Estimates of Transient Behavior for Linear Systems with Input Delay and External Disturbance

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Abstract: Estimates of transient behavior for linear systems with input delay and external disturbance are established. Our results provide effective methods for quantiative stability analysis of delayed systems.

Key words: time delay; disturbance; decay estimate; transient behavior

具有输入时滞和外部扰动的线性系统的过渡过程估计

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摘要：给出了具有输入时滞和外部扰动的一类线性系统的过渡过程的估计.理论分析和仿真结果表明，文中的主要结论为时滞系统稳定性的定量分析提供了一种有效的分析方法.

关键词：时滞；扰动；衰变估计；过渡过程

1 Introduction

Delays and random external disturbances are ubiquitous and often become sources of instability. In the past few years, many contributions have been devoted to the study of delayed systems[1~7]. An important fact in engineering is that uncertain delayed perturbations are generally imposed on the control signals[7]. So, it is necessary to obtain the stability information of practical closed loop systems. Since the information about both stability properties and transient responses of retarded functional differential equations (RFDE) is important to practical engineering systems design, it is necessary to achieve the decay estimate of transient responses of retarded systems with external disturbances. There are some works for obtaining the information about transient responses of special classes of RFDE. Mori et al.[1] calculated estimates on the decay rate of linear time invariant (LTI) point delay systems. Lehman and Shujaee[2] presented estimates on the rate of decay of solutions of time-varying RFDE and generalized the results in [3] or [4]. During the process of deriving the robust stability criteria for dynamical systems including delayed perturbations, Wu and Mizukami[5] get a decay estimate for such special class of RFDE. Here, we will present our results that ensure the trajectory of retarded linear systems with external disturbances within certain bounds.

In this paper, we present an extension of the ones reported by Halanay[3], and estimate the decay rate of transient behavior for linear systems with input delay and external disturbance.

2 Problem formulation

If $u:\mathbb{R}\to\mathbb{R}$ is a continuous function, then $D^+(u)$ denotes the right-hand derivative of u and

$$\|u_t\|_r := \sup_{\theta\in[-r,0]}\{|u(t+\theta)|\}. \tag{1}$$

Let $\|\cdot\|$ denotes the Euclidean norm. For any $A\in\mathbb{R}^{n\times n}$, $\|A\|$ denotes the matrix norm of A induced by the vector norm $\|\cdot\|$. $\lambda_i(\cdot)$ denotes the ith eigenvalue of the square matrix $(\cdot)$, Re $\lambda_i(\cdot)$ its real part $(1\leqslant i\leqslant n)$. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are defined as follows.

$$\lambda_{\max}(\cdot) = \max_{1\leqslant i\leqslant n}\{\mathrm{Re}\lambda_i(\cdot)\},$$
$$\lambda_{\min}(\cdot) = \min_{1\leqslant i\leqslant n}\{\mathrm{Re}\lambda_i(\cdot)\}. \tag{2}$$

Consider the following linear system,

$$\begin{cases} D^+x(t) = Ax(t) + Bu(t-h(t)) + \varepsilon(t); \\ x(t_0) = x_0 \end{cases} \tag{3}$$

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where x is the state of the system (assumed to be available for feedback), the vector u represents a control siganl, and $\varepsilon(t)(\|\varepsilon(t)\| \leqslant \varepsilon)$ denotes uncertain external disturbance at time t. The time delay $h(t)$ is any nonnegative and bounded function. Let $0 \leqslant h(t) \leqslant \bar{h}/2$ such that $\bar{h}$ is any constant. The matrices A, B have compatible dimensions and the pair (A, B) is assuned to be stabilizable. So, a memoryless feedback law can be described as

$$u(t) = Fx(t). \tag{4}$$

Therefore, the closed loop system (3) and (4) can be described by the following differential difference equation of the form

$$D^+ x(t) = Ax(t) + \bar{B}x(t - h(t)) + \varepsilon(t) \tag{5}$$

where $\bar{B} = BF$ and $A + \bar{B}$ is an asymptotically stable matrix. We assume that the solution of system (5) is continuous with initial condition given by

$$x(t) = \psi(t),\ t \in [t_0 - \bar{h}, t_0]. \tag{6}$$

3 Main results

Main results are obtained in the following theorems. In order to derive results more conveniently, we first present a lemma.

Lemma Suppose that $u(t)$ is continuous nonnegative function on $[t_0 - \bar{h}, \beta)$, and satisfies the following inequality

$$D^+ (u(t)) \leqslant - \mu_1 u(t) + \mu_2 \sqrt{u(t)} + \mu_3 \| u_t \|_{\bar{h}},\ \text{for } t \in [t_0, \beta), \tag{7}$$

where $\mu_1 > \mu_3 \geqslant 0, \mu_2 \geqslant 0$ and $\bar{h} \geqslant 0$. Then there exists a positive number $\theta_0 > 0$ such that, for every $t \in [t_0 - \bar{h}, \beta)$,

$$u(t) \leqslant \alpha_0 + \| u_{t_0} \|_{\bar{h}} \exp\{-\theta_0(t - t_0)\}, \tag{8}$$

$$\alpha_0 = \frac{\mu_2^2}{4\theta_0(\mu_1 - \mu_3 e^{\theta_0 \bar{h}} - \theta_0)} \tag{9}$$

and $\theta_0 > 0$ satisfies the following inequality

$$g(\theta_0) = \mu_1 - \mu_3 e^{\theta_0 \bar{h}} - \theta_0 > 0. \tag{10}$$

Proof Since $g(0) = \mu_1 - \mu_3 > 0$, there exists a positive number $\theta_0 > 0$ such that $g(\theta_0) > 0$. Now, let $\alpha > \alpha_0$ be arbitrary and, for every $t \in [t_0 - \bar{h}, \beta)$, define

$$l(t) = u(t) - \alpha - \| u_{t_0} \|_{\bar{h}} \exp\{-\theta_0(t - t_0)\}. \tag{11}$$

Assume that $T = \{t \mid l(t) > 0, t \in [t_0 - \bar{h}, \beta)\}$. If $T \neq \varnothing$, then we can get $\bar{t} = \inf\{T\}$. It must be that $\bar{t} \in [t_0, \beta), l(\bar{t}) = 0$ and $l(t) \leqslant 0$ for $t \in [t_0 - \bar{h}, \bar{t}]$. So, for every $t \in [\bar{t}_0 - \bar{h}, \bar{t}]$,

$$\begin{aligned} u(t) &\leqslant \alpha + \| u_{t_0} \|_{\bar{h}} \exp\{-\theta_0(t - t_0)\} \leqslant \\ &(\alpha + \| u_{t_0} \|_{\bar{h}} \exp\{-\theta_0(\bar{t} - t_0)\}) e^{\theta_0 \bar{h}} \leqslant \\ &u(\bar{t}) e^{\theta_0 \bar{h}}. \end{aligned} \tag{12}$$

Therefor, we obtain that

$$\begin{aligned} &D^+ (l(\bar{t})) \leqslant \\ &- \mu_1 u(\bar{t}) + \mu_2 \sqrt{u(\bar{t})} + \mu_3 u(\bar{t}) e^{\theta_0 \bar{h}} + \\ &\theta_0 \| u_{t_0} \|_{\bar{h}} \exp\{-\theta_0(\bar{t} - t_0)\} \leqslant \\ &- \mu_1 u(\bar{t}) + \mu_2 \sqrt{u(\bar{t})} + \mu_3 u(\bar{t}) e^{\theta_0 \bar{h}} + \\ &\theta_0 (u(\bar{t}) - \alpha) \leqslant \\ &- g(\theta_0) u(\bar{t}) + \mu_2 \sqrt{u(\bar{t})} - \alpha\theta_0 \leqslant \\ &- \alpha\theta_0 + \frac{\mu_2^2}{4g(\theta_0)} \leqslant - (\alpha - \alpha_0)\theta_0 < 0. \end{aligned} \tag{13}$$

This implies that there exists $\delta > 0$ such that $l(t) < l(\bar{t}) = 0$ for every $t \in (\bar{t}, \bar{t} + \delta) \subseteq (\bar{t}, \beta)$. This contradicts the definition of $\bar{t}$. Therefore, $T = \varnothing$ or

$$u(t) \leqslant \alpha + \| u_{t_0} \|_{\bar{h}} \exp\{-\theta_0(t - t_0)\}. \tag{14}$$

Finally, let $\alpha \to \alpha_0$ to find that (8) is established.

Q.E.D.

Remark 1 The above lemma is a generalization of [3,4].

For system (5), since the matrix $A + \bar{B}$ is asymptotically stable, in the light of the Lyapunov stability theory, Lyapunov-type

$$(A + \bar{B})^{\mathrm{T}} P + P(A + \bar{B}) = - Q \tag{15}$$

holds for any symmetric positive definite matrix Q; the solution P is also a symmetric positive definite matrix. We take the following positive definite function as our Lyapunov function

$$V(x) = x^{\mathrm{T}} P x \tag{16}$$

and let

$$\varepsilon = \| \varepsilon(t) \|,\ \mu_1 = \lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}),$$

$$\mu_2 = \varepsilon(\bar{h} \| P^{\frac{1}{2}} \bar{B} \| + 2\sqrt{\lambda_{\max}(P)}),$$

$$\mu_3 = \bar{h}(\| P^{\frac{1}{2}} \bar{B} A P^{-\frac{1}{2}} \| + \| P^{\frac{1}{2}} \bar{B}^2 P^{-\frac{1}{2}} \|).$$

Thus, we have the following theorem.

Theorem For closed loop system (3), (4), if $\mu_1 > \mu_3 \geqslant 0$, then there exists a positive number $\theta_0 > 0$ such that, for every $t \geqslant t_0 - \bar{h}$, (8) ~ (10) are established.

Proof From (16), we obtain that

$$
\begin{aligned}
& D^+ V(x(t)) = \\
& D^+ x^{\mathrm{T}}(t) P x(t) + x^{\mathrm{T}}(t) P D^+ x(t) = \\
& -x^{\mathrm{T}}(t) Q x(t) - 2x^{\mathrm{T}}(t) P\bar{B}(x(t) - \\
& x(t - h(t))) + 2x^{\mathrm{T}}(t) P\varepsilon(t) \leqslant \\
& -x^{\mathrm{T}}(t) P P^{-1} Q x(t) + \\
& 2\int_{t-\frac{\bar{h}}{2}}^{t} \| x^{\mathrm{T}}(t) P^{\frac{1}{2}} P^{\frac{1}{2}} \bar{B} D^+ (x(s)) \| \mathrm{d}s + \\
& 2 \| P^{\frac{1}{2}} x(t) \| \; \| P^{\frac{1}{2}} \| \varepsilon \leqslant \\
& -x^{\mathrm{T}}(t) P P^{-1} Q x(t) + 2 \| P^{\frac{1}{2}} x(t) \| \; \| P^{\frac{1}{2}} \| \varepsilon + \\
& 2 \| P^{\frac{1}{2}} x(t) \| \int_{t-\frac{\bar{h}}{2}}^{t} \| P^{\frac{1}{2}} \bar{B} (Ax(s) + \\
& \bar{B}x(s - h(s)) + \varepsilon(s)) \| \mathrm{d}s \leqslant \\
& -\mu_1 V(x(t)) + 2 \| P^{\frac{1}{2}} x(t) \| \; \| P^{\frac{1}{2}} \| \varepsilon + \\
& 2 \| P^{\frac{1}{2}} x(t) \| \int_{t-\frac{\bar{h}}{2}}^{t} (\| P^{\frac{1}{2}} \bar{B} A P^{-\frac{1}{2}} P^{\frac{1}{2}} x(s) \| + \\
& \| P^{\frac{1}{2}} \bar{B}^2 P^{-\frac{1}{2}} x(s - h(s)) \| + \| P^{\frac{1}{2}} \bar{B} \| \varepsilon) \mathrm{d}s \leqslant \\
& -\mu_1 V(x(t)) + \mu_2 \sqrt{V(x(t))} + \mu_3 \| V_t \|_{\bar{h}}.
\end{aligned}
$$

Directly from lemma. Q.E.D.

Remark 2 The above theorem presents an estimate of the trajectory bounds of the solution of the linear system with state delay and external disturbances. Moreover, the decay estimate is dependent on the time delay and the bounds of random disturbance.

4 Illustrative example

Example Consider the following system described by (3),(4), where

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix},$$

then we can obtain:

$$P = \begin{bmatrix} 1.4515 & 0.0989 \\ 0.0989 & 0.2240 \end{bmatrix}, Q = \begin{bmatrix} 2.2510 & 0.3135 \\ 0.3135 & 1.1607 \end{bmatrix},$$

$$F = [-1.1937 \quad -0.1396].$$

For (5) and (6), let $\bar{h} = 0.01, \varepsilon(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} 0.2 \sin 1000t$, from(18) ~ (20), we have $\mu_1 = 1.0769, \mu_2 = 0.4840, \mu_3 = 0.0085$, that is $\mu_1 > \mu_3$. Here we have $\alpha_0 = 0.2084, \theta_0 = 0.6, \psi^{\mathrm{T}}(t) = [1 \quad 2], t \in [t_0 - \bar{h}, t_0]$, such that, for every $t \geqslant t_0 - \bar{h}$, (8) ~ (10) are established. Fig. 1 describes $V(x)$ and the decay estimate of its transient responses.

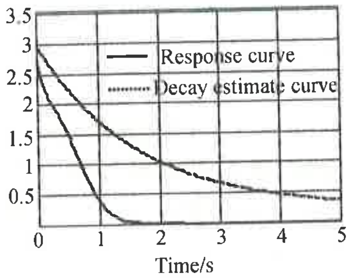

Fig. 1 Transient response and the decay estimation of $V(x)$

5 Conclusion

Decay estimates, which are dependent on the time delay and the bound of external disturbances, are derived in this paper for the linear system with input delay and external disturbance. Our results may provide an effective quantitative analytical method for the transient response of practical engineering systems with delays.

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