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Existence of Positive Definite Solution to Periodic Riccati Differential Equation *

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Abstract: This paper deals with the standard periodic Riccati differential equation. A complete necessary and sufficient condition is presented for the existence of Hermitian periodic positive definite (HPPD) solution. Precisely, after a proper change of basis in the state space the condition can be expressed in terms of the notions of stabilizability and detectability. Moreover, it is shown that when an HPPD solution exists, it is either unique, or else there are uncountably many such solutions. The result of the paper can be considered as a valid extension of Richardson and Kwong's result to the periodic version.

Key words: periodic Riccati differential equation; Hermitian periodic positive definite (HPPD) solution; state space basis transform; stabilizability and detectability

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周期黎卡提微分方程正定解的存在性

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摘要: 讨论了标准的周期黎卡提微分方程.给出了其存在埃尔米特周期正定(HPPD)解的一个完整的充分必要条件.准确地说,在经过一个适当的状态空间基底变换后该条件通过能稳性和能检测性概念表述.结果表明,当 HPPD 解存在时,它或者是唯一的,或者有无限多个.这一结果可以看作是 Richardson 和 Kwong 的结果对周期时变情况的扩展.

关键词: 周期黎卡提微分方程; 埃尔米特周期正定(HPPD)解; 状态空间基底变换; 能稳性和能检测性

1 Introduction

In this paper, we are concerned with the periodic Riccati differential equation (PRDE) (see Eq.(1) below). In recent years, the importance of PRDE in the optimal control, filtering and many other problems of periodically time-varying linear systems has led to the development of a considerable research activity on the subject (see, e.g., [1 ~ 6] etc. and references quoted there). Many authors have considered the possibility of extending notions and results of algebraic Riccati equation into PRDE. For instance, various characterizations of the notions for the periodic version, including stabilizability, detectability, controllability and observability, have been completed (see [7, 8] etc.). The well-known Lyapunov lemma and Wonham-Kučera theorem have been also extended to the periodic version (see [9 ~ 11]). In addition, the necessary and sufficient conditions have been obtained for the existence of a stabilizing symmetric periodic solution, a symmetric periodic positive semidefinite solution, or a unique symmetric periodic positive definite solution to the standard PRDE, respectively. For more details, the interested reader is referred to existing survey articles on PRDEs[5].

This paper is devoted to the problem of what is the necessary and sufficient condition for the existence of a Hermitian periodic positive definite (HPPD) solution to the standard PRDE with complex coefficients. Namely, we consider the existence of an HPPD solution with peri-

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od T to the following standard PRDE

$$\dot{P}(t) = -C^*(t)C(t) - A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t), \quad (1)$$

where $A(\cdot), B(\cdot)$, and $C(\cdot)$ are $n \times n, n \times m$ and $p \times n$ matrices of complex, continuous and T-periodic functions, respectively. $\dot{P}(t)$ is the derivative with respect to the time t, and the superscript $*$ stands for the conjugate transpose of a matrix or vector.

For the time-invariant version the Eq. (1) becomes a standard algebraic Riccati equation (ARE). For the ARE Richardson and Kwong have presented a remarkable result[12]. In our paper, for Eq. (1) we will give an analogous one, i.e., a necessary and sufficient condition for the existence of an HPPD solution. Like Richardson-Kwong theorem, via a nonsingular and periodic state-space transformation the condition is expressed in terms of the notions of stabilizability and detectability. It is also shown that when an HPPD solution exists, it is either unique, or else there are uncountably many such solutions.

Before introducing our main result, we first recall some basic notions relative to periodic systems and, in particular, the state-space transformation performed for Eq.(1).

Associated with Eq. (1) is the periodically time-varying linear system

$$\dot{x} = A(t)x + B(t)u, \ y = C(t)x, \quad (2)$$

where $x \in \mathbb{C}^n, u \in \mathbb{C}^m, y \in \mathbb{C}^p$, are the state. the input and the output of the system, respectively. Let $\Phi_A(t, \tau)$ denote the system transition matrix. In literature $\Phi_A(T,0)$ is named the monodromy matrix of the matrix $A(\cdot)$, and its eigenvalues are called the characteristic multipliers. It is well known that $A(\cdot)$ is asymptotically stable if and only if all the characteristic multipliers of $A(\cdot)$ lie inside the open unit disk of the complex plane[13]. Furthermore, partial multiplicities of a characteristic multiplier μ of $A(\cdot)$ are the dimensions of the Jordan blocks corresponding to μ in the Jordan form of $\Phi_A(T,0)$. It is also well known that the necessary and sufficient condition for $A(\cdot)$ being stable (in the sense of Lyapunov) is all the characteristic multipliers of $A(\cdot)$ lie on the closed unit disk and the partial multiplicities of those lying on the unit circle are all 1 (see, e.g., [13]). The pair $(A(\cdot), B(\cdot))$ is referred to as stabilizable if there exists a T-periodic function matrix $K(\cdot)$ such that $A(\cdot) + B(\cdot)K(\cdot)$ is asymptotically stable; the pair $(C(\cdot), A(\cdot))$ is detectable if $(A^*(\cdot), C^*(\cdot))$ is stabilizable. Note that a number of if different yet equivalent characterizations of these notions can be found in [8]. A T-periodic function matrix $P(\cdot)$ is called positive definite (resp. semidefinite), denoted by $P(\cdot) > 0$ (resp. $P(\cdot) \geqslant 0$), if $P(t)$ is positive definite (resp semidefinite) for all $t \in [0, T]$.

A state-space transformation

$$x(t) = F(t)\hat{x}(t) \quad (3)$$

with a nonsingular, T-periodic and differentiable function matrix $F(\cdot)$ transfers system (2) into

$$\frac{d\hat{x}}{dt} = \hat{A}(t)\hat{x} + \hat{B}(t)u, y = \hat{C}(t)\hat{x}, \quad (4)$$

where

$$\begin{cases} \hat{A}(t) = F^{-1}(t)A(t)F(t) - F^{-1}(t)\dot{F}(t), \\ \hat{B}(t) = F^{-1}(t)B(t), \\ \hat{C}(t) = C(t)F(t). \end{cases} \quad (5)$$

Correspondingly, Eq. (1) becomes

$$\frac{d\hat{P}(t)}{dt} = -\hat{C}^*(t)\hat{C}(t) - \hat{A}^*(t)\hat{P}(t) - \hat{P}(t)\hat{A}(t) + \hat{P}(t)\hat{B}(t)\hat{B}^*(t)\hat{P}(t), \quad (6)$$

where $\hat{P}(t) = F^*(t)P(t)F(t)$.

It is obvious that stabilizability and detectability hold under the transformation (3), i.e., $(A(\cdot), B(\cdot))$ is stabilizable if and only if so is $(\hat{A}(\cdot), \hat{B}(\cdot))$.

At the end of this section, we specially mention a type of important state-space transformations, that is Floquet transformation. Through Floquet transformation, one can transfer system (2) into (4) with constant matrix $\hat{A}(t)$ (see, e.g., [13]).

2 Main Results

As in the time-invariant version, we shall first show that if there exists an HPPD solution to Eq. (1). then via a state-space transformation system (2) can be decomposed in such a way as to separate out the unstabilizable, undetectable part from the remainder, which is stabilizable. To be exact, we shall prove the following theorem.

Theorem 1 The existence of an HPPD solution to Eq. (1) implies that there exists a state-space transformation (3) such that Eq. (1) becomes (6) with matrices

$\hat{A}(t)$, $\hat{B}(t)$ and $\hat{C}(t)$ of the form

$$\hat{A}(t)=\begin{bmatrix}A_{11}(t) & 0\\ 0 & A_{22}\end{bmatrix},$$

$$\hat{B}(t)=\begin{bmatrix}B_1(t)\\ 0\end{bmatrix},\ \hat{C}(t)=[C_1(t)\quad 0],$$

where $(A_{11}(\cdot), B_1(\cdot))$ is stabilizable, A_{22} is a constant and diagonalizable matrix and only of imaginary axis eigen-values. Furthermore, the solution $\hat{P}(t)$ of Eq. (6) takes, with respect to the above transformation, the form

$$\hat{P}(t)=\begin{bmatrix}P_1(t) & 0\\ 0 & P_2\end{bmatrix},$$

where P_2 is a constant matrix, and $A_{11}(t)-B_1(t)B_1^*(t)P_1(t)$ is a constant and asymptotically stable matrix.

The following lemma will be used in the proof of Theorem 1. One easily verifies it if taking $V(t,x)=x^*P(t)x$ as the Lyapunov function of the closed-loop system (2) performed by the control

$$u(t)=-B^*(t)P(t)x(t).$$

Lemma 1 If there exists an HPPD solution $P(\cdot)$ to Eq. (1), then all the characteristic multipliers of the matrix

$$A_p(t)=:A(t)-B(t)B^*(t)P(t) \qquad (7)$$

belong to the closed unit disk of the complex plane, and the partial multiplicities of these lying on the unit circle are all 1.

Proof Assume that Eq. (1) has an HPPD solution $P(\cdot)$. Then one can write (1) into

$$\dot{P}(t)=-C_P^*(t)C_P(t)-A_P^*(t)P(t)-P(t)A_P(t), \qquad (8)$$

where

$$C_P(t)=\begin{bmatrix}C(t)\\ -B^*(t)P(t)\end{bmatrix}, \qquad (9)$$

and $A_P(t)$ is defined in (7). By Lemma 1, using a Floquet transformation $x(t)=F(t)\hat{x}$, where $F(\cdot)$ is a nonsingular, T-periodic and differentiable complex function matrix, one can transfer equation (8) into

$$\frac{\mathrm{d}\hat{P}(t)}{\mathrm{d}t}=-\hat{C}_P^*(t)\hat{C}_P(t)-A_0^*\hat{P}(t)-\hat{P}(t)A_0, \qquad (10)$$

such that A_0 is a complex constant matrix, Re $\sigma(A_1)<0$, Re $\sigma(A_2)=0$ and A_2 is diagonalizable, where

$$\begin{cases}A_0=\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}=F^{-1}(t)A_P(t)F(t)-F^{-1}(t)\dot{F}(t),\\ \hat{C}_P(t)=C_P(t)F(t),\ \hat{P}(t)=F^*(t)P(t)F(t).\end{cases} \qquad (11)$$

Here $\sigma(\cdot)$ denotes the set eigenvalues of a matrix. Relations expressed by using this notation are understood to apply it to each eigenvalue individually. Re denotes the real part of a complex number. Assume that matrices A_1 and A_2 have dimensions $k\times k$ and $(n-k)\times(n-k)$, respectively. Partition

$$\hat{P}(t)=\begin{bmatrix}P_1(t) & P_3(t)\\ P_3^*(t) & P_2(t)\end{bmatrix},$$

such that P_1 and P_2 have the same dimensions as A_1 and A_2, respectively. Thus, (10) becomes

$$\begin{bmatrix}\dot{P}_1+A_1^*P_1+P_1A_1 & \dot{P}_3+A_1^*P_3+P_3A_2\\ \dot{P}_3^*+A_2^*P_3^*+P_3^*A_1 & \dot{P}_2+A_2^*P_2+P_2A_2\end{bmatrix}=-\hat{C}_P^*(t)\hat{C}_P(t). \qquad (12)$$

We now prove several claims:

Claim 1

$$P_2(t)=\text{constant},\ A_2^*P_2(t)+P_2(t)A_2\equiv 0. \qquad (13)$$

Proof Let v_j be an eigenvector associated with the eigenvalue $i\omega_j$ of A_2. Then we have that

$$v_j^*(A_2^*P_2(t)+P_2(t)A_2)v_j\equiv 0, \qquad (14)$$

$$v_j^*\dot{P}_2(t)v_j\leqslant 0. \qquad (15)$$

because (12) implies that $\dot{P}_2(t)+A_2^*P_2(t)+P_2(t)A_2\leqslant 0$. On the other hand, being P_2 a T-periodic differentiable function matrix, it follows form (15) that $v_j^*P_2(t)v_j=$ constant and thus

$$v_j^*\dot{P}_2(t)v_j\equiv 0, \qquad (16)$$

Hence, (14) and (16) yield the required identities in (13), because by the assumed properties of A_2 the vectors $\{v_j\}$ constitute a linearly independent set with number of elements equal to the dimension of A_2.

Claim 2

$$\dot{P}_3(t)+A_1^*P_3(t)+P_3(t)A_2\equiv 0. \qquad (17)$$

Proof We can write (sometimes the time t is omitted for the simplicity of writing)

$$\hat{C}_P^*\hat{C}_P=D^*D=$$

$$\begin{bmatrix}D_1^* & D_3^*\\ D_2^* & D_4^*\end{bmatrix}\begin{bmatrix}D_1 & D_2\\ D_3 & D_4\end{bmatrix}=$$

$$\begin{bmatrix} D_1^* D_1 + D_3^* D_3 & D_1^* D_2 + D_3^* D_4 \\ D_2^* D_1 + D_4^* D_3 & D_2^* D_2 + D_4^* D_4 \end{bmatrix}, \quad (18)$$

where D is an $n \times n$ matrix and its subdivision is chosen so that the dimensions of the two subdivisons of $\hat{C}_P^* \hat{C}_P$ in (12) and (18) correspond. By Claim 1 we conclude $D_2^* D_2 + D_4^* D_4 \equiv 0$, and thus $D_2 \equiv 0$ and $D_4 \equiv 0$. From (12) and (18) we obtain the identity (17).

Claim 3 $P_3(t) \equiv 0$.

Proof For a matrix M let $\boldsymbol{M}$ denote the vector obtained by composing in a single column all columns of M taken in their natural ordering. It is not difficult to verify that (17) can be rewritten into

$$\frac{\mathrm{d}\boldsymbol{P}_3(t)}{\mathrm{d}t} + (A_1^* \otimes I_k + I_{n-k} \otimes A_2^*)\boldsymbol{P}_3(t) = 0, \quad (19)$$

where $\otimes$ denotes the Kronecker product, and I_j is a $j \times j$ identity matrix. One can prove that $A_1^* \otimes I_k + I_{n-k} \otimes A_2^*$ is asymptotically stable because $\mathrm{Re}\sigma(A_1) < 0$, $\mathrm{Re}\sigma(A_2) = 0$ and A_2 is diagonalizable. It follows from well-known Floquet theorem (see, e.g., [13]) that the periodic solution $\boldsymbol{P}_3(t)$ of (19) must be zero.

We now continue the proof of Theorem 1. Using Claim 1 and Claim 3, one obtains from (9), (11) and (12) that there exist some matrices $M(\cdot) \geqslant 0$ and $N(\cdot) \geqslant 0$ such that

$$\{F^* C^* CF\}(t) = \begin{bmatrix} M(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad (20)$$

$$\{F^* PBB^* PF\}(t) = \begin{bmatrix} N(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad (21)$$

and

$$F^*(t)P(t)F(t) = \begin{bmatrix} P_1(t) & 0 \\ 0 & P_2(t) \end{bmatrix}. \quad (22)$$

(20) yields that

$$C(t)F(t) = [C_1(t) \quad 0],$$

for a matrix $C_1(\cdot)$ with the same column dimension as $P_1(\cdot)$. Next, from (21) and (22) we obtain that

$$(F^* PFF^{-1}B)(B^* F^{-*} F^* PF) = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix},$$

$$(F^{-1}B)(F^{-1}B)^* = \begin{bmatrix} P_1^{-1}NP_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

and thus

$$F^{-1}(t)B(t) = \begin{bmatrix} B_1(t) \\ 0 \end{bmatrix},$$

for a matrix B_1 with row dimension equal to that of $P_1(\cdot)$. Finally, from (11) we have that

$$F^{-1}AF - F^{-1}\dot{F} =$$
$$A_0 + (F^{-1}B)(F^{-1}B)^*(F^* PF) =$$
$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 B_1^* P_1 & 0 \\ 0 & 0 \end{bmatrix} =$$
$$\begin{bmatrix} A_1 + B_1 B_1^* P_1 & 0 \\ 0 & A_2 \end{bmatrix}. \quad (23)$$

Thus, the proof of Theorem 1 is complete.

Theorem 1 presents necessary conditions for the existence of an HPPD solution to Eq. (1). We now turn our attention to sufficient conditions. From Theorem 1, if Eq. (1) admits an HPPD solution $P(\cdot)$, then $P(\cdot)$ takes the form diag $(P_1(\cdot), P_2)$ (through an appropriate change of basis in state space) such that $P_1(\cdot)$ satisfies the PRDE

$$\dot{P}_1(t) = -C_1^*(t)C_1(t) - A_{11}^*(t)P_1(t) - P_1(t)A_{11}(t) + P_1(t)B_1(t)B_1^*(t)P_1(t) \quad (24)$$

and P_2 meets the following equation

$$A_{22}^* P_2 + P_2 A_{22} = 0, \quad (25)$$

where $A_{11}(\cdot)$, A_{22}, $B_1(\cdot)$ and $C_1(\cdot)$ possess the properties stated in Theorem 1.

The following lemma originates from Lemma 2 of [12].

Lemma 2 If in (25) A_{22} is a constant and diagonalizable matrix with property $\mathrm{Re}\sigma(A_{22}) = 0$, then there exist uncountable many positive definite solutions to equation (25).

Now in order to obtain the sufficient conditions for the existence of an HPPD solution to (1) we need only to show when Eq. (24) will admit an HPPD solution. The following theorem is well known and can be found in [5].

Theorem 2 Eq. (1) admits a unique HPPD solution $P(\cdot)$ such that $A_P(\cdot) = A(\cdot) - B(\cdot)B^*(\cdot)P(\cdot)$ is asymptotically stable if and only if the pair $(A(\cdot), B(\cdot))$ is stabilizabe, the pair $(C(\cdot), -A(\cdot))$ is detectable.

Combining the above results, we obtain the following necessary and sufficient conditions for the existence of an HPPD solution to Eq. (1).

Theorem 3 Eq. (1) admits an HPPD solution if and only if, through an appropriate nonsingular, T-periodic and differentiable state-space transformation, matrices $A(t)$, $B(t)$ and $C(t)$ take the form

$$A(t) = \begin{bmatrix} A_{11}(t) & 0 \\ 0 & A_{22} \end{bmatrix},$$

$$B(t) = \begin{bmatrix} B_1(t) \\ 0, \end{bmatrix}, \quad C(t) = [C_1(t) \quad 0],$$

where the pair $(A_{11}(\cdot), B_1(\cdot))$ is stabilizable, the pair $(C_1(\cdot), -A_{11}(\cdot))$ is detectable, and A_{22} is a constant and diagonalizable matrix and only of imaginary axis eigenvalues. Furthermore, the solution $P(\cdot)$ takes, with respect to the above transformation, the form

$$P(t) = \begin{bmatrix} P_1(t) & 0 \\ 0 & P_2 \end{bmatrix},$$

where $P_1(\cdot)$ is unique and

$$A_{11}(t) - B_1(t)B_1^*(t)P_1(t)$$

is a constant and asymptotically stable matrix, P_2 is a constant matrix and there exist in fact uncountably many such P_2.

3 Conclusions

This paper extends the result of Richardson and Kwong[12] on positive definite solutions of algebraic Riccati equations to periodic Riccati differential equations. For the periodic Riccati differential equation a complete necessary and sufficient condition is presented for the existence of Hermitian periodic positive definite solution. Moreover, it is shown that when an HPPD solution exists, it is either unique, or else there are uncountably many such solutions.

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