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Moment Stability of Linear Stochastic Systems with Coloured Noises *

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Abstract: Moment stability of linear stochastic systems with coloured noises is investigated, an algebraic criterion with matrix eignvalues without computation for square root of matrices is obtained, an example is given to illustrate the method of this paper.

Key words: stochastic systems; coloured noise; moment stability; Lie algebra; algebraic criteria
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具有有色噪声的线性随机系统的矩稳定性

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摘要: 研究具有有色噪声的线性随机系统的矩稳定性,得到了无需计算矩阵方根的矩阵特征值代数判据,并用实例演示了文中的方法.

关键词: 随机系统; 有色噪声; 矩稳定性; 李代数; 代数判据

1 Introduction

To describe the stability of stochastic systems, various stability conceptions is introduced. For engineering problems, a suitable conception is the moment stability. The moment stability mostly used is the mean-square stability, i.e. the 2-moment stability[1]. It is obvious that moment stability with higher order implies better convergence of the states of systems. General moment stability was investigated in some literature, e.g. in [2, 3] etc., but few result on moment stability of stochastic systems with coloured noises is obtained. In [3] the Lie algebra approach was introduced to investigate the moment stability of the coloured noise systems. In [3], under certain simple conditions, the moment stability of the systems was finally decided by the existence of positive definite solution matrices of some Lyapunov-like matrix equations. While no method was given to verify whether there exist positive definite solutions to the Lyapunov-like matrix equations. While no method was given to verify whether there exist positive definite solutions to the Lyapunov-like matrix equations. In this paper, we will give some algebraic criteria to determine the existence of the positive definite solutions and give an algebraic algorithm for the problem.

2 Conception

Consider linear stochastic system with coloured noise parameters

$$\dot{x} = Ax + \sum_{k=1}^{m} B_k x \eta_k(t), \tag{1}$$

where $x \in \mathbb{R}^n$, $A, B_k \in \mathbb{R}^{n\times n}$ are constant matrices, $\{\eta_k(t)\}$, $k = 1,2,\cdots,m$, are zero-mean stationary Gauss processes with correlation matrix

$$R_{\eta\eta}(\tau) = \mathrm{E}[\eta(t)\eta^{\mathrm{T}}(t-\tau)], \tag{2}$$

where $\eta(t) = [\eta_1(t), \eta_2(t), \cdots, \eta_m(t)]^{\mathrm{T}}$.

Define the spectral density matrix of the stochastic process $\eta(t)$ as

$$S_{\eta\eta}(\mathrm{j}\omega) = F[R_{\eta\eta}(\tau)], \tag{3}$$

where F refers to the Fourier transformation. It is assumed that the elements of $S_{\eta\eta}(0)$ are bounded.

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Let p be a positive number, $t_0 \in \mathbb{R}^+$, for $x_0 \in \mathbb{R}^n$, the solution process of (1) is denoted by $x(t, t_0, x_0)$, or simply by $x(t)$.

Definition 1 The equilibrium $x = 0$ of system (1) is said to be exponentially p-moment stable in large, or briefly p-MESL, if for any $x_0 \in \mathbb{R}^n$, there exist positive numbers $\alpha > 0, \beta > 0$ such that for any rear-rangement $\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n$, of $x_1, x_2, \cdots, x_n$, and positive numbers $p_1, p_2, \cdots, p_n$ with $p_1 + p_2 + \cdots + p_n = p$, $|\mathrm{E}[\hat{x}_1^{p_1} \hat{x}_2^{p_2} \cdots \hat{x}_n^{p_n}]| \leqslant \alpha \mathrm{e}^{-\beta(t - t_0)}$, for $t \geqslant t_0$.

Definition 2 A subset λ of $\mathbb{R}^{n \times n}$ is said to be a Lie algebra if for any $A, B \in \lambda$, $[A, B] = AB - BA \in \lambda$.

Set $\lambda^0 = \{A, B_1, B_2, \cdots, B_m\}$, let $\lambda(A, B_1, B_2, \cdots, B_m)$ the Lie algebra generated by λ_0, i.e. the minimal Lie algebra containing $A, B_1, B_2, \cdots, B_m$.

Definition 3 Given a Lie algebra λ, define its derived series by

$$\lambda^0 = \lambda,$$

$$\lambda^{(n+1)} = [\lambda^n, \lambda^n] = \{[A, B] \mid A, B \in \lambda^n\}.$$

λ is said to be Abelian if $\lambda^{(1)} = \{0\}$; λ is said to be solvable if $\lambda^{(n)} = \{0\}$ for some n.

By [2], a matrix Lie algebra λ is solvable if and only if there exits a nonsingular matrix P such that PAP^{-1} is an upper triangular matrix for all $A \in \lambda$.

3 Lemmas

Consider also the linear stochastic system with white noises

$$\mathrm{d}x = Ax\mathrm{d}t + \sum_{k=1}^{m} \sigma_k B_k x \mathrm{d}W_k, \tag{4}$$

where $\sigma_k, k = 1, 2, \cdots, m$, are non-negative constants describing the noise intensities, $W = [W_1, W_2, \cdots, W_m]^{\mathrm{T}}$ is an m-dimensional Wiener process with independent components satisfying

$$\mathrm{E}[W_k(t)] = 0,\ \mathrm{E}[W_k(t) W_k(s)] = \min\{t, s\}.$$

Lemma 1[4,5] 1) The equilibrium $x = 0$ of Itô stochastic system (4) is mean-square asymptotically stable if and only if for any given positive definite matrix $Q \in \mathbb{R}^{n \times n}$, there exists a positive definite solution matrix to the Lyapunov-like matrix equation

$$A^{\mathrm{T}}P + PA + \sum_{k=1}^{m} \sigma_k^2 B_k^{\mathrm{T}} P B_k = -Q, \tag{5}$$

2) For given positive definite matrix $Q \in \mathbb{R}^{n \times n}$, there exists a positive definite solution matrix to the Lyapunov-like matrix equation (5) if and only if the matrix

$$M = A \oplus A + \sum_{k=1}^{m} \sigma_k^2 B_k \otimes B_k, \tag{6}$$

is a stable matrix, i.e.. $\mathrm{Re}\,\lambda(M) < 0$, where $A \oplus A = I_n \otimes A + A \otimes I_n$ and $\otimes$ is the Kronecker's matrix product.

Lemma 2[2,3] If the Lie algebra $\lambda(A, B_1, B_2, B_m)$ is solvable, then the equilibrium $x = 0$ of system (1) is p-MESL if for any given positive definite matrix $Q \in \mathbb{R}^{n \times n}$, there exists a positive definite solution matrix to the Lyapunov-like matrix equation

$$\left[A + \frac{p}{4} \sum_{r=1}^{m} \sum_{s=1}^{m} S_{\eta\eta}(0)_{rs} B_r B_s\right] P + P\left[A + \frac{p}{4} \sum_{r=1}^{m} \sum_{s=1}^{m} S_{\eta\eta}(0)_{rs} B_r B_s\right]^{\mathrm{T}} + \frac{p}{2} \sum_{r=1}^{m} \sum_{s=1}^{m} S_{\eta\eta}(0)_{rs} B_r P B_s^{\mathrm{T}} = -Q. \tag{7}$$

4 Algebraic criterion for moment stability

Although criteria were given to determine the moment exponential stability of system (1) in Lemma 2 with Lyapunov-like matrix equations, and this kind of matrix equations are not standard Lyapunov matrix equations or Lyapunov-like matrix equation of the form of (5), one can not verify the existence of the positive definite solutions with standard tools, e.g. MATLAB.

Since $S_{\eta\eta}(0)_{rs}, r, s = 1, 2, \cdots, m$ may be negative, we can not directly write $S_{\eta\eta}(0)_{rs}$ as $[S_{\eta\eta}(0)_{rs}]^{\frac{1}{2}}[S_{\eta\eta}(0)_{rs}]^{\frac{1}{2}}$ and then change (7) into the form of (5). Here we rewrite equation (7) in the form of (5) and thus obtain the following result.

Theorem 1 Assume that the Lie algebra $\lambda(A, B_1, B_2, \cdots, B_m)$ is solvable, let

$$[S_{\eta\eta}(0)]^{\frac{1}{2}} = [g_{rs}]_{m \times m},\ \tilde{B}_k = \sum_{r=1}^{m} g_{rk} B_r,$$

$$M_{(1)} = \left[A + \frac{p}{4} \sum_{r=1}^{m} \sum_{s=1}^{m} S_{\eta\eta}(0)_{rs} B_r B_s\right] \oplus \left[A + \frac{p}{4} \sum_{r=1}^{m} \sum_{s=1}^{m} S_{\eta\eta}(0)_{rs} B_r B_s\right] + \frac{p}{2} \sum_{k=1}^{m} \tilde{B}_k \otimes \tilde{B}_k, \tag{8}$$

then the equilibrium $x = 0$ of system (1) is p-MESL if and only if $M_{(1)}$ defined by (8) is stable.

Proof Since $[S_{\eta\eta}(0)_{rs}]^{\frac{1}{2}} = [g_{rs}]_{m \times m}$ is symmetric, we have

$$\sum_{k=1}^{m} \tilde{B}_k P \tilde{B}_k^{\mathrm{T}} = \sum_{k=1}^{m} \sum_{r=1}^{m} \sum_{s=1}^{m} g_{rk} g_{sk} B_r P B_s^{\mathrm{T}} = \sum_{r=1}^{m} \sum_{s=1}^{m} \sum_{k=1}^{m} g_{rk} g_{ks} B_r P B_s^{\mathrm{T}} = \sum_{r=1}^{m} \sum_{s=1}^{m} S_{\eta\eta}(0)_{rs} B_r P B_s^{\mathrm{T}},$$

thus Eq. (7) can be rewritten as

$$\left[A+\frac{p}{4}\sum_{r=1}^{m}\sum_{s=1}^{m}S_{\eta\eta}(0)_{rs}B_rB_s\right]P+$$
$$P\left[A+\frac{p}{4}\sum_{r=1}^{m}\sum_{s=1}^{m}S_{\eta\eta}(0)_{rs}B_rB_s\right]^{\mathrm{T}}+\frac{p}{2}\sum_{k=1}^{m}\tilde{B}_kP\tilde{B}_k^{\mathrm{T}}=-Q. \tag{9}$$

By Lemma 1, the existence of positive definite solutions of (9) is equivalent to the stability of the matrix

$$G_{(1)}=\left[A+\frac{p}{4}\sum_{r=1}^{m}\sum_{s=1}^{m}S_{\eta\eta}(0)_{rs}B_rB_s\right]^{\mathrm{T}}\oplus \left[A+\frac{p}{4}\sum_{r=1}^{m}\sum_{s=1}^{m}S_{\eta\eta}(0)_{rs}B_rB_s\right]^{\mathrm{T}}+\frac{p}{2}\sum_{k=1}^{m}\tilde{B}_k^{\mathrm{T}}\otimes\tilde{B}_k^{\mathrm{T}}. \tag{10}$$

In fact, we have $G_{(1)}=M_{(1)}^{\mathrm{T}}$, from this and by Lemma 2 we get the result of Theorem 1.

Remark 1 With the definition of $\tilde{B}_k$, one can rewrite $M_{(1)}$ into

$$M_{(1)}=\left[A+\frac{p}{4}\sum_{r=1}^{m}\sum_{s=1}^{m}S_{\eta\eta}(0)_{rs}B_rB_s\right]\oplus \left[A+\frac{p}{4}\sum_{r=1}^{m}\sum_{s=1}^{m}S_{\eta\eta}(0)_{rs}B_rB_s\right]+\frac{p}{2}\sum_{r=1}^{m}\sum_{s=1}^{m}S_{\eta\eta}(0)_{rs}B_r\otimes B_s, \tag{11}$$

which is convenient for the practical applications, for we do not have to compute the square root of the matrix $S_{\eta\eta}(0)$.

5 Example

Let $n=2, m=2$, and

$$A=\begin{bmatrix}-10 & 5\\ -5 & -9\end{bmatrix},\ B_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ B_2=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},$$

$$\mathrm{E}[\eta_1(t)\eta_1(t-\tau)]=\mathrm{E}[\eta_2(t)\eta_2(t-\tau)]=\exp(-|\tau|),$$
$$\mathrm{E}[\eta_1(t)\eta_2(t-\tau)]=0.$$

One can show that $\lambda(A, B_1, B_2)$ is solvable.

A computation shows that the spectral desity matrix of the noises at zero is $S_{\eta\eta}(0)=\mathrm{diag}(2,2)$.

For the case $p=3$, with MATLAB we easily get

$$M_{(1)}=\begin{bmatrix}-2 & 5 & 5 & 0\\ 4 & -8 & 0 & 5\\ 4 & 0 & -8 & 5\\ 3 & 4 & 4 & -11\end{bmatrix},$$

$$\sigma(M_{(1)})=\{3.6406;\ -8.3094;\ -16.3310;\ -8.0000\}.$$

Thus, by Theorem 1, system (1) is not exponentially 3-moment stable.

For the case $p=2$, we have

$$M_{(1)}=\begin{bmatrix}-8 & 0 & 5 & 0\\ 1 & -11 & 0 & 5\\ 1 & 0 & -11 & 5\\ 2 & 1 & 1 & -12\end{bmatrix},$$

$$\sigma(M_{(1)})=\{-5.1739;\ -12.9130\pm0.7946\mathrm{j};\ -11.0000\}.$$

By Theorem 1, system (1) is exponentially mean-square stable.

The simulation curves of the system are shown in Fig. 1.

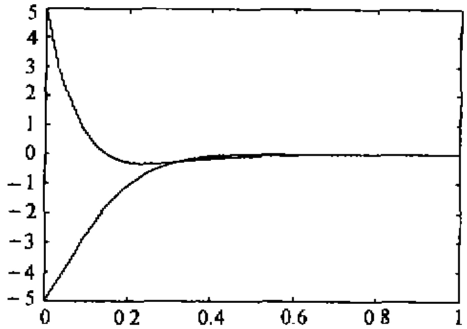

Fig. 1 Simulation curves of the system

6 Conclusion

In this paper, we have investigated the moment stability of linear stochastic systems with coloured noises, induced the condition with non-standard matrix equations into a criterion with computations of eigenvalues of associated matrices with the systems. The criterion does not involve computation of square root of any matrix, so, it is very convenient for engineering applications.

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