Robust Stability of Systems with Any Unknown Constant Delay*

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Abstract: In this paper, some robust stability criteria for uncertain systems with any single unknown but constant delay are established by Lyapunov functional method together with a vector inequality. The obtained results are independent of delay. The illustrative examples show that the obtained criteria are less conservative than the existing ones in the literature.

Key word: robust stability; time-delay systems; independent of delay

1 Introduction

Over past thirty years, there has been a great amount of literature discussing stability of time-delay systems. The research approaches are either via frequency-domain or via time-domain[1~3]. The main time-domain methods are Lyapunov methods[4]. The recent results have involved Lyapunov functional method[5~9], Lyapunov-Razumikhin method[10~12] and vector Lyapunov function method[13~16].

In this paper, some robust stability criteria for uncertain systems with any single unknown but constant delay are presented. The obtained results are delay-independent. The illustrative examples show that the obtained criteria are less conservative than the existing ones in the literature[12, 17~19]. In the following, Section 2 establishes the main results, Section 3 provides some illustrative examples, and Section 4 is the conclusion.

Notation x^{T} and M^{T} denote the transpose of a vector $x \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, respectively. $\lambda_M(M)$ and $\lambda_m(M)$ denote the maximum and minimum eigenvalue of M, respectively. $\| x \|_1 = \sum_{i=1}^{n} |x_i|$, $\| x \|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$ and $\| x \|_\infty = \max_i |x_i|$. $\| M \|_1 = \max_j \left\{\sum_{i=1}^{n} |m_{ij}|\right\}$, $\| M \|_2 = [\lambda_M(M^{\mathrm{T}}M)]^{\frac{1}{2}}$ and $\| M \|_\infty = \max_i \left\{\sum_{j=1}^{n} |m_{ij}|\right\}$. $\mu_k(M)$ are the matrix measures derived from the matrix norms $\| M \|_k$ for $k = 1, 2, \infty$, i.e. $\mu_1(M) = \max_j \left\{\mathrm{Re}(m_{jj}) + \sum_{i=1, i\neq j}^{n} |m_{ij}|\right\}$, $\mu_2(M) = \lambda_M[(M + M^{\mathrm{T}})/2]$ and $\mu_\infty(M) = \max_i \left\{\mathrm{Re}(m_{ii}) + \sum_{j=1, j\neq i}^{n} |m_{ij}|\right\}$.

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2 Main Results

Consider a nonlinear time delay dynamic system with a linear delay-free term in the form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t,x(t-\tau)), & t \geqslant 0, \\ x(t) = \phi(t), & t \in [-r,0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n, A(t) \in \mathbb{R}^{n\times n}$ is continuous on $[0,\infty), 0 < \tau \leqslant r < \infty$ is any unknown but constant delay, $\phi(t)$ is an initial function, and F denotes the uncertainty satisfying

$$\| F(t,\xi) \|_2 \leqslant \beta \| \xi \|_2, \quad \forall\ (t,\xi) \in [0,\infty) \times \mathbb{R}^n, \tag{2}$$

and β is a positive number. We also consider two special cases of system (1) as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t,x(t-\tau)), & t \geqslant 0, \\ x(t) = \phi(t), & t \in [-r,0], \end{cases} \tag{3a}$$

and

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geqslant 0, \\ x(t) = \phi(t), & t \in [-r,0], \end{cases} \tag{3b}$$

where $A, B \in \mathbb{R}^{n\times n}$ are constant matrices.

Lemma 1[20] Assume that $A(t)$ is continuous and bounded on $[0,\infty)$ and $\dot{x}(t) = A(t)x(t)$ is exponentially stable. If $Q(t)$ is an $n \times n$ bounded symmetric matrix on $[0,\infty)$ and $\lambda I_n \leqslant Q(t)$ for all $t \in [0,\infty)$, where $\lambda > 0$ is a constant and I_n denote the $n \times n$ identity matrix, then the following Lyapunov differential equation

$$\dot{P}(t) + P(t)A(t) + A^{\mathrm{T}}(t)P(t) = -Q(t), \tag{4}$$

has an $n \times n$ symmetric matrix solution $P(t)$ on $[0,\infty)$ and $P(t)$ satisfies $\eta_1 I_n \leqslant P(t) \leqslant \eta_2 I_n$ for all $t \in [0,\infty)$, where $\eta_1 > 0$ and $\eta_2 > 0$ are constants.

Theorem 1 Assume that the assumptions of Lemma 1 hold and λ and η_2 are definde as in Lemma 1. System (1) is asymptotically stable if

$$\beta < \frac{\lambda}{2\eta_2}. \tag{5}$$

Proof Let

$$V(t,x(t)) = x^{\mathrm{T}}(t)P(t)x(t) + \varepsilon\int_{t-\tau}^{t} x^{\mathrm{T}}(s)x(s)\mathrm{d}s, \tag{6}$$

where $P(t)$ is the solution of equation (4) and $\varepsilon > 0$ is a constant. Along the trajectory of system (1), we obtain

$$\begin{aligned} \dot{V}(t,x(t)) = & x^{\mathrm{T}}(t)(\dot{P}(t) + P(t)A(t) + A^{\mathrm{T}}(t)P(t))x(t) \\ & + 2x^{\mathrm{T}}(t)P(t)F(t,x(t-\tau)) + \varepsilon[x^{\mathrm{T}}(t)x(t) - x^{\mathrm{T}}(t-\tau)x(t-\tau)] \\ \leqslant & -\lambda \| x(t) \|_2^2 + 2\eta_2\beta \| x(t) \|_2 \| x(t-\tau) \|_2 + \varepsilon(\| x(t) \|_2^2 - \| x(t-\tau) \|_2^2). \end{aligned} \tag{7}$$

It is easy to show that

$$2u^{\mathrm{T}}v \leqslant \frac{1}{\varepsilon}u^{\mathrm{T}}u + \varepsilon v^{\mathrm{T}}v, \quad u,v \in \mathbb{R}^m, \tag{8}$$

holds for any constant $\varepsilon > 0$. Let $\varepsilon = \eta_2\beta$. By (8), we further obtain

$$\begin{aligned} \dot{V}(t,x(t)) & \leqslant -\lambda \| x(t) \|_2^2 + \frac{1}{\varepsilon}\eta_2^2\beta^2 \| x(t) \|_2^2 + \varepsilon \| x(t) \|_2^2 \\ & \leqslant -(\lambda - 2\eta_2\beta) \| x(t) \|_2^2 \end{aligned}$$

$$= -\rho \| x(t) \|_2^2. \tag{9}$$

If (5) holds, then $\rho = \lambda - 2\eta_2\beta > 0$ and $\dot{V}(t,x(t)) < 0$. This gives the proof. Q. E. D.

For system (3), assume that A is stable so that the following Lyapunov algebraic equation

$$PA + A^{\mathrm{T}}P = -Q, \tag{10}$$

has the unique symmetric and positive definite solution $P \in \mathbb{R}^{n\times n}$, where $Q \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix. We have the following corollary.

Corollary 1 Assume that A is stable. System (3a) and (3b) are asymptotically stable if

$$\beta < \frac{\lambda_m(Q)}{2\lambda_M(P)}, \tag{11a}$$

and

$$\| PB \|_2 < \frac{\lambda_m(Q)}{2}, \tag{11b}$$

respectively.

Proof Immediate from Theorem 1. Q. E. D.

Remark 1 For system (1) and (3), using Razumikhin-type theorem as given in [12] yields the following stability conditions

$$\beta < \frac{\lambda}{2\eta_2}\left(\frac{\eta_1}{\eta_2}\right)^{1/2}, \tag{12}$$

$$\beta < \frac{\lambda_m(Q)}{2\lambda_M(P)}\left(\frac{\lambda_m(P)}{\lambda_M(P)}\right)^{1/2}, \tag{13a}$$

and

$$\| PB \|_2 < \frac{\lambda_m(Q)}{2}\left(\frac{\lambda_m(P)}{\lambda_M(P)}\right)^{1/2}, \tag{13b}$$

respectively. Obviously, the results given in Theorem 1 and Corollary 1 are better than those obtained by the method of [12] since $(\eta_1/\eta_2)^{1/2} \leqslant 1$ and $(\lambda_m(P)/\lambda_M(P))^{1/2} \leqslant 1$.

Remark 2 It sould be pointed out that the results given in [12] can include the time-varying delay case. But the results given here show that those results are conservative when they are used for testing the systems with any single unknown but constant delay.

Remark 3 In [17], the following criteria are given for system (3b):

$$\| B \|_k < -\mu_k(A). \tag{14}$$

This implies that only the class of systems which satisfy $\mu_k(A) < 0$ can be considered.

Theorem 2 System (3a) and (3b) are asymptotically stable if

$$\beta < \lambda_m^{1/2}(\gamma P^{-1}QP^{-1} - \gamma^2 P^{-2}), \tag{15a}$$

and

$$\| B \|_2 < \lambda_m^{1/2}(\gamma P^{-1}QP^{-1} - \gamma^2 P^{-2}), \tag{15b}$$

respectively, where $0 < \gamma < \lambda_m(Q)$.

Proof We only give the proof of (15a) in the following, but the condition (15b) can be obtained in the same way. Let

$$V(t,x(t)) = x^{\mathrm{T}}(t)Px(t) + \gamma\int_{t-\tau}^{t} x^{\mathrm{T}}(s)x(x)ds, \tag{16}$$

where P is the solution of equation (10) and $\gamma > 0$ is a constant. Along the trajectory of system (3a) and by inequality (8), we obtain

$$\begin{aligned}\dot{V}(t,x(t)) &\leqslant 2x^{\mathrm{T}}(t)PAx(t) + 2\beta\,\|Px(t)\|_2\,\|x(t-\tau)\|_2 \\ &\quad + \gamma(\|x(t)\|_2^2 - \|x(t-\tau)\|_2^2) \\ &\leqslant -x^{\mathrm{T}}(t)Qx(t) + \frac{\beta^2}{\gamma}\|Px(t)\|_2^2 + \gamma\|x(t-\tau)\|_2^2 \\ &\quad + \gamma(\|x(t)\|_2^2 - \|x(t-\tau)\|_2^2) \\ &= -x^{\mathrm{T}}(t)P(P^{-1}QP^{-1} - \gamma P^{-2} - \frac{\beta^2}{\gamma}I_n)Px(t) \\ &\leqslant -\frac{\lambda_m^2(P)}{\gamma}[\lambda_m(\gamma P^{-1}QP^{-1} - \gamma^2P^{-2}) - \beta^2]\,\|x(t)\|_2^2. \end{aligned} \tag{17}$$

It is not difficult to show that $\lambda_m(\gamma P^{-1}QP^{-1} - \gamma^2P^{-2}) > 0$ for $0 < \gamma < \lambda_m(Q)$, i.e.

$$\begin{aligned}\lambda_m(\gamma P^{-1}QP^{-1} - \gamma^2P^{-2}) &= \lambda_m[P^{-1}Q^{1/2}(\gamma I_n - \gamma^2Q^{-1})Q^{1/2}P^{-1}] \\ &= \lambda_m[Q^{1/2}P^{-2}Q^{1/2}(\gamma I_n - \gamma^2Q^{-1})] \\ &\geqslant \lambda_m(Q^{1/2}P^{-2}Q^{1/2})\lambda_m(\gamma I_n - \gamma^2Q^{-1}) \\ &= \lambda_m(Q^{1/2}P^{-2}Q^{1/2})(\gamma - \gamma^2\lambda_M(Q^{-1})) \\ &= \lambda_m(Q^{1/2}P^{-2}Q^{1/2})(\gamma - \gamma^2/\lambda_m(Q)) \\ &> 0. \quad [0 < \gamma < \lambda_m(Q)] \end{aligned} \tag{18}$$

If (15a) holds, then we have

$$\dot{V}(x(t)) \leqslant -\mu\,\|x(t)\|_2^2, \quad \mu > 0, \tag{19}$$

where

$$\mu = \frac{\lambda_m^2(P)}{\gamma}[\lambda_m(\gamma P^{-1}QP^{-1} - \gamma^2P^{-2}) - \beta^2]. \tag{20}$$

We complete the proof. Q. E. D.

Remark 4 As we have seen, a key technique used in the proof of Theorem 1, 2 and Corollary 1 is that two terms with the unknown constant delays cancel out each other. This is different from the existing other methods such as that in [18].

Remark 5 By appropriately choosing γ and/or Q, we can derive some less conservative upper bounds for β or $\|B\|_2$ from (15).

3 Illustrative Examples

Example 1 Consider system (1) with $F(t,x(t-\tau))$ satisfying (2) and

$$A(t) = \begin{bmatrix} -t-3 & \mathrm{e}^t \\ -\mathrm{e}^t & -t-4 \end{bmatrix}.$$

Choosing

$$Q(t) = \begin{bmatrix} 2t+6 & 0 \\ 0 & 2t+8 \end{bmatrix},$$

and solving (4) yield $P(t) = I_2, \eta_1 = \eta_2 = 1$ and $\lambda = 6$. By (5), we have $\beta < 3$.

Example 2 Consider system (3a) with a constant matrix

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}.$$

The obtained bounds for asymptotic stability of the system are

$\beta < 0.1459$ [by(13a)with $Q = I_2$],

$\beta < 0.3820$ [by(11a)with $Q = I_2$],

$\beta < 0.5400$ [by(15a) with $Q = \begin{bmatrix} 4.6 & 2.7 \\ 2.7 & 10.2 \end{bmatrix}$ and $\gamma = 3.1 < \lambda_m(Q) = 3.5103$].

The upper bound of β is improved up to about 270.11%. For this example, a recent result given in [19] is $\beta < 0.5245$. Besides, $\mu_1(A) = 2$, $\mu_2(A) = 0.0811$, and $\mu_\infty(A) = 1$. this implies the method given in [17] is useless.

4 Conclusion

Some robust stability criteria for uncertain systems with any unknown but constant delay are derived by using Lyapunov functional method and a vector inequality. The established results are independent of delay. The illustrative examples show that the obtained stability criteria are less conservative than the existing ones in the literature. It is not difficult to see that the technique used in this paper can also be used to deal with systems with multiple unknown constant delays.

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具有任意未知常时滞系统的鲁棒稳定性

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摘要：本文采用李雅普诺夫泛函法和一个矢量不等式建立了若干具有任意未知常时滞系统的鲁棒稳定性判据. 所获得结果是时滞无关的、文中示例说明了所得稳定性判据减少了现存结果的保守性.

关键词：鲁棒稳定性；时滞系统；时滞无关

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