

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