Adaptive Terminal Sliding Mode Control of Uncertain Nonlinear Systems——Backstepping Approach*

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Abstract: An adaptive finite time sliding mode control design scheme for a family of uncertain nonlinear systems with parametric uncertainties and unknown nonlinearities is presented by introducing slow and fast switching lines. The backstepping approach is used in the adaptive sliding mode control scheme. The global stability is guaranteed under the scheme developed and the system state reaches the origin in finite time. Simulation results are presented to show the effectiveness of the scheme.

Key words: adaptive control; variable structure control; terminal sliding mode

1 Introduction

Adaptive control of nonlinear systems in parametric-strict-feedback form can be solved by employing the well known backstepping approach[1~4]. A common assumption in the backstepping procedure is that all the nonlinearities are known. This assumption can be relaxed for the systems in which the nonlinear uncertain dynamics satisfy the triangular bounds condition[5].

Sliding mode control systems have been studied extensively and used in many applications. Recently, the sliding mode control using the backstepping approach has been dealt with for the uncertain linearizable nonlinear systems[6,7]. This combination enables generalization of the backstepping approach to more general nonlinear systems. Traditional switching manifolds are usually linear hyperplanes which guarantee the asymptotic stability. And the speed of convergence is slow in the small neighbourhood of the origin. Advanced industrial applications sometimes require to realise the accurate and fast tracking in finite time, for example, a position and its velocity of a robotic manipulator is required to reach the target in finite time.

In this paper, we develop an adaptive finite time terminal sliding mode control design using the backstepping approach for the nonlinear uncertain systems. The second order uncertain nonlinear systems will be used to inform the discussion. Two switching lines, the asymptotic switching line (ASL) and the terminal switching line (TSL), are used in order to give better convergence performance. The terminal sliding mode control proposed in [8] which exhibits finite time convergence is realised. This is in contrast to the conventional linear hyperplane switching manifolds which guarantee asymptotic convergence.

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2 Control Design for Second Order System

We firstly discuss the finite time control design for the second order nonlinear systems with a specific structure. This design will be used in the successive sections.

Consider the following second order system

$$\begin{cases} \dot{x}_1 = x_2 + \theta_1^{\mathrm{T}}(t)\phi_1(x_1), \\ \dot{x}_2 = -a_1x_1 - a_2x_2 + u, \\ \dot{\theta}_1(t) = -x_1\phi_1(x_1) \end{cases} \tag{1}$$

where $\phi_1(x_1)$, a_1 and a_2 are known, $\phi_1(x_1)$ is continuously differentiable with respect to x_1 and satisfies $\| \phi_1(x_1) \| = O(|x_1|^{\alpha})$, $\phi_1(0) = 0$, $\alpha > 0$.

The control target is to achieve finite time convergence. The terminal switching line (TSL) can be used for this purpose[8]. From Fig. 1, one can see that the convergence speed near the origin is greatly improved in comparison to the asymptotic (linear) switching line (ASL). However, the convergence speed in the TSL is slower when far away from the origin. This inspires us to combine these two switching lines together to achieve fastest convergence by taking advantage of their different convergence performances.

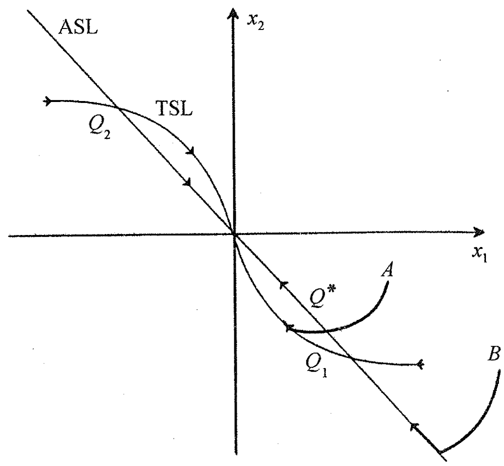

Fig. 1 Phase plane portrait

For the system (1), we denote the ASL as

$$s_1 = x_2 + \lambda_1 x_1, \tag{2}$$

and the TSL as

$$s_2 = x_2 + \lambda_2 x_1^{q/p}, \tag{3}$$

where $\lambda_1, \lambda_2 > 0$, p and q are odd positive integers and are assumed to satisfy

$$q < p < 2q, \quad \alpha > (q/p).$$

For the system (1) to reach the ASL, the control law is taken as

$$u(t) = -\lambda_1 x_2(t) - \theta_1^{\mathrm{T}}(t)\phi_1(x_1(t)) + a_1x_1(t) + a_2x_2(t) - K\mathrm{sgn}(s_1), \quad K > 0, \tag{4}$$

such that the condition

$$s_1\dot{s}_1 = -K|s_1| < 0 \tag{5}$$

is satisfied. This shows that the trajectory of the system (1) will reach the TSL $s_1 = 0$ in finite time. In the sliding mode $s_1 = 0$, it follows from (1) and (2) that

$$\begin{cases} \dot{x}_1 = -\lambda_1 x_1 + \theta_1^{\mathrm{T}}(t)\phi_1(x_1), \\ \dot{\theta}_1(t) = -x_1\phi_1(x_1). \end{cases} \tag{6}$$

Take a Lyapunov function as

$$V(x_1, \theta_1) = \frac{1}{2}x_1^2(t) + \frac{1}{2}\theta_1^{\mathrm{T}}(t)\theta_1(t). \tag{7}$$

The time derivative of V along (6) is $\dot{V} = -\lambda_1 x_1^2(t) \leqslant 0$, that is

$$\int_0^t \lambda_1 x_1^2(\tau)\mathrm{d}\tau + V(t) \leqslant V(0), \tag{8}$$

which means that $x_1(t)$ and $\theta_1(t)$ are bounded and $x_1(t)$ will move toward the origin along the ASL $s_1 = 0$.

We then design the controller to reach the TSL. As in the TSL $s_2 = 0$, we have

$$\dot{s}_2 = \dot{x}_2 + \frac{q}{p}\lambda_2 x_1^{(q-p)/p}\dot{x}_1 = -a_1x_1 - a_2x_2 + u + \frac{q}{p}\lambda_2 x_1^{(q-p)/p}(x_2 + \theta_1(t)\phi_1(x_1)). \tag{9}$$

We select the controller

$$u(t) = a_1x_1 + a_2x_2 - \frac{q}{p}\lambda_2 x_1^{(q-p)/p}(x_2 + \theta_1(t)\phi_1(x_1)) - K_1\mathrm{sgn}(s_2), \tag{10}$$

such that

$$s_2\dot{s}_2 = -K_1|s_2|, \tag{11}$$

which shows that once $x(t)$ reaches $s_2 = 0$, it will stay at zero forever.

In the TSL $s_2 = 0$, from the system (1), it follows

$$\dot{x}_1 = -\lambda_2 x_1^{q/p} + \theta_1^{\mathrm{T}}(t)\phi_1(x_1). \tag{12}$$

Now we illustrate how $x_1(t)$ reaches zero in finite time. Taking the Lyapunov function as

$$V(x_1,\theta_1) = \frac{1}{2}x_1^2 + \frac{1}{2}\theta_1^{\mathrm{T}}\theta_1, \tag{13}$$

the time derivative of $V(x_1,\theta_1)$ along (1) and (12) is

$$\dot{V} = -\lambda_2 x_1^{(q+p)/p} \leqslant 0. \tag{14}$$

This implies that $x_1(t)$, $\theta_1(t)$ are bounded and $x_1(t)$ tends to zero. Since $(q/p) < \alpha$, from the assumption condition of $\phi_1(x_1)$, there exists a positive number M such that

$$|x_1(t)\theta_1^{\mathrm{T}}(t)\phi_1(x_1(t))| \leqslant M|x_1(t)|^{1+\alpha} \leqslant M|x_1(t)|^{\alpha-p/q}|x_1(t)|^{(p+q)/p}.$$

This implies that

$$\begin{aligned} x_1(t)\dot{x}_1(t) &= x_1(t)\theta_1^{\mathrm{T}}(t)\phi(x_1(t)) - \lambda_2 x_1^{(q+p)/p}(t) \\ &\leqslant M|x_1(t)|^{1+\alpha} - \lambda_2 x_1^{(q+p)/p}(t) \leqslant -M^* x_1^{(q+p)/p}, \end{aligned} \tag{15}$$

where $0 < M^* < \lambda_2 - M|x_1(t)|^{\alpha-p/q}$. Here the choice of M^* is easy due to $\lambda_2 > 0$ and $x_1(t)$ thending to zero. Because $(q+p)/p < 2$, from (15), it is easily proved that $x_1(t)$ will approach zero in finite time along $s_2 = 0$[8]. This, together with $x_2 = -\lambda_2 x_1^{q/p}$ and $\phi_1(0) = 0$ when $s_2 = 0$, implies that $x_2(t) = 0$ if $x_1(t) = 0$. Hence we conclude that along $s_2 = 0$ the state $x(t)$ will reach the origin in finite time.

The next question is how to combine these two controllers together to give rise to a better finite time controller. There is a singularity problem that has to be taken into consideration, that is, the TSL controller has the term $x_1^{(q-p)/p}x_2$ which, when $x_1 = 0$ but $x_2 \neq 0$, will tend to infinity. To avoid this, the ASL should be used to escape from the neighborhood of $x_1 = 0, x_2 \neq 0$. The sketch of the TSL and ASL is shown in Fig. 1 where for any fixed $\theta_1(t)$, the ASL $s_1 = 0$ and the TSL $s_2 = 0$ will intersect at two points Q_1 and Q_2. The switch scheme is as follows.

For an initial state that is far away from Q_1, Q_2, choose the ASL controller such that the state $x(t)$ reaches $s_1 = 0$ first then moves along $s_1 = 0$. There exists a moment t_2 such that $x(t_2)$ arrives at Q_1 (or Q_2). That is the state $x(t)$ reaches the TSL $s_2 = 0$ in finite time. Once $x(t)$ arrives at Q_1 (or Q_2), we switch to the TSL controller such that $x(t)$ will reach the origin along $s_2 = 0$ in finite time. For an initial state which is close to the origin, say, the state $x(t)$ under the control law (4) hits the point Q^* on $s_1 = 0$ which is located on the line between Q_1 and Q_2, a simple strategy is to let $x(t)$ reach Q^*, and then switch to the TSL controller such that $x(t)$ from $s_1 = 0$ will reach $s_2 = 0$ in finite time. Hence it will reach the origin along $s_2 = 0$ in finite time. Observe that in the motion of $x(t)$ from $s_1 = 0$ to $s_2 = 0$ started from Q^*, the state $x(t)$ does not cross the x_2- axis because on the x_2- axis when $x_2 > 0$ then $\dot{x}_1 > 0$ (or when $x_2 < 0$ then $\dot{x}_1 < 0$).

Above analysis is summarized into the following theorem.

Theorem 1 For the system (1), if the switching lines are chosen as in (2) (ASL) and (3) (TSL), and the controllers are designed as in (4) and (10), then the state $x(t)$ will reach the origin in finite time.

3 Control Design for Uncertain Systems

The family of the uncertain second order nonlinear dynamic systems is given by

$$\dot{x}_1 = x_2 + \theta_1^{\mathrm{T}}\phi_1(x_1), \quad \dot{x}_2 = \theta_2^{\mathrm{T}}\phi_2(x_1, x_2) + \Delta(x,t) + \beta(x)u, \tag{16}$$

where $\theta_i (i = 1, 2)$ is unknown, but $\phi_1(x_1), \phi_2(x_1, x_2), \beta(x)$ are known and $|\beta(x)| \geqslant \beta_0 > 0$. $\Delta(x,t)$ and $\phi(x)$ satisfy

$$|\Delta(x,t)| \leqslant h(x), \quad |\phi_1(x_1) = O(|x_1|^{\alpha}),$$

where upper bound function $h(x)$ is known and α is as given in Section 2.

For system (16), the backstepping procedure is defined as

$$\begin{cases} z_1 = x_1, \quad z_2 = x_2 - \alpha_1(x_1, \theta_1), \\ \alpha_1(x_1, \theta_1) = y - \theta_1^{\mathrm{T}}(t)\phi_1(x_1), \quad \dot{\theta}_1(t) = z_1(t)\phi_1(x_1(t)), \end{cases} \tag{17}$$

where $\tilde{\theta}_1(t) = \theta_1(t) - \theta_1, \theta_1, p, q$ and α are defined as in section 2, $y(t)$ is generated by

$$\dot{y} = -a_1 z_1 - a_2 y + v(t), \tag{18}$$

where $v(t)$ will be chosen according to Theorem 1 so that on $z_2 = 0$ the states $z_1(t), y(t)$ satisfy

$$\dot{z}_1 = y - \tilde{\theta}_1^{\mathrm{T}}(t)\phi(z_1), \quad \dot{y} = -a_1 z_1 - a_2 y + v(t), \tag{19}$$

and reach the origin in finite time. From Theorem 1, such a $v(t)$ can be implemented.

From the transformation (17), the system (16) can be transformed into the following form

$$\begin{cases} \dot{z}_1 = z_2 + y(t) - \tilde{\theta}_1^{\mathrm{T}}(t)\phi_1(z_1), \\ \dot{z}_2 = \theta_2^{\mathrm{T}}\phi_2(x_1, x_2) + \Delta(x,t) + \beta(x)u - \dot{y}(t) \\ \qquad + [\theta_1^{\mathrm{T}}(t)\dfrac{\partial\phi_1}{\partial z_1}](z_2(t) + y + \tilde{\theta}_1^{\mathrm{T}}\phi_1(z_1)) + z_1\phi_1^{\mathrm{T}}(z_1)\phi_1(z_1). \end{cases} \tag{20}$$

The adjusting law for unknown parameter θ_2 is given by

$$\dot{\theta}_2(t) = z_2\phi_2(x_1, x_2). \tag{21}$$

The sliding mode control $u(t)$ for system (20) is defined as

$$\beta(x)u(t) = -\theta_2^{\mathrm{T}}(t)\phi_2(x_1, x_2) - \operatorname{sgn}(z_2)[h(x) + 1] + \dot{y}(t)$$
$$- [\theta_1^{\mathrm{T}}(t)\frac{\partial\phi_1}{\partial z_1}](z_2(t) + y(t)) - z_1\phi_1^{\mathrm{T}}(z_1)\phi(z_1)$$

$$-\operatorname{sgn}(z_2)K_1 \| \phi_2(x_1,x_2) \| - K_2\operatorname{sgn}(z_2) \| \theta_1^{\mathrm{T}}(t) \frac{\partial \phi_1}{\partial z_1}\phi_1(z_1) \| , \tag{22}$$

where K_1, K_2 are certain positive constants, and $\dot{y}(t)$ is given by (19). The performance of the controller is analysed as follows: By an appropriate choice of K_1 and K_2 and using same manipulation as done in Section 2, it is easy to see from (21) that

$$z_2\dot{z}_2 \leqslant -|z_2|. \tag{23}$$

Hence, $z_2(t)$ will reach zero in finite time.

On $z_2 = 0$, from (19) and (20) we obtain

$$\begin{cases} \dot{z}_1 = y(t) - \tilde{\theta}_1^{\mathrm{T}}(t)\phi_1(z_1), \\ \dot{y} = -a_1z_1 - a_2y + v(t), \end{cases} \tag{24}$$

which is similar to (1). By means of Theorem 1, we can design $v(t)$ such that $y(t), z_1(t)$ reach zero in finite time along the TSL $y + z_1^{q/p} = 0$.

By the definitions of $z_1(t), z_2(t)$ in the backstepping approach (17), it is easily seen that once $z_1(t), z_2(t)$ reach zero, $x_1(t), x_2(t)$ will approach zero in finite time as well. Therefore all the signals of the closed-loop system are bounded.

The above analysis is summarized into the following theorem.

Theorem 2 For the nonlinear uncertain dynamic system (16) in the parametric-strict-feedback form, the sliding mode control law (22) enables the closed-loop system stable and all the states reach the origin in finite time.

Note that the family of second order uncertain systems (16) represents a very broad class of second order uncertain systems. In practice, as shown in [4], vehical active suspension is an example. By using the presented method, we can make the fluid flow to reach zero in finite time. Another example is the dynamics of the n- joint robotic manipulators discribed by the following equation

$$H(q)\ddot{q} + C(q,\dot{q}) + G(q) = u + \sigma(t), \tag{25}$$

where $q(t)$ is a $n \times 1$ vector of joint angular positions as the system output; $H(q)$ is an $n \times n$ symmetric positive definite inertia matrix; $C(q,\dot{q})$ is the $n \times 1$ vector of Coriolis and centrifugal forces; $G(q)$ is the $n \times 1$ vector of gravitational torques and $u(t)$ is the control torques; $\sigma(t)$ is the input disturbances. We assume that the robot has uncertainties, such that

$$H(q) = H_0(q) + \Delta H(q),\ C(q,\dot{q}) = C_0(q,\dot{q}) + \Delta C(q,\dot{q}),\ G(q) = G_0(q) + \Delta G(q), \tag{26}$$

where $H_0(q), C_0(q,\dot{q})$, and $G_0(q)$ are known; and $H_0(q)$ is a symmetric positive definite matrix. Denote

$$\rho(t) = \sigma(t) - \Delta H(q)\ddot{q} - \Delta C(q,\dot{q}) - \Delta G(q). \tag{27}$$

Substituting (26) and (27) into (25) yields

$$H_0(q)\ddot{q} + C_0(q,\dot{q}) + G_0(q) = u + \rho(t). \tag{28}$$

In practice, usually $H(q)$ is invertible and bounded by positive constant, that is, there exist a positive constant β such that $|H(q)| \leqslant \beta$. And

$$\| C(q,\dot{q}) \| \leqslant \alpha_1 + \alpha_2 \| q \| + \alpha_3 \| q \|^2, \quad \| G(q) \| \leqslant \beta_1 + \beta_2 \| q \| .$$

Hence it can be proved[9] that

$$|\rho(t)| \leqslant b_1 + b_2 \| q \| + b_3 \| q \|^2. \tag{29}$$

Let $H_0^{-1}(q) = B(q)$, and $B(q)u = v$, then from (28) we have

$$\ddot{q} = - B(q)C_0(q,\dot{q}) - B(q)G_0(q) + v + B(q)\rho(t). \tag{30}$$

For the ith joint, the dynamical equation is

$$\ddot{q}_i = - \sum_{j=1}^{n} B_{ij}(q)C_{0j}(q,\dot{q}) - \sum_{j=1}^{n} B_{ij}(q)G_{0j}(q) - \sum_{j=1}^{n} B_{ij}(q)\rho_j(t) + v_j. \tag{31}$$

Let $x_1 = q, x_2 = \dot{q}$. Since position q and velocity $\dot{q}$ are measurable, equation (31) can be transformed into a state observable system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = - \sum_{j=1}^{n} B_{ij}(x_1)C_{0j}(x_1,x_2) - \sum_{j=1}^{n} B_{ij}(x_1)G_{0j}(x_1) - \sum_{j=1}^{n} B_{ij}(x_1)\rho_j(t) + v_i. \end{cases} \tag{32}$$

The control objective is to make $x(t)$ track accurately the bounded reference signal $y_1(t)$, $y_2(t)$ in finite time. Define error function $e_{i1} = - x_{11} - y_{11}, e_{i2} = x_{21} - y_{21}$, where x_{ij}, y_{ij} is the j th components of x_i, y_i respectively, then

$$\begin{cases} \dot{e}_{i1} = e_{i2}, \\ \dot{e}_{i2} = - \sum_{j=1}^{n} B_{ij}(x_1)C_{0j}(x_1,x_2) - \sum_{j=1}^{n} B_{i1}(x_1)G_{0j}(x_1) - \sum_{j=1}^{n} B_{ij}(x_1)\rho_j(t) - \dot{y}_{21}(t) + v_i. \end{cases} \tag{33}$$

This system is a special case of system (16) ($\phi_1(\cdot) = 0$), so we can design a variable structure adaptive control law v_i as done in Theorems 1 and 2 as well as $u(t) = B(q)v(t)$ to guarantee that $e_{i1}(t), e_{i2}(t)$ reach zero in finite time, that is the completely tracking can be implemented in finite time, which means as the position reaches target, the velo city vector tends to zero. Here along the switching line, both the errors of the position and velocity reach zero in finite time.

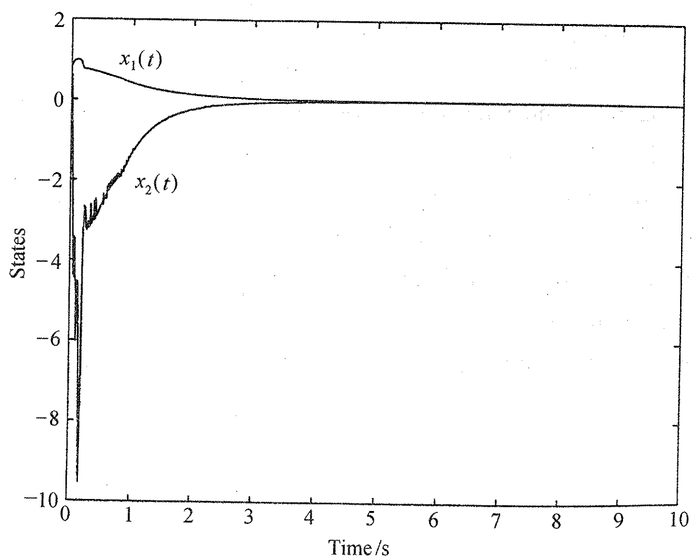

Fig. 2 Time responses

4 Simulation Results

The following system was considered in the simulation

$$\begin{cases} \dot{x}_1(t) = x_2(t) + \theta_1 x_1^2(t), \\ \dot{x}_2(t) = - x_2 + \theta_2(x_1(t)x_2(t) + x_2^2(t)) + \exp(x_1)\sin(x_1x_2)(x_1(t) + x_2(t)) + u(t). \end{cases} \tag{34}$$

The real values of θ_1 and θ_2 were set to be $\theta_1 = 5, \theta_2 = 3$. The setting for the controller was $p = 5, q = 3, a_1 = 1, a_2 = 1, K_1 = 8, K_2 = 5$.

Fig. 2 shows that the system states $x_1(t)$, $x_2(t)$, are bounded and reach to zero. Fig. 3 depicts the bounded control law $u(t)$. Fig. 4 illustrates the adaptation of adaptive parameters $\theta_1(t)$ and $\theta_2(t)$.

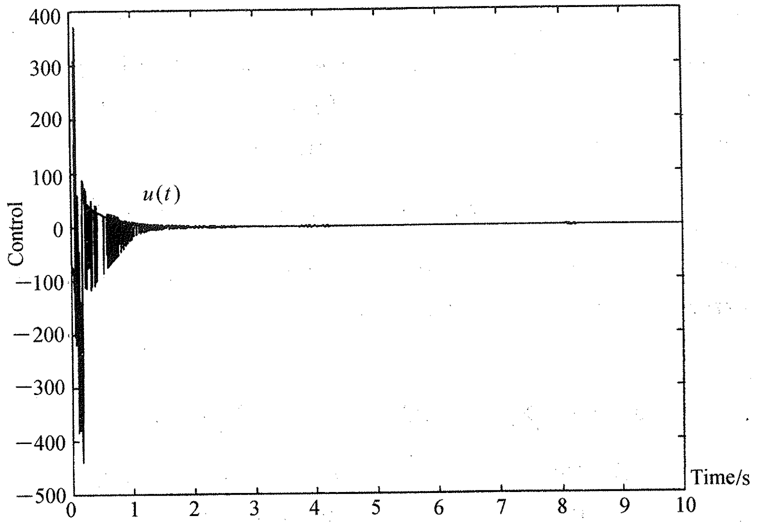

Fig. 3 Control law $u(t)$

5 Conclusion

In this paper, an adaptive sliding mode control scheme has been proposed for the second order uncertain nonlinear systems with parametric and dynamic uncertainties. The ASL and TSL have been used to give rise to a fastest convergence performance. Simulation results have been presented to show the effectiveness of the approach.

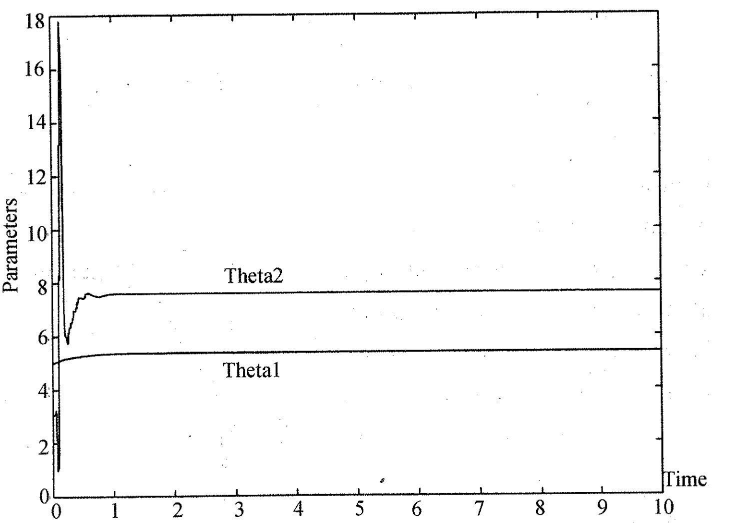

Fig. 4 Parameter estimation

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不确定非线性系统的自适应最终滑模控制——Backstepping 方法

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摘要：对具有参数不确定与未知非线性的一类非线性系统，本文通过引入快慢两种切换线给出了一种自适应有限时间滑模控制机制. Backstepping 方法被应用到设计中. 此种控制机制保证了闭环系统的稳定性并使状态在有限时间内收敛到原点. 仿真结果表明该控制机制的有效性.

关键词：自适应控制；变结构控制；最终滑动模态

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