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Robust Region Stability for Uncertain Generalized Systems *

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Abstract: This paper deals with D-stability for generalized linear systems, and robust D-stability for generalized uncertain systems, where D is defined by a linear matrix inequality (LMI), and called the LMI stability region[1]. The relevant criteria are derived in terms of LMIs. Moreover, a convex optimization algorithm is provided in order to calculate the perturbation radius of uncertain parameter matrix in a system matrix.

Key words: generalized systems; LMI region; stability; robustness
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不确定广义系统鲁棒区域稳定性

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摘要: 本文涉及广义线性系统的 D-稳定性及不确定广义系统的鲁棒 D-稳定性,其中 D 是由线性矩阵不等式(LMI)来定义的且称为 LMI 稳定性区域[1],给出 LMI 形式的有关判据及用于计算系统矩阵中不确定参量阵摄动"半径"的凸优化算法.

关键词: 广义系统; LMI 区域; 稳定性; 鲁棒性

1 Introduction

Generalized systems, also called singular systems or descriptor systems, have some special characteristics and natures which are not seen in the traditional systems[2~4]. Recently the LMI approach has been widely used in the analysis and synthesis of control systems. Ref. [1] introduced LMI stability regions in the complex plane with the help of an LMI formula. Since the intersection of finite LMI regions is also an LMI region. The LMI stability regions include almost all of the important stability regions. Ref. [1] gave a criterion for the LMI region stability of traditional linear systems with the help of an LMI. This paper intends to discuss the LMI region stability of generalized systems, give a corresponding LMI-type criterion, analyze the LMI region stability robutness of uncertain generalized systems, and also give the steps for calculating the admissible perturbation radius of uncertain perturbed systems.

In this paper, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}^-$ *denote the real axis, the* complex plane, and the open left-half complex plane, respectively. $\bar{z}$ stands for the conjugate number of complex z, superscript $*$ for the conjugate transpose of a vector or matrix, and superscript T for the transpose of a vector or matrix. Let I be an identity matrix, whose dimension is kept on subscript in a necessary case. The symbol $\otimes$ denotes the Kronecker product of matrices. It can be proved that $(M \otimes N)^T = M^T \otimes N^T$, and $(AB) \otimes (MN) = (A \otimes M)(B \otimes N)$. $W > 0$ ($W < 0$) means W is a symmetric and positive (negative) definite matrix. If $A - B > 0$ ($A - B < 0$), then simply denoted it with $A > B$ ($A < B$).

2 Preliminaries

Consider the following generalized system:

$$E\dot{x} = (A + F\Sigma G)x, \tag{1}$$

where E, A, F, and G are $n \times n, n \times n, n \times p, p \times n$ real matrices, respectively, and $0 < \operatorname{rank}E = r < n$, the generalized state vector $x \in \mathbb{R}^n$. Σ denotes a $p \times p$ uncertain parametric matrix. It belongs to the following *perturbation class*

$$\Omega_\eta \doteq \{\Sigma : \Sigma\Sigma^T \leqslant \eta^2 I\}. \tag{2}$$

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$(E, A+F\Sigma G)$ simply denotes the uncertain system (1), and (E,A) its nominal one. $\Lambda(E,A) \doteq \{s \in \mathbb{C}: \det(sE - A) = 0\}$, and called the set of finite eigenvalues of (E,A). Obviously, $\Lambda(I,A) = \Lambda(A)$ is just the set of eigenvalues of A. A number of definitions and properties are listed below for generalized systems considered in this paper.

1) (E,A) is called to be regular if $\det(sE - A)$ is not identically zero; to be impulse-free if deg det $(sE - A) = \text{rank } E$.

2) (E,A) is called to be stable if $\Lambda(E,A) \subset \mathbb{C}^-$; to be D-stable if $\Lambda(E,A) \subset D$.

3) (E,A) is called to be admissible if it is regular impulse-free and stable; to be D-admissible if it is regular impulse-free and D-stable.

4) (I,A) is admissible if A is stable, i.e., $\Lambda(A) \subset \mathbb{C}^-$. (I,A) is D-admissible if A is D-stable, i.e., $\Lambda(A) \subset D$.

5) $(E, A + F\Sigma G)$ is called robustly D-stable (admissible) if $(E, A + F\Sigma G)$ is D-stable (admissible) for every $\Sigma \in \Omega_\eta$.

6) For any $n \times n$ nonsingular matrices P and Q, (E, A) is D-admissible if and only if (PEQ, PAQ) is D-admissible; also $(E, A + F\Sigma G)$ is robustly D-admissible if and only if $(PEQ, P(A + F\Sigma G)Q)$ is robustly D-admissible.

7) Suppose (E,A) is regular and impulse-free, there must exist $n \times n$ nonsingular matrices P and Q so that

$$PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\ PAQ = \begin{bmatrix} A_r & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

Without the loss of generality we can assume that

$$\begin{cases} E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} A_r & 0 \\ 0 & I_{n-r} \end{bmatrix}, \\ F\Sigma G = \begin{bmatrix} F_1\Sigma G_1 & F_1\Sigma G_2 \\ F_2\Sigma G_1 & F_2\Sigma G_2 \end{bmatrix}, \end{cases} \tag{3}$$

where $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ and $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ have appropriate subblocks corresponding with E.

Lemma 2.1 Suppose

$$\eta < \frac{1}{\bar{\sigma}(G_2F_2)}, \tag{4}$$

where $\bar{\sigma}(\cdot)$ means the maximal singular value of a matrix, then the system defined by (1) ~ (3) is regular and impulse-free for every $\Sigma \in \Omega_\eta$.

Proof Condition (4) implies that $(I_{n-r} + F_2\Sigma G_2)$ is nonsingular for $\forall \Sigma \in \Omega_\eta$; the latter insures the impulse-free of $(E, A + F\Sigma G)$; of course the regularity is insured. Q.E.D.

The following refers to the LMI region.

Definition 2.1[1] We called an open region D to be LMI region if it is defined by

$$D \doteq \{z \in \mathbb{C}: U + zV + \bar{z}V^T < 0\}, \tag{5}$$

where $U = U^T$ and V are both $m \times m$ real matrices.

Remark Since the intersection of finite LMI regions is also an LMI region, so the whole of all LMI stability regions contains almost every of usual important stability regions, for instance, the open left complex plane, left sector, circle region, etc.

Concerning the LMI region stability of traditional linear systems, the following is a powerful criterion.

Lemma 2.2[1] Given an $n \times n$ real matrix M and the LMI region D defined by (5), then M is D-stable if and only if there exists an $n \times n$ real $X > 0$ so that

$$M_D(M,X) < 0, \tag{6}$$

where

$$M_D(M,\ X) \doteq U \otimes X + V \otimes (MX) + V^T \otimes (MX)^T. \tag{7}$$

3 LMI region stability

For the D-stability of pair (E,A), a corresponding criterion is given below.

Theorem 3.1 Given a regular pair (E,A) and LMI region D defined in Definition (2.1), then (E,A) is D-stable if there exists such an $n \times n$ real $X > 0$ that

$$(I_m \otimes w^T)M_D(E,A,X)(I_m \otimes w) < 0, \tag{8}$$

for any real $w \in \{w: E^Tw \neq 0\}$, where

$$M_D(E,A,X) \doteq$$
$$U \otimes (EXE^T) + V \otimes (AXE^T) + V^T \otimes (EXA^T). \tag{9}$$

Conversely, if pair (E,A) as in (3) is D-stable, then there exists an $n \times n$ real $X > 0$ which satisfies (8) for any real $w \in \{w: E^Tw \neq 0\}$.

Proof The regularity of (E,A) implies the nonemptiness of $\Lambda(E,A)$. Suppose $\lambda \in \Lambda(E,A)$ and that $w^* = u^T + jv^T$ is a corresponding left eigenvector i.e., $w^*A = \lambda w^*E$. Note that both E^Tu and E^Tv can not be zero simultaneously, so $w^*EXE^Tw = u^TEXE^Tu$

$+ v^{\mathrm{T}} EXE^{\mathrm{T}} v > 0$. From (8) and $(I_m \otimes w^*) M_D(E, A, X)(I_m \otimes w) = (U + \lambda V + \bar{\lambda} V^{\mathrm{T}}) w^* EXE^{\mathrm{T}} w$, we derive $(U + \lambda V + \bar{\lambda} V^{\mathrm{T}}) < 0$, hence the D-stability of (E, A).

Conversely, from the D-stability of (E, A) as in (3) we know the D-stability of A_r. Recalling Lemma 2.2 derives that there exists an $r \times r$ real matrix X_1 such that

$$M_D(A_r, X_1) < 0, \tag{10}$$

where

$$M_D(A_r, X_1) \doteq U \otimes X_1 + V \otimes (A_r X_1) + V^{\mathrm{T}} \otimes (A_r X_1)^{\mathrm{T}}. \tag{11}$$

Let $X = \mathrm{diag}\{X_1, X_4\}$ where real square matrix $X_4 > 0$ with $(n - r)$-dim can be selected as one pleases. For any real vector $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^n$ with r-dim sub-block $w_1 \neq 0$, it is certain that $E^{\mathrm{T}} w \neq 0$. Note that

$$w^{\mathrm{T}} AXE^{\mathrm{T}} w = w_1^{\mathrm{T}} A_r X_1 w_1,$$
$$w^{\mathrm{T}} EXA^{\mathrm{T}} w = w_1^{\mathrm{T}} X_1 A_r^{\mathrm{T}} w_1,$$

then

$$(I_m \otimes w^{\mathrm{T}}) M_D(E, A, X)(I_m \otimes w) =$$
$$U(w_1^{\mathrm{T}} X_1 w_1) + V(w_1^{\mathrm{T}} A_r X_1 w_1) + V^{\mathrm{T}}(w_1^{\mathrm{T}} X_1 A_r^{\mathrm{T}} w_1) =$$
$$(I_m \otimes w_1^{\mathrm{T}}) M_D(A_r, X_1)(I_m \otimes w_1).$$

Ineq. (8) then follows immediately from this and (10).

Q.E.D

Sometimes we can use the following result to test the D-stability, the proof will be omitted here since it is similar to that given in Theorem (3.1).

Theorem 3.2 Given a regular pair (E, A) and an LMI region D defined in Definition (2.1), then (E, A) is D-stable if there exists an $n \times n$ real matrix Y that

i) $EY = (EY)^{\mathrm{T}} \geqslant 0$,

ii) $N_D(E, A, Y) < 0$,

where

$$N_D(E, A, Y) \doteq U \otimes (EY) + V \otimes (AY) + V^{\mathrm{T}} \otimes (AY)^{\mathrm{T}}.$$

Remark

i) The conditions given above may not be necessary, for example, $(E, A) = (\Sigma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix})$. Let D be defined by (5) with $U = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $V = \begin{bmatrix} -0 & 1 \\ 0 & 0 \end{bmatrix}$ i.e., D is the circle region centered at $(-2, j0)$ with radius 1. It is easily seen that only $Y = \begin{bmatrix} y_1 & 0 \\ y_3 & y_4 \end{bmatrix}$ with $y_1 > 0$ can satisfy conditions i). Simply calculating results in

$$N_D(E, A, Y) = \begin{bmatrix} -y_1 & 0 & 0 & 0 \\ 0 & 0 & y_3 & y_4 \\ 0 & y_3 & -y_1 & 0 \\ 0 & y_4 & 0 & 0 \end{bmatrix}.$$

This means $N_D(E, A, Y) \not< 0$ for any y_1, y_3 and y_4.

ii) In some cases the regularity and impulse-free may not be necessary.

4 Robust D-stability

If the uncertain generalized system, defined by (1) ~ (4), satisfies the two conditions of Theorem 3.1 for every $\Sigma \in \Omega_\eta$, then we call it to be robustly D-stable. For all Σ in Ω_η if there exists a common $X > 0$ satisfying (8), then it possesses a stronger property than robust D-stability. For this we give the following result.

Theorem 4.1 The system defined by (1) ~ (4) is robustly D-stable if it satisfies both of conditions below:

i) The nominal system (E, A) is D-admissible,

ii) There exist an $n \times n$ real matrix $X > 0$ and a real number $\varepsilon > 0$ that

$$(I_m \otimes w^{\mathrm{T}}) \mathscr{M}(X, \varepsilon)(I_m \otimes w) < 0, \tag{12}$$

for $\forall w \in \{w \in \mathbb{R}^n : E^{\mathrm{T}} w \neq 0\}$, where

$$\mathscr{M}(X, \varepsilon) \doteq M_D(E, A, X) + \frac{\eta^2}{\varepsilon^2}(V \otimes F)(V \otimes F)^{\mathrm{T}} + \varepsilon^2 (I_m \otimes EXG^{\mathrm{T}})(I_m \otimes EXG^{\mathrm{T}})^{\mathrm{T}}.$$

Proof Both of conditions i) and (4) deduce the regularity and impulse-free of systems (1) ~ (4). Recalling the useful matrix inequality

$$MN + N^{\mathrm{T}} M^{\mathrm{T}} \leqslant \varepsilon^2 MM^{\mathrm{T}} + \frac{1}{\varepsilon}^2 N^{\mathrm{T}} N, \ \forall \varepsilon > 0,$$

we gradually deduce

$$M_D(E, A + F\Sigma G, X) =$$
$$M_D(E, A, X) + V \otimes (F\Sigma GXE^{\mathrm{T}}) + V^{\mathrm{T}} \otimes (F\Sigma GXE^{\mathrm{T}})^{\mathrm{T}} \leqslant$$
$$M_D(E, A, X) + \varepsilon^2 (I_m \otimes EXG^{\mathrm{T}})(I_m \otimes EXG^{\mathrm{T}})^{\mathrm{T}} +$$
$$\frac{1}{\varepsilon^2}(V \otimes F)(I_m \otimes \Sigma\Sigma^{\mathrm{T}})(V \otimes F)^{\mathrm{T}} \leqslant$$
$$\mathscr{M}(x, \varepsilon).$$

From this and (12) we get

$$(I_m \otimes w^{\mathrm{T}}) M_D(E, A + F\Sigma G, X)(I_m \otimes w) < 0, \ \forall \Sigma \in \Omega_\eta$$

by Theorem 3.1, hence the result. Q.E.D.

Under the admissibility of (E, A), and algorithm is given below for calculating the admissible perturbation bound η of perturbation class Ω.

Algorithm 4.1

Optimization index: $\max\{\eta\}$.

Constraint condition: Ineq. (12), $E^{T}w \neq 0, \eta > 0, \varepsilon > 0, X = X^{T} > 0$.

As $\eta^{*} = \max\{\eta\}$ is reached, then

$$\eta^{2} = \min\{\eta^{*}, \frac{1}{\bar{\sigma}(G_2 F_2)}\}$$

is the bound we want.

Explanation i) Theorem 3.1 tells us that there may exist many $X > 0$ satisfying (8) provided that (E, A) is D-admissible. After selecting an X, generally speaking, there always exist η and ε so long as they are small enough, that is to say, an initial perturbation bound is available. From this we can use Algorithm 4.1 to find a bigger bound.

ii) For robust D-stability, the bound we reached by using Algorithm 4.1 may be conservative.

iii) Algorithm 4.1 is a convex optimization problem with an LMI constraint, which can be solved with the help of LMI toolbox in Matlab software, and the main sentence seems "mincx".

5 Conclusions

The paper introduced LMI region stability, discussed robust region stability of generalized linear systems, and finally gave the relevant LMI criteria and convex optimization algorithm for calculating the admissible perturbation bound. Of course, many problems of uncertain generalized systems are still open to be discussed and need further, in-depth study.

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