G-cactus and Some New Results on Structural Controllability of Composite Systems*

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Abstract: "G-cactus" (or called "cactus-like diagraph") is a useful notion in the study of structural properties of dynamical systems. In this paper, this notion is used to derive several new results on the structural controllability of composite systems.

Key words: G-cactus; cascade connection of G-cactus; structural controllability; composite system

1 Introduction

A large-scale system such as a complex industrial process is generally composed of a set of units or subsystems interlinked through several connections to form an interconnected one. Structural analysis plays an important role in the design of such a composite system for which the properties of each unit is known. It is also important for the synthesis of control structures of such a process.

The structural controllability is an important notion for structural analysis, and after its first introduction by Lin C. T. [1], some others [2～5] have made much effort to simplify the proof of the controllability theorem or to demostrate its complementary properties. However, the studies concerning the structural field relating to a complex composite system which is considered as a global one are still far from perfection, and the problem we have undertaken has not been investigated thoroughly.

"G-cactus" or called "cactus-like diagraph" is a useful notion in the study of structural properties of dynamical systems. [6,7], and this diagraph is the generalized form of "cactus" which was introduced by Lin C. T. [1]. In this paper, the new graph—"g-cactus" is used to derive some conditions for the structural controllability of composite systems with different forms of interconnections.

2 G-cactus and Structural Controllability

We first briefly recall several graphic notions as preliminary knowledge (one may refer to LI Kang, et al. [6,7]).

1) Stem. A stem is a directed noncyclic path of edges, where the origin is required to be a control input node.

2) Cycle. A cycle is a directed path whose origin concides to its extremity.

3) Bud. A bud is a directed edge at the end of which a cycle is attached, where the number of nodes in the cycle is not restricted.

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4) Cactus. A single-input cactus is a graph

$$C = T \cup B_1 \cup B_2 \cdots \cup B_p, \tag{1}$$

where T is the stem, and $B_i, i = 1,2,\cdots,p$ are buds. The origin of the distinguished edge of B_i is also the origin of an edge in the graph $T \cup B_1 \cup B_2 \cdots \cup B_{i-1}$, $\cup$ denotes union.

A multi-input cactus is a disjoint union of some single-input cactus.

5) G-cactus. A single-cactus is the union of a stem and some buds,

$$C_l(A,B) = T \cup B_1 \cup B_2 \cdots \cup B_p, \tag{2}$$

where T is the stem, $B_i, i = 1,2,\cdots,p$ are buds, the origin of the distinguished edge of B_i $\forall$ $i = 1,2,\cdots,p$ is only required to be the only node belonging to both B_i and graph $T \cup B_1 \cdots \cup B_{i-1}$ (see Fig. 1).

A multi-input g-cactus is a disjoint union of some single-input g-cactus.

In this paper, a single-input g-cactus is called elementary g-cactus.

6) Spray. A spray is the union of a group of buds,

$$S_p = B_1 \cup B_2 \cdots \cup B_p, \tag{3}$$

where $B_i, i = 1,2,\cdots,p$ are buds, and the origin of the distinguished edge of B_i $\forall$ $i = 1,2,\cdots,p$ is the only node belonging to both B_i and graph $B_1 \cup \cdots \cup B_{i-1}$. The origin of the distinguished edge of B_l is also called the origin of the spray (see Fig. 2).

Apparently, a bud is the simplest form of a spray.

Remark 2.1 Eqn. (2) is the general form of g-cactus. However, some graphs such as sprays, they possess the same properties of g-cactus relating to the structural controllability, therefore, they are also g-cactus—— special g-cactus. In order to coincide with the general description of g-cactus of (2), if a spray is consider as a g-cactus, its origin is treated as the "stem"—— a special stem that has only one node and its origin and extermity is the same. In this paper, if we do not point out, a g-cactus is referred to one of any form that can be taken as g-cactus (including sprays), and if we call a g-cactus a general g-cactus, its construction strictly follows (2) (e. g. its stem is a noncyclic directed path of edges), and if we call a g-cactus is a special g-cactus, we refer it to be a spray.

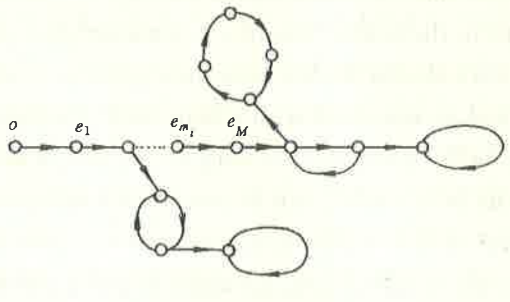

Fig. 1 A g-cactus

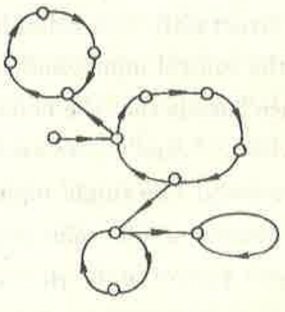
Fig. 2 A spray

Now, we consider a linear time-invariant system:

$$S: \quad \dot{x} = Ax + Bu, \tag{4}$$

where, A and B are structured matrices with dimension $n \times n$ and $n \times m$ respectively.

Let $G(A,B)$ be the diagraph (or called coats graph) of structured matrices pair (A,B) . Then we have the followng.

Theorem 2. 1[6,7] Structured matrices pair (A,B) is structurally controllable if and only if its diagraph $G(A,B)$ is spanned by a g-cactus $C_l(A,B)$.

Corollary 2. 2[6,7] System $S(A,B)$ is structurally controllable if and only if there exists a group of cycles $C_{yi},i=1,2,\cdots,p$ and a group of stems $T_i,i=1,2,\cdots,k$ ($k \leqslant m$, m is the number of control inputs) satisfying all the following conditions:

1) $C_{yi},i=1,2,\cdots,p$ and $T_i,i=1,2,\cdots,k$ are mutually disjoint;

2) $\bigcup_{i=1}^{p} C_{yi} \bigcup_{j=1}^{k} T_j$ covers all state nodes;

3) $C_{yi}\ \forall\, i=1,2,\cdots,p$ is reachable through an edge by a node in its rest part of the graph.

3 About the Structural Controllability of Composite Systems

When a composite system with the form of interconnections of subsystems is taken into consideration, the verification turns out to be complicated. Consider a general composite system,

$$\Sigma:\quad \dot{x}=Ax+Bu, \tag{5}$$

where A and B are structured matrices with dimension $n\times n$ and $n\times m$ respectively,

$$A=\begin{bmatrix} A_1 & & A_{ij} \\ & \ddots & \\ A_{ij} & & A_N \end{bmatrix},\quad B=\begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix} \tag{6}$$

and subsystems described as:

$$\Sigma_i: \dot{x}_i = A_i x_i + B_i u + \sum_{j<i} A_{ij}x_j + \sum_{j>i} A_{ij}x_j, \tag{7}$$

where $$A_i \in \mathbb{R}^{N_i\times N_i},\quad B_i\in \mathbb{R}^{N_i\times m},\quad i=1,2,\cdots,N,\quad \sum_{i=1}^{N} N_i = n.$$

Firstly, a general composite system with only two subsystems (Σ_1,Σ_2) is considered. We suppose Σ_1 structurally controllable relative to the control inputs, Σ_2 structurally controllable relative to the control inputs and all state variables of Σ_1 that have direct influence on Σ_2 ("Direct influence" means that the node concerned is connected with a node in Σ_2 through a directed edge), while, C_{l1} and C_{l2} are assumed to be the g-cactus spanning the graphs of subsystems Σ_1,Σ_2 respectively. The single-input g-cactus in C_{l1} and C_{l2} can be partitioned into two distinct types —— general single-input g-cactus and special single-input g-cactus (c. f. Remark 2. 1).

Let o_{1i}, $i=1,2,\cdots,n_1$ be the origins of the stems of general single-input g-cactus in C_{l1} , and define a set $O_1=\{o_{1i},i=1,2,\cdots,n_1\}$, it is required that $o_{li}\in\{u_j,j=1,2,\cdots,m\}$, $u_j,j=1,2,\cdots,m$ are control inputs.

Let $e_i,i=1,2\cdots,n_1$ be the terminal nodes of the stems of general single-input g-cactus in C_{l1} , and define a set $E_1=\{e_i,i=1,2,\cdots,n_1\}$.

With respect to Σ_2, according to the hypothesis that Σ_2 is structurally controllable relative to the control inputs and all state variables of Σ_1 that have direct influence on Σ_2, the origins of stems of general single-input g-cactus in C_{t2} can be classified into two distinct types. The first group includes those origins that are control nodes, and the second group includes those origins that are state nodes of subsystem Σ_1, define two sets, $O_2^1=\{o_{2i}^1, i=1,2,\cdots,n_2^1\}$, and $O_2^2=\{o_{2i}^2, i=1,2,\cdots,n_2^2\}$, where $o_{2i}^1 \in \{u_j, j=1,2,\cdots,m\}$, o_{2i}^2 are state nodes of subsystem Σ_1.

Definition 3.1 Cascade connection of g-cactus. Consider a general composite system Σ with two subsystems (Σ_1, Σ_2), Σ_1 is structurally controllable relative to the control inputs, Σ_2 is structurally controllable relative to the control inputs and all state variables of Σ_1 that have direct influence on Σ_2, and two multi-input g-cactus C_{t1} and C_{t2} are found to span graphs of Σ_1 and Σ_2 respectively, then the interconnection of C_{t1} and C_{t2} is a cascade relative to the control inputs if and only if both the following two conditions hold for all the general single-input g-cactus in C_{t1} and C_{t2}:

1) o_{2i}^2 are different from each other, and $o_{2i}^2 \in E_1, \forall\, i=1,2,\cdots,n_2^2$;

2) $O_1 \cap O_2^1 = \varnothing$.

Cascade connection of two general g-cactus is illustrated in Fig. 3.

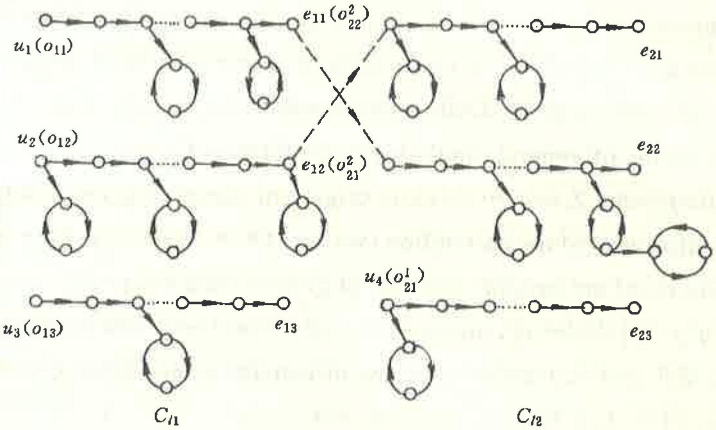

Fig. 3 Cascade connection of two general g-cactus

Remark 3.1 As one might have noted in Definition 3.1 that some special conditions are only imposed on all general single-input g-cactus, not on the special single-input g-cactus, that is to say, if a single-input g-cactus in C_{t1} is actually a spray (or simply a bud), its origin is only required to be a control input, and if a single-input g-cactus in C_{t2} is a spray (or simply a bud), its origin is only required to be a state node in Σ_1 or to be a control node, and the interconnection of g-cactus C_{t1} and C_{t2} is a cascade only if all general single-input g-cactus in C_{t1} and C_{t2} satisfy the two conditions in Definition 3.1. This case is illustrated in Fig. 4.

Theorem 3.1 Given a general composite system with two subsystems $\Sigma(\Sigma_1, \Sigma_2)$, Σ_1 is structurally controllable relative to the control inputs, Σ_2 is structurally controllable relative to control inputs and all state variables of Σ_1 that have direct influence on Σ_2, C_{t1}, C_{t2} are g-cactus spanning the graphs of Σ_1, Σ_2 respectively, then, the global system $\Sigma(\Sigma_1, \Sigma_2)$ is structurally controllable if the interconnection of C_{t1} and C_{t2} is a cascade relative to the control inputs.

The proof of Theorem 3.1 is apparent according to the definition of g-cactus and cascade connection of g-cactus.

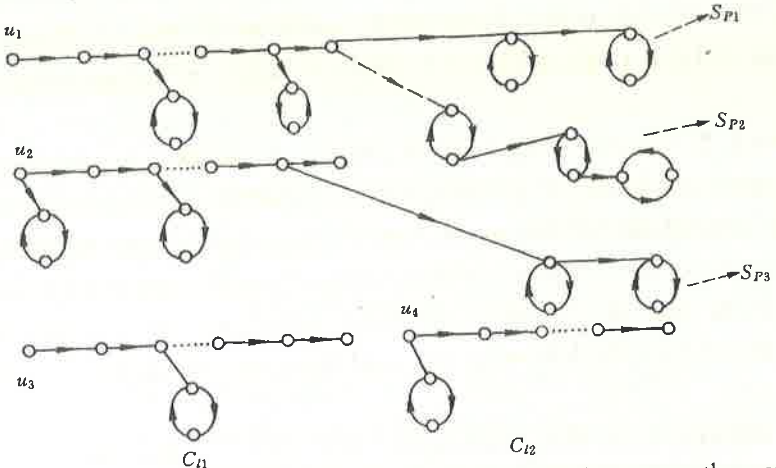

Fig. 4 Cascade connection of two g-cactus. In the second g-cactus, there are three sprays, and no special restriction is imposed on them

Now, we consider a general composite system Σ with N subsystems $(\Sigma_1, \Sigma_2, \cdots, \Sigma_N)$. The graph of Σ_1 is supposed to be spanned by a multi-input g-cactus C_{l1}, define a set $O_1^1 = \{o_{1i}, i = 1,2,\cdots,n_1\}$, where $o_{1i}, i = 1,2,\cdots,n_1$ are origins of stems of general single-input g-cactus in C_{l1}, and they are all control nodes. Define another set $E_1 = \{e_{1i}, i = 1,2,\cdots,n_1\}$, where e_{1i} are terminal nodes of stems of general single-input g-cactus in C_{l1}.

For other subsystems $\Sigma_i = 2,\cdots,N$, the origins of stems of general single-input g-cactus in C_{li} are divided into two groups and define two sets $O_i^1 = \{o_{ij}^1, j = 1,2,\cdots,n_i^1\}$, $O_i^2 = \{o_{ij}^2, j = 1,2,\cdots,n_1^2\}$, where o_{ij}^1, o_{ij}^2 are origins of stems of general single-input g-cactus in C_{li}, $o_{ij}^1 \in \{u_k, k = 1,2,\cdots,m\}$ $\forall$ $j = 1,2,\cdots,n_i^1$, and $o_{ij}^2 (j = 1,2,\cdots,n_i^2)$ are state nodes of Σ_j $j \in \{1,2,\cdots, i-1\}$. Let e_{ij} be the terminal nodes of stems of general single-input g-cactus in C_{li}, and define a set $E_i = \{e_{ij}, j = 1,2,\cdots,n_i\}$, where $n_i = n_i^1 + n_i^2$.

Definition 3.2 Cascade connection of g-cactus. Given composite system $\Sigma(\Sigma_1, \Sigma_2\cdots, \Sigma_N)$, where Σ_i $\forall$ $i = 1,2,\cdots,N$ is structurally controllable relative to the control inputs and all state variables of $\Sigma_j (j=1,2,\cdots,i-1)$ that have direct influence on Σ_i. Graph of Σ_i $\forall$ $i = 1,2,\cdots,N$ is spanned by a multi-input g-cactus C_{li}. The interconnection of $C_{li}, i = 1,2,\cdots,N$ is a cascade relative to the control inputs if and only if the general single-input g-cactus in $C_{li}, i = 1,2,\cdots,N$ satisfy all the following conditions:

1) $$o_{ij}^2 \in \bigcup_{k=1}^{i-1} E_k, \quad \forall\ j = 1,2,\cdots n_i^2, \quad i = 1,2,\cdots,N.$$

2) $$O_i^1 \cap O_j^1 = \varnothing, \quad \forall\ i,j \in \{1,2,\cdots,N\}, \quad i \neq j.$$

3) $$O_i^2 \cap O_j^2 = \varnothing, \quad \forall\ i,j \in \{1,2,\cdots,N\}, \quad i \neq j.$$

Theorem 3.2 Given a general composite system $\Sigma(\Sigma_1, \Sigma_2\cdots, \Sigma_N)$, Σ_i $\forall$ $i = 1,2,\cdots,N$ is structurally controllable relative to the control inputs and all state variables of $\Sigma_j (j=1,2,\cdots,i-1)$ that have direct influence on Σ_i, $C_{li}, i=1,2,\cdots,N$ are g-cactus spanning the graphs of subsystems respectively, then the global system Σ is structurally controllable if the interconnection of $C_{li}, i = 1, 2, \cdots, N$ is a cascade relative to the control inputs.

Theorem 3.2 is apparent according to the definition of cascade connection of g-cactus.

Theorem 3.2 offers a sufficient condition for general composite systems to be structurally controllable, moreover, for a composite system with certain interconnections of subsystems, this condition is also a necessary one.

Consider the following interconnected system:

$$\Sigma: \quad \dot{x} = Ax + Bu, \tag{8}$$

where, A and B are structured matrices with dimension $n \times n$ and $n \times m$ respectively,

$$A = \begin{bmatrix} A_1 & & O \\ & \ddots & \\ A_{ij} & & A_N \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix}, \tag{9}$$

and subsystems described as:

$$\Sigma_i: \dot{x}_i = A_i x_i + B_i u + \sum_{j<i} A_{ij} x_j, \tag{10}$$

where $$A_i \in \mathbb{R}^{N_i \times N_i}, \quad B_i \in \mathbb{R}^{N_i \times m}, \quad i = 1, 2, \cdots, N, \quad \sum_{i=1}^{N} N_i = n.$$

Theorem 3.3 Consider an interconnected system $\Sigma(\Sigma_1, \Sigma_2, \cdots, \Sigma_N)$ of (8), (9), (10), the global system Σ is structurally controllable if and only if subsystem $\Sigma_i \ \forall\, i = 1, 2, \cdots, N$ is structurally controllable relative to the control inputs and all state variables of $\Sigma_j (j = 1, 2, \cdots, i-1)$ that have direct influence on Σ_i, while the interconnection of g-cactus $C_{li}\ i = 1, 2, \cdots, N$ spanning the graphs of subsystems is a cascade relative to the control inputs.

In the following, we also consider the structural controllability of composite systems with identical units. Consider the following large-scale system Σ that may be described as an interconnection of an external system Σ_0 and N subsystems $\Sigma_0, \Sigma_1, \cdots, \Sigma_N (N > 1)$:

$$\Sigma: \dot{x} = Ax + Bu, \tag{11}$$

where, A and B are structured matrices with dimension $n \times n$ and $n \times m$ respectively,

$$A = \begin{bmatrix} A_0 & e_{01}H_{01} & e_{02}H_{02} & \cdots & e_{0N}H_{0N} \\ e_{10}H_{10} & A_1 & e_{12}H_{12} & \cdots & e_{1N}H_{1N} \\ e_{20}H_{20} & e_{21}H_{21} & A_2 & \cdots & e_{2N}H_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{N0}H_{N0} & e_{N1}H_{N1} & e_{N2}H_{N2} & \cdots & A_N \end{bmatrix}, \quad B = \mathrm{diag}[B_0 \quad B_1 \quad \cdots \quad B_N], \tag{12}$$

and subsystems described as:

$$\Sigma_i: \dot{x}_i = A_i x_i + B_i u_i + \sum_{\substack{j=0 \\ j \neq i}}^{N} e_{ij} H_{ij} x_j, \tag{13}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}, i = 0, 1, \cdots, N, \sum_{i=0}^{N} n_i = n, \sum_{i=0}^{N} m_i = m, n_1 = n_2 \cdots = n_N, m_1 = m_2 = \cdots = m_N$, and usually $e_{i0} = e_{0i} = 1$, and

$$e_{ij} = \begin{cases} 1, & (\Sigma_j \text{ acts on } \Sigma_i \text{ through } H_{ij}), \\ 0, & (\Sigma_j \text{ does not acts on } \Sigma_i). \end{cases} \quad i,j = 1,2,\cdots,N; i \neq j.$$

We shall hereby refer to the system described in (11),(12),(13) as a large-scale system with identical units. As for the structural controllability of such composite system, we may also present some sufficient conditions.

Theorem 3.4 System Σ as described in (11),(12),(13) is structurally controllable if subsystem Σ_i $\forall\, i=0,1,2,\cdots,N$ is structurally controllable relative to both the control inputs u_i and the state variables of $\Sigma_j(j=0,1,\cdots,i-1)$ that have direct influence on Σ_i while the interconnection of g-cactus C_{li} $i=0,1,\cdots,N$ spanning the graphs of subsystems is a cascade relative to the control inputs $u_0,u_1,\cdots,u_N$.

Theorem 3.5 Consider the large-scale system Σ as described in (11),(12),(13), it is supposed that all state nodes are covered by a group of disjoint cycles (e.g. Condition A in their paper of Yang Guanghong and Zhang Siying [5]), then the system Σ is structurally controllable if and only if each state node is reachable by a control input.

Example 1 The graph of a composite system with three subsystems is illustrated in Fig. 5. We are required to determine its structural controllability.

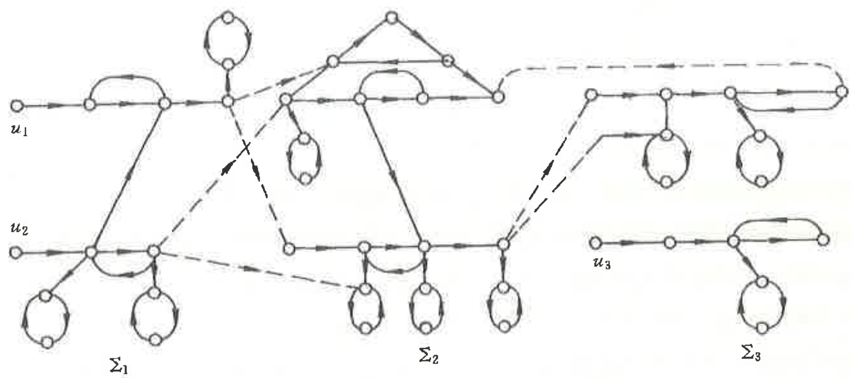

Fig. 5 An interconnected system

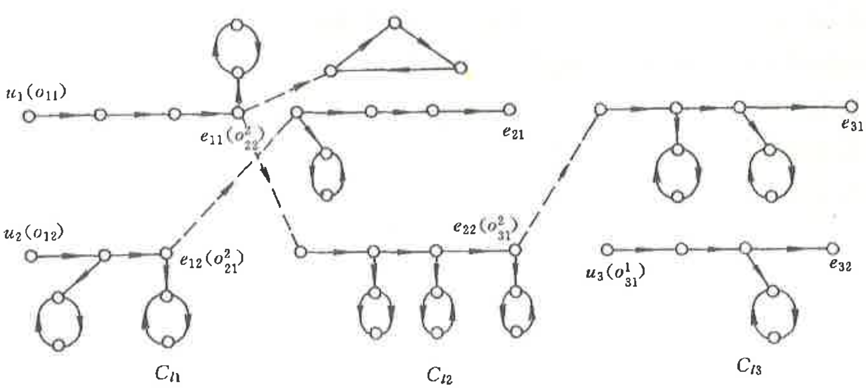

Fig. 6 Cascade connection of C_{l1}, C_{l2} and C_{l3}

Step1 A g-cactus C_{l1} spanning the graph of Σ_1 is found, then according to the terminal nodes of stems in g-cactus C_{l1} together with rest control nodes, g-cactus C_{l2} is found out for subsystem Σ_2, similarly, g-cactus C_{t3} is also found for subsystem Σ_3. The three g-cactus and their interconnection is illustrated in Fig. 6.

Step2 Apparently, the interconnection of these three g-cactus is a cascade relative to the control inputs, and the composite system is structurally controllable.

4 Conclusion

Several important results have been drawn with respect to the structural controllability of composite systems with different forms of interconnection of subsystems. The approach presented in this paper is quite suitable to the design of a composite system for which the properties of each unit are known, because it satisfies in a simple way both the accessibility and generic rank criterion. This approach is also quite helpful for the design of control systems of complex industrial processes.

Futhermore, the notions of "g-cactus" and "cascade connection of g-cactus" may deepen our understanding of the structural controllability theory, and they are also shown to be graphic tools which are really worth manipulating.

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类仙人掌及组合大系统结构能控的一些新结果

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摘要：本文讨论了组合大系统的结构能控性. 首先介绍了类仙人掌的图形概念，然后，构造了一种新的图形概念——类仙人掌的串级联结，进而给出了检验不同组合下大系统结构能控的充分条件. 获得的结果比原有的结果无论在操作上还是在表达上都有更大的便利性，同时它还加深了我们对结构能控概念的理解.

关键词：类仙人掌；类仙人掌的串级联结；结构能控；动态关联系统

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