# Model Reference Variable Structure Control for Non-Holonomic Mechanical Control Sysytems *

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**Abstract:** A variable structure model reference tracking controller for non-holonomic mechanical systems is designed under parametric perturbation and external disturbance. A design procedure with three-stage control is developed to solve the tracking problem for perturbed non-holonomic mechanical control systems based on a proper matrix decomposition, the concept of input/output decoupling in nonlinear control theory, and the theory of variable structure control. Finally, a typical non-holonomic example —— a vertical wheel moving on a given plane —— is given with computer simulation to illustrate the significant advantage of the proposed method.

**Key words:** non-holonomic mechanical system; decoupling; tracking; variable stucture control

# 非完整机械控制系统的模型参考变结构控制

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**摘要:** 在参数扰动和外部干扰情况下,对非完整机械控制系统设计了变结构模型参考跟踪控制器.基于适当的矩阵分解、非线性控制理论中的输入-输出解耦概念及变结构控制理论,为解决干扰非完整机械控制系统的跟踪问题提出了三级控制设计过程.最后通过一非完整机械控制系统的例子(在给定平面上运动的垂直轮)的计算机仿真说明了提出方法的优越性.

**关键词:** 非完整机械系统; 解耦; 跟踪; 变结构控制

## 1 Introduction

Problems associated with the tracking, stabilization and stability of control systems with non-holonomic constraints (i.e, control systems with non-integrable constraints) have become increasingly important during the past decade and have consequently received the attention of many investigators. This system has extensive practical background, such as mobile robots, automatic pilot vehicles, a knife edge moving in point contact on a plane surface, a vertical wheel rolling without slipping on a plane surface. Bloch et al[2] treats a special system, the Caplygin non-holonomic mechanical system (i.e., a non-holonomic system with certain symmetry properties). A so-called normal form is introduced to model the dynamics completely. Based on the normal form, a feedback control strategy to perform input/output decoupling is proposed by You and Chen[3]. McClamroch and Wang[1] considered the holonomic case. In these designs, the dynamic models are assumed to be perfect, exactly known, and free of external disturbances. However, in many practical situations, the system parameters may be perturbed due to unmodelled dynamics, model trunca-

* This work is supported by National Science Foundation (69874010) and Postdoctoral Foundation of China.
Manuscript received Sep.25, 1996, revised Oct.20, 1997.

tion, uncertainty in friction force and load, flexibility of the mechanical body and deformation of the constrained surface, etc. In addition, external disturbance due to variations of environment and load may also occur. Hence, a control design for control systems to achieve the purpose of robust tracking under parametric perturbation and external disturbance is important for practical application. This paper considers the problem.

## 2 Main results

Consider the following non-holonomic mechanical system with parametric perturbation and external disturbance

$$M(q)q^{(2)} + C(q,\dot{q})\dot{q} + G(q) = B(q)\tau + J^{\mathrm{T}}(q)\lambda + d_1, \tag{1}$$

$$J(q)\dot{q} = d_2 \tag{2}$$

where the variable $q \in \mathbb{R}^n$ denotes the generalized coordinate and determines the geometric configuration of the mechanical system. The time derivative $\dot{q} \in \mathbb{R}^n$ is the generalized velocity. $M: \mathbb{R}^n \to \mathbb{R}^{n\times n}$ represents the generalized moment of inertia, assumed to be a symmetric and positive definite matrix. $\tau \in \mathbb{R}^r$ is the control input. $B: \mathbb{R}^n \to \mathbb{R}^{n\times r}$ denotes the input matrix with full column rank, and $B(q)\tau \in \mathbb{R}^n$ is known as a nonconservative generalized force along the direction of its corresponding generalized coordinate $q$. $G(q)$ denotes the gravitational forces and the term $C(q,\dot{q})\dot{q}$ includes centrifugal and Coriolis forces (even the frictional forces), $J: \mathbb{R}^n \to \mathbb{R}^{m\times n}$ determines and models kinematic constraints $J(q)\dot{q} = 0$ in (2), and $\lambda \in \mathbb{R}^m$ denotes the contact force due to the reaction of the non-holonomic constraint in (2), $d_1, d_2$ denote uncertainties, $J(q)$ has full row rank $m(m < n)$.

**Remark 1** The equation $J(q)\dot{q} = 0$ consists of $m(m < n)$ non-integrable and independent constraints. Unlike the holonomic case, it is impossible to obtain an equivalent algebraic constraint equation $\Phi(q) = 0$, for some mapping $\Phi$ satisfying $\frac{\partial \Phi(q)}{\partial q} = J(q)$, by integrating $J(q)\dot{q} = 0$ on both sides.

Analogous to the method of reducing order in [6], without loss of generality, assume (2) can be written into the form $(J_1(q)\ J_2(q))\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} = d_2$, so that $J_1: \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is non-singular whereas $q_1 \in \mathbb{R}^m$ is regarded as the passive position variable, and $q_2 \in \mathbb{R}^{n-m}$ is regarded as the active position variable. Then, the independent constraint $J(q)\dot{q} = d_2$ becomes

$$\dot{q}_1 = J_{12}(q)\dot{q}_2 + J_1^{-1}(q)d_2 \tag{3}$$

where $J_{12}(q) = -J_1^{-1}(q)J_2(q)$. In terms of the above non-holonomic constraint, the velocity $\dot{q}$ is expressed in the form

$$\dot{q} = \begin{pmatrix} J_{12}(q) \\ I_{n-m}\end{pmatrix}\dot{q}_2 + \begin{pmatrix} J_1^{-1}(q)d_2 \\ 0\end{pmatrix} = T(q)E_2\dot{q}_2 + \omega_1 \tag{4}$$

where the matrices $E_1$ and $E_2$ are defined to be partitions of the identity matrix $I_n = (E_1\ E_2)$ as $E_1 = \begin{pmatrix} I_m \\ 0\end{pmatrix} \in \mathbb{R}^{n\times m}$, $E_2 = \begin{pmatrix} 0 \\ I_{n-m}\end{pmatrix} \in \mathbb{R}^{n\times(n-m)}$, $T(q) = \begin{pmatrix} I_m & J_{12}(q) \\ 0 & I_{n-m}\end{pmatrix}$. So

$$q_1^{(2)} = J_{12}(q)q_2^{(2)} + F(q_2,\dot{q}_2)\dot{q}_2 + \omega_2 \tag{5}$$

where

$$F(q_2,\dot{q}_2) = \frac{\mathrm{d}J_{12}(q)}{\mathrm{d}q}T(q)E_2\dot{q}_2, \quad \omega_2 = \frac{\mathrm{d}J_{12}(q)}{\mathrm{d}q}\omega_1\dot{q}_2 + \frac{\mathrm{d}J_1^{-1}(q)}{\mathrm{d}q}T(q)E_2\dot{q}_2 d_2 + \frac{\mathrm{d}J_1^{-1}(q)}{\mathrm{d}q}\omega_1 d_2.$$

Decompose the equations of (1) into the following form

$$\begin{pmatrix} M_{11}(q) & M_{12}(q) \\ M_{21}(q) & M_{22}(q)\end{pmatrix}\begin{pmatrix} q_1^{(2)} \\ q_2^{(2)}\end{pmatrix} + \begin{pmatrix} C_1(q,\dot{q})\dot{q} \\ C_2(q,\dot{q})\dot{q}\end{pmatrix} + \begin{pmatrix} G_1(q) \\ G_2(q)\end{pmatrix} = \begin{pmatrix} B_1(q)\tau \\ B_2(q)\tau\end{pmatrix} + \begin{pmatrix} d_{11} \\ d_{12}\end{pmatrix} + J^{\mathrm{T}}\lambda. \tag{6}$$

From (3) ~ (6), the following dynamic equations are obtained

$$\bar{M}_{12}(q)q_2^{(2)} + \bar{C}_{12}(q,\dot{q}_2)\dot{q}_2 + G_1(q) = B_1(q)\tau + J_1^{\mathrm{T}}(q)\lambda + \bar{d}_{11}, \tag{7}$$

$$\bar{M}_{22}(q)q_2^{(2)} + \bar{C}_{22}(q,\dot{q}_2)\dot{q}_2 + \bar{G}_2(q) = \bar{B}_2(q)\tau + \bar{d}_{12}, \tag{8}$$

$$\dot{q}_1 = J_{12}(q)\dot{q}_2 + J_1^{-1}(q)d_2 \tag{9}$$

where

$\bar{M}_{12}(q) = M_{11}(q)J_{12}(q) + M_{12}(q)$,

$\bar{C}_{12}(q,\dot{q}_2) = M_{11}(q)F(q,\dot{q}_2) + C_1(q,\dot{q})T(q)E_2$,

$$\bar{d}_{11} = d_{11} + (C_1(q,\dot{q}_2) - C_1(q,T(q)\cdot E_2\dot{q}_2 + \omega_1))(T(q)E_2\dot{q}_2 + \omega_1) - M_{11}(q)\omega_2 - C_1(q,\dot{q}_2)\omega_1,$$
$$\bar{M}_{22}(q) = M_{21}(q)J_{12}(q) + M_{22}(q) + J_{12}^{\mathrm{T}}\bar{M}_{12}(q),$$
$$\bar{B}_2(q) = B_2(q) + J_{12}^{\mathrm{T}}B_1(q),$$
$$\bar{C}_{22}(q,\dot{q}_2) = M_{21}(q)F(q,\dot{q}_2) + C_2(q,\dot{q})T(q)E_2 + J_{12}^{\mathrm{T}}\bar{C}_2,$$
$$\bar{G}_2(q) = G_2(q) + J_{12}^{\mathrm{T}}G_1(q),$$
$$\bar{d}_{12} = d_{12} + (C_2(q,\dot{q}_2) - C_2(q,T(q)\cdot E_2\dot{q}_2 + \omega_1))(T(q)E_2\dot{q}_2 + \omega_1) - M_{21}(q)\omega_2 - C_2(q,\dot{q}_2)\omega_1 + J_{12}^{\mathrm{T}}\bar{d}_{11}.$$

Select the first-stage feedback control to be
$$\bar{B}_2(q)\tau = \bar{M}_{22}(q)u_1^* + \bar{C}_{22}(q,\dot{q}_2)\dot{q}_2 + \bar{G}_2(q) \tag{10}$$
where $u_1^*$ is a new external input that is specified later. With the above feedback law substituted into (8), the following dynamic equation is obtained
$$\bar{M}_{22}(q)(q_2^{(2)} - u_1^*) - \bar{d}_{12} = 0. \tag{11}$$
As $M(q)$ symmetric positive definite, $\bar{M}_{22}(q) = (J_{12}^{\mathrm{T}}(q)\ \ I_{n-m}^{\mathrm{T}})\ M(q)\begin{pmatrix} J_{12}(q) \\ I_{n-m}\end{pmatrix}$ so $\bar{M}_{22}(q)$ non-singular. From (11), the following dynamic equation is obtained
$$q^{(2)} - u_1^* - d = 0 \tag{12}$$
where $d = \bar{M}_{22}^{-1}(q)\bar{d}_{12}$. (12) is the reduced form of the non-holonomic mechanical system. For the force $\lambda$, we have an expression
$$\lambda = \bar{J}_1^{-\mathrm{T}}(q)\{(C_{12}(q,\dot{q}) - \bar{M}_{12}(q)\bar{M}_{22}^{-1}\cdot \bar{C}_{22}(q,\dot{q}))\dot{q}_2 + (\bar{M}_{12}(q)\bar{M}_{22}^{-1}\bar{d}_{12} - \bar{d}_{11} + (\bar{M}_{12}(q)\bar{M}_{22}^{-1}\bar{B}_2(q) - B_1(q))\tau + (G_1(q) - \bar{M}_{12}(q)\bar{M}_{22}^{-1}(q)\bar{G}_2(q))\}.$$
From the constrained dynamics (9) and the reduced form (12), a generalized normal form of the perturbed non-holonomic mechanical systems is described according to
$$\begin{cases} \dot{q} = \begin{pmatrix} J_{12}(q) \\ I_{n-m}\end{pmatrix} v + \begin{pmatrix} J_1^{-1}(q)d_2 \\ 0 \end{pmatrix} = D(q)v + \omega_1, \\ \dot{v} = u_1^* + d \end{cases} \tag{13}$$
where we have set the new variable $v = \dot{q}_2$ and $D(q) = \begin{pmatrix} J_{12}(q) \\ I_{n-m}\end{pmatrix} = \begin{pmatrix} I_m & J_{12}(q) \\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} 0 \\ I_{n-m}\end{pmatrix} = T(q)E_2$. Our purpose is to make the first $(n-m)$ configuration variables tracking the desired reference signal. Choose the first $(n-m)$ configuration variables as output variables $y \in \mathbb{R}^{(n-m)}$ and the reduced statespace non-holonomic mechanical system is established as
$$\dot{x} = F(x) + Gu_1^* + d^*,$$
$$y = H(x) = E_3^{\mathrm{T}}q \tag{14}$$
where $x = \begin{pmatrix} q \\ v \end{pmatrix} \in \mathbb{R}^{2n-m}$, $F(x) = \begin{pmatrix} D(q)v \\ 0_{n-m}\end{pmatrix}$, $G = \begin{pmatrix} 0_{n\times(n-m)} \\ I_{n-m}\end{pmatrix}$, $E_3 = \begin{pmatrix} I_{n-m} \\ 0_{m\times(n-m)}\end{pmatrix}$, $d^* = \begin{pmatrix} \omega_1 \\ d \end{pmatrix}$. Let $G = (g_1^{\mathrm{T}}\ \ g_2^{\mathrm{T}}\ \cdots\ g_{n-m}^{\mathrm{T}})^{\mathrm{T}}$, the $(n+i)$ th element of $g_i$ is 1, the other is zero; $H(x) = (H_1^{\mathrm{T}}(x)\ \ H_2^{\mathrm{T}}(x)\ \cdots\ H_{n-m}^{\mathrm{T}}(x))^{\mathrm{T}}$,

**Assumption 1** $L_{gj}L_{F(x)}^k H_i(x) = 0$ for all $k = 0, 1, \cdots, \beta_i - 2; i = 1,2,\cdots,(n-m); j = 1,2,\cdots,(n-m)$;

**Assumption 2** rank $M(x) = n-m$, $M(x)$ is $(n-m)\times(n-m)$ matrix, the element $i$th row and $j$th column is $m_{ij}(x) = L_{gj}L_{F(x)}^{\beta_i-1}H_i(x)$.

Then under the Assumption 1 and 2, we have
$$\dot{y}_i = L_{F(x)}H_i(x) + L_{d^*}H_i(x),$$
$$y_i^{(2)} = L_{F(x)}^2 H_i(x) + \sum_{j=1}^{n-m} L_{gj}(L_{F(x)}H_i(x))u_{1j}^* + L_{d^*}L_{F(x)}H_i(x) + L_{d^*}^2 H_i(x) + \sum_{j=1}^{n-m} L_{gj}(L_{d^*}H_i(x)) + L_{F(x)}L_{d^*(x)}H_i(x) = L_{F(x)}^2 H_i(x) + d_{i2}^*,$$
$$\vdots$$
$$y_i^{(\beta_i-1)} = L_{F(x)}^{\beta_i-1}H_i(x) + d_{i\beta_i-1}^*,$$
$$y_i^{\beta_i} = L_{F(x)}^{\beta_i}H_i(x) + \sum_{j=1}^{n-m} m_{ij}(x)u_j + d_{i\beta_i}^*.$$
Suppose that for channal $i(i = 1,\cdots,n-m)$, the reference signal is generated by the following desired model with certain appropriate initial conditions
$$y_{d_i}^{(\beta_i)}(x) + \alpha_{1i}y_{d_i}^{(\beta_i-1)}(x) + \cdots + \alpha_{\beta_i-1i}\dot{y}d_i + \alpha_{\beta_{ii}}y_{d_i} = 0,\ i = 1,2,\cdots,n-m. \tag{15}$$
Without loss of generality, it is assumed that $\beta_1 \leqslant \beta_2 \leqslant \beta_3 \leqslant \cdots \leqslant \beta_{n-m}$. From Assumption 2, $M(x)$ is invertible, the control $u_1^*$ can be designed as

$$
\begin{aligned}
u_1^*(x) = & -M^{-1}(x)\Bigg[A(x) - \begin{pmatrix} y_{d_1}^{(\beta_1)}(x) \\ y_{d_2}^{(\beta_2)}(x) \\ \vdots \\ y_{d_{n-m}}^{(\beta_{n-m})}(x) \end{pmatrix} + \\
& A_1 \begin{pmatrix} y_1^{(\beta_1-1)}(x) - y_{d_1}^{(\beta_1-1)}(x) \\ y_2^{(\beta_2-1)}(x) - y_{d_2}^{(\beta_2-1)}(x) \\ \vdots \\ y_{n-m}^{(\beta_{n-m}-1)}(x) - y_{d_{n-m}}^{(\beta_{n-m}-1)}(x) \end{pmatrix} + \cdots + \\
& A_{\beta_1} \begin{pmatrix} y_1(x) - y_{d_1}(x) \\ y_2^{(\beta_2-\beta_1)}(x) - y_{d_2}^{(\beta_2-\beta_1)}(x) \\ \vdots \\ y_{n-m}^{(\beta_{n-m}-\beta_1)}(x) - y_{d_{n-m}}^{(\beta_{n-m}-\beta_1)}(x) \end{pmatrix} + \\
& A_{\beta_1+1} \begin{pmatrix} y_1(x) - y_{d_1}(x) \\ y_2^{(\beta_2-\beta_1-1)}(x) - y_{d_2}^{(\beta_2-\beta_1-1)}(x) \\ \vdots \\ y_{n-m}^{(\beta_{n-m}-\beta_1-1)}(x) - y_{d_{n-m}}^{(\beta_{n-m}-\beta_1-1)}(x) \end{pmatrix} + \\
& \cdots + A_{\beta_2} \begin{pmatrix} y_1(x) - y_{d_1}(x) \\ y_2(x) - y_{d_2}(x) \\ \vdots \\ y_{n-m}^{(\beta_{n-m}-\beta_1-\beta_2)}(x) - y_{d_{n-m}}^{(\beta_{n-m}-\beta_1-\beta_2)}(x) \end{pmatrix} + \\
& \cdots + A_{\beta_{n-m}} \begin{pmatrix} y_1(x) - y_{d_1}(x) \\ y_2(x) - y_{d_2}(x) \\ \vdots \\ y_{n-m}(x) - y_{d_{n-m}}(x) \end{pmatrix} \Bigg] + v^*
\end{aligned}
\tag{16}
$$

where $A(x) = (A_1^{\mathrm{T}}(x) \quad A_2^{\mathrm{T}}(x) \quad \cdots \quad A_{n-m}^{\mathrm{T}}(x))^{\mathrm{T}}$, $A_i(x) = L_{F(x)}^{\beta_i} H_i(x) (i = 1,2,\cdots,n-m)$; $v^*$ is some new external control; $A_l = \mathrm{diag}\{\bar{a}_{l1}, \bar{a}_{l2}, \cdots, \bar{a}_{ln-m}\}$, $l = 1,2,\cdots,\beta_{n-m}$; $\bar{a}_{li}(l = 1,2,\cdots,\beta_{n-m}; i = 1,2,\cdots,n-m)$ satisfies condition: $\bar{a}_{li} = 0$ if $l > \beta_i$, else $\bar{a}_{li} = a_{li}$ if $0 < l \leqslant \beta_i$; where $a_{li}$ is the same as in (15). Under the control (16), the state tracking error dynamic equation which the decoupled system (14) tracks (15) is given by

$$
e_i^{(\beta_i)}(x) + a_{1i} e_i^{\beta_i - 1}(x) + \cdots + a_{\beta_i i} e_i - v_i^* - d_{i\beta_i}^* = 0, \quad i = 1,2,\cdots,n-m \tag{17}
$$

where $e_i(x) = y_i(x) - y_{d_i}(x)$, $e_i^{(j)}(x) = y_i^{(j)}(x) - y_{d_i}^{(j)}(x)$, $i = 1,2,\cdots,(n-m)$; $1 \leqslant j \leqslant \beta_i$.

Therefore, robust tracking can be designed channel by channel independently, we express these tracking error dynamics in state space form

$$
\dot{z}_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ -a_{\beta_i i} & -a_{\beta_i - 1 i} & -a_{\beta_i - 2 i} & \cdots & -a_{1i} \end{pmatrix}_{\beta_i \times \beta_i} \cdot
z_i + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} v_i^* + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} d_{i\beta_i}^* =
E_i z_i + F_i v_i^* + F_i d_{i\beta_i}^*, \; i = 1,2,\cdots,(n-m) \tag{18}
$$

where $z_i = [e_i \; \dot{e}_i \; \cdots \; e_i^{(\beta_i - 1)}]^{\mathrm{T}}$. Next we proceed to discuss the variable stucture tracking control problem based on the above state equation. First we choose switching function $S_i = C_i z_i$, $i = 1,2,\cdots,n-m$; $C_i$ is left for determined. On the manifold $S_i = 0$, suppose $C_i = [C_{i1} \; C_{i2} \cdots C_{i\beta_i - 1} \; 1]^{\mathrm{T}}$. Select $C_{ij}$ such that all the roots of $\lambda^{\beta_i - 1} + C_{i\beta_i - 1}\lambda^{\beta_i - 2} + \cdots + C_{i1} = 0 (i = 1,2,\cdots,n-m)$ have negative real-part. Then on $S_i = 0$, $\lim_{t\to\infty} e_i = 0$. Next we design control $v_i^* (i = 1,2,\cdots,n-m)$ such that $\lim_{t\to\infty} S_i = 0$. Suppose $\| d_{\beta_i}^* \| < \eta_i$, $v_i^*$ is designed as

$$
v_i^* = \sum_{j=1}^{\beta_i - 1} (-C_{ij} + a_{\beta_i - ji}) e_i^{(j)} + a_{\beta_i i}) e_i - \varepsilon \,\mathrm{sgn}\, S_i - f_i(S_i) \tag{19}
$$

where $f_i(S_i)$ is continuous function vector and satisfies $S_i f_i(S_i) > 0$, $\varepsilon > 0$. Then

$$
S_i^{\mathrm{T}} \dot{S}_i = S_i^{\mathrm{T}} d_{\beta_i}^* - \varepsilon S_i^{\mathrm{T}} \mathrm{sgn} S_i - S_i^{\mathrm{T}} f_i(S_i) < -\varepsilon S_i^{\mathrm{T}} \mathrm{sgn} S_i,
$$

so, $\lim_{t\to\infty} S_i = 0$.

In summary, the following steps need to be taken to arrive at a stable implemention of the control laws:

Step 1 The system is decomposed into (7) and (8), and control $\tau$ is designed as (10);

Step 2 On the basis of the concept of input/output decoupling in nonlinear control theory, a decoupled sysem is established and external control $u_1^*$ is designed as (16).

Step 3 Variable structure control $v^* = ((v_1^*)^{\mathrm{T}} (v_2^*)^{\mathrm{T}} \cdots (v_{n-m}^*)^{\mathrm{T}})^{\mathrm{T}}$ is designed as (19).

**Remark 2** when it is difficult to estimate or to measure the corresponding disturbance $d^*_{\beta_i i}$, we can use the method of $H_\infty$ control (that is described in the other paper) such that the effect of $d^*_{\beta_i i}$ on the tracking performance is as small as possible or at least below a certain desired level.

## 3 Simulation

**Example —— Vertical wheel** Let $x$ and $y$ denote the coordinates of the contact point of the vertical wheel on the plane. Denote $\phi$ as the heading angle of the vertical wheel (measured from the $x$ axis), and $\theta$ the rotation angle of the vertical wheel due to rolling (measured from a fixed reference), the dynamic equations of the vertical wheel, with all numerical constants set to unity, are given by

$$\begin{cases} x^{(2)} = \lambda_1, \; y^{(2)} = \lambda_2, \\ \theta^{(2)} = \tau_1 - \lambda_1 \cos\phi - \lambda_2 \sin\phi, \\ \phi^{(2)} = \tau_2 - \lambda_1 \sin\phi + \lambda_2 \cos\phi \end{cases} \tag{20}$$

with the non-holonomic constraint

$$\dot{x} = \dot{\theta}\cos\phi + \dot{\phi}\sin\phi, \; \dot{y} = \dot{\theta}\sin\phi - \dot{\phi}\cos\phi \tag{21}$$

where $\tau_1$ denotes the control torque about the rolling axis of the wheel, $\tau_2$ is the control torque about the vertical axis through the point of control, and $\lambda_1$ and $\lambda_2$ denote the forces of the constraint that arise from the non-holonomic constraints of the vertical wheel in (21). Suppose that we want $x$ and $y$ to track the desired reference signals $x_d(t) = \sin t + 1, y_d(t) = 2\cos t$ respectively. The following reference model is adapted to generate $x_d$ and $y_d$: $y_d^{(2)} + \dot{y}_d + y_d + 2\sin t = 0, x_d^{(2)} + \dot{x}_d + x_d - \cos t - 1 = 0$ with the desired initial conditions, $y_d(0) = 2$, $\dot{y}_d(0) = 0, x_d(0) = 1, \dot{x}_d(0) = 1$. Select

$$\tau = -\begin{pmatrix} 2\cos\phi & 2\sin\phi \\ 2\sin\phi & -2\cos\phi \end{pmatrix} \left( \begin{pmatrix} -\dot{\theta}\dot{\phi}\sin\phi + \dot{\phi}^2\cos\phi \\ \dot{\theta}\dot{\phi}\cos\phi + \dot{\phi}^2\sin\phi \end{pmatrix} - \begin{pmatrix} x_d^{(2)} \\ y_d^{(2)} \end{pmatrix} + \begin{pmatrix} \dot{x} - \dot{x}_d \\ \dot{y} - \dot{y}_d \end{pmatrix} + \begin{pmatrix} x - x_d \\ y - y_d \end{pmatrix} + \begin{pmatrix} -\cos t - 1 \\ 2\sin t \end{pmatrix} + \begin{pmatrix} e_1 - \cos t - 1 - f(S_1) - \varepsilon \operatorname{sgn} S_1 \\ e_2 + \sin t - 1 - f(S_2) - \varepsilon \operatorname{sgn} S_2 \end{pmatrix} \right) \tag{22}$$

where $e_1 = x - x_d, e_2 = y - y_d, S_i = e_i + \dot{e}_i, i = 1, 2$. The simulation results are shown in Fig.1 and Fig.2.

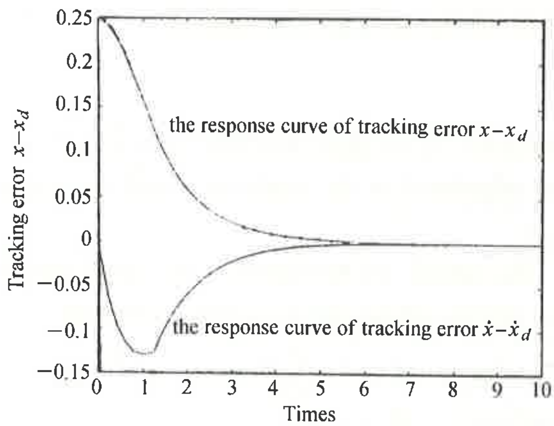


Fig. 1 The response curve of tracking error $x-x_d, \dot{x}-\dot{x}_d$

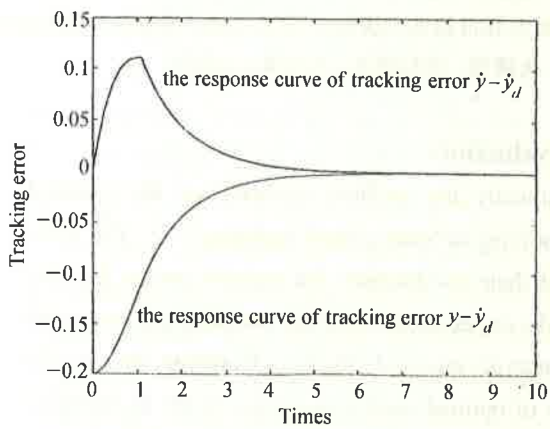


Fig. 2 The response curve of tracking error $y-y_d, \dot{y}-\dot{y}_d$

## 4 Conclusion

A variable structure model reference tracking controller for non-holonomic mechanical systems is designed under parametric perturbation and external disturbance. A design procedure with three-stage control is developed to solve the tracking problem for perturbative non-holonomic mechanical control systems. In the first stage, by a proper decomposition matrix, the perturbative non-holonomic mechanical system is transformed into a reduced form of non-holonomic mechanical system for the convenience of control design. In the second stage, on the basis of the concept of "input/output decoupling" in nonlinear control theory, a decoupled system with parameter perturbation and external disturbance is established via a nonlinear dynamic state feedback control. In the third stage, a switching manifold on which desired model reference tracking performance without parametric perturbation and external disturbance is achieved for the decoupled system. Variable structure control that realizes sliding mode is also explicitly constructed. The computer simulations illustrate the validity of the design approach.

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