Article ID: 1000-8152(2001)06-0827-06

Nonlinear Stability of (ρ,σ)-Methods for Stiff Delay-Differential-Algebraic Systems*

ZHANG Chengjian
(Department of Mathematics, Huazhong University of Science and Technology · Wuhan, 430074, P.R.China)
LIAO Xiaoxin
(Department of Control Science and Engineering, Huazhong University of Science and Technology · Wuhan, 430074, P.R.China)

Abstract: This paper deals with the stability of (ρ,σ)-methods for stiff delay-differential-algebraic systems with one-index. In particular, we prove that (resp. strong) $G(c,p,q)$-algebraic stability of the (ρ,σ)-methods for ordinary differential equations (ODEs) leads to (resp. asymptotic) global stability of the corresponding methods for stiff delay-differential-algebraic systems.

Key words: (ρ,σ)-methods; stiff delay-differential-algebraic systems; nonlinear stability
Document code: A

(ρ,σ)-方法关于刚性延迟微分代数系统的非线性稳定性

张诚坚　　廖晓昕
(华中科技大学数学系·武汉,430074)　(华中科技大学控制科学与工程系·武汉,430074)

摘要: 本文涉及(ρ,σ)-方法应用于1-指标的非线性刚性延迟微分代数系统的稳定性.证明了求解常微分方程(ODEs)的(ρ,σ)-方法的(强)$G(c,p,q)$-代数稳定性导致相应延迟微分代数系统方法的(渐近)整体稳定性.

关键词: (ρ,σ)-方法; 刚性延迟微分代数系统; 非线性稳定

1 Introduction

In the last twenty years, the research for the uumerical solutions of delay differential equations (DDEs) and differential algebraic equations (DAEs) has made great advances (cf.[1~12]). In particular, papers [3~7] extended the above topic to nonlinear stiff problems. However, up to now, only partial results have contributed to the numerical solutions of delay-differential-algebraic equations (DDAEs) (cf.[11,12]) and deal mainly with *linear and nonstiff problems*. The systems, with delay and algebraic constraint, often arise in some eugineering fields such as automatic control, electrocircuit analysis and chemical engineering. When the classical Lipschitz constants of the systems are very large, it will suffer from stiff phenomenon in the computational procedure. To solve this problem, in the presented paper, we deal with the stability of (ρ,σ)-methods for a class of nonlinear stiff systems of DDAEs. Some algebraic criteria on global stability and asymptotic stability are obtained.

2 The statement of the system

Consider nonlinear systems of DDAEs

$$\begin{cases} x'(t) = f(x(t),x(t-\tau),y(t),y(t-\tau)),\ t \geqslant 0, \\ g(x(t),y(t)) = 0,\ -\tau \leqslant t < +\infty, \\ x(t) = \varphi(t),\ -\tau \leqslant t \leqslant 0, \end{cases} \tag{1}$$

and

$$\begin{cases} u'(t) = f(u(t),u(t-\tau),v(t),v(t-\tau)),\ t \geqslant 0, \\ g(u(t),v(t)) = 0,\ -\tau \leqslant t < +\infty, \\ u(t) = \psi(t),\ -\tau \leqslant t \leqslant 0, \end{cases} \tag{2}$$

where, $\tau > 0, f: D_1 \to \widetilde{D}_1, g: D_2 \to \widetilde{D}_2 (D_1 \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}, D_2 \subseteq \mathbb{R}^{n_1 \times n_2}, \widetilde{D}_1 \subseteq \mathbb{R}^{n_1}, \widetilde{D}_2 \subseteq \mathbb{R}^{n_2})$ are continuous functions, and the Jacobi matrix $\dfrac{\partial g(x,y)}{\partial y} (\forall x,y \in D_2)$ is invertible, which means systems (1), (2) are of 1-index. By the existence the-

* Foundation item: supported by Natural Science Foundation of China (69974018).
Received date: 2000-03-15; Revised date: 2000-11-27.

orem of implicit function, we know that the algebraic constraint equations in (1) and (2) have a unique solution, respectively,

$$y(t) = \Gamma(x(t)),\ v(t) = \Gamma(u(t)),\ -\tau \leqslant t < +\infty. \tag{3}$$

Substituting (3) into (1) and (2), respectively, we obtain

$$x'(t) = f(x(t), x(t-\tau), \Gamma(x(t)), \Gamma(x(t-\tau))). \tag{4}$$

$$u'(t) = f(u(t), u(t-\tau), \Gamma(u(t)), \Gamma(u(t-\tau))). \tag{5}$$

In the following, we assume that (4) and (5) satisfy the non-classical Lipschitz condition

$$\begin{aligned}&\langle f(x_1, \bar{x}_1, \Gamma(x_1), \Gamma(\bar{x}_1)) - \\ &f(x_2, \bar{x}_2, \Gamma(x_2), \Gamma(\bar{x}_2)),\ x_1 - x_2\rangle \leqslant \\ &\alpha \| x_1 - x_2 \|^2 + \beta \| \bar{x}_1 - \bar{x}_2 \|^2 + \\ &\gamma \| f(x_1, \bar{x}_1, \Gamma(x_1), \Gamma(\bar{x}_1)) - \\ &f(x_2, \bar{x}_2, \Gamma(x_2), \Gamma(\bar{x}_2)) \|^2, \\ &\forall x_1, x_2, \bar{x}_1, \bar{x}_2 \in \mathbb{R}^{n_1},\end{aligned} \tag{6}$$

where, $\langle \cdot, \cdot \rangle$ is certain inner product in $\mathbb{R}^{n_1}$, $\| \cdot \|$ is the induced norm, and α, β and γ are some constants independent of stiff. All the problems (1) and (6) are called the class $D_A(\alpha, \beta, \gamma)$.

As an example, we consider a real d-dimension linear system

$$\begin{cases} x'(t) = Ax + Bx(t-\tau) + Cy(t) + \\ \qquad Dy(t-\tau) + E,\ t \geqslant 0, \\ y(t) = Mx(t) + N,\ -\tau \leqslant t \leqslant \infty, \end{cases} \tag{7}$$

where $A, B, C, D, E, M, N \in \mathbb{R}^{d\times d}$, $x(t) = \varphi(t)(-\tau \leqslant t \leqslant 0)$ is known. (7) can be reduced to state space form

$$\begin{aligned} x'(t) = &(A + CM)x(t) + (B + DM)x(t-\tau) + \\ &CN + DN + E. \end{aligned} \tag{8}$$

For $\forall x_i, \bar{x}_i \in \mathbb{R}^d (i = 1,2)$ we have

$$\begin{aligned}&\langle (A + CM)(x_1 - x_2) + (B + DM)(\bar{x}_1 - \\ &\bar{x}_2), x_1 - x_2\rangle - \alpha \| x_1 - x_2 \|^2 - \beta \| \bar{x}_1 - \bar{x}_2 \|^2 - \\ &\gamma \| (A+CM)(x_1 - x_2) + (B+DM)(\bar{x}_1 - \bar{x}_2) \|^2 = \\ &\ll \begin{pmatrix} x_1 - x_2 \\ \bar{x}_1 - \bar{x}_2 \end{pmatrix}, H\begin{pmatrix} x_1 - x_2 \\ \bar{x}_1 - \bar{x}_2 \end{pmatrix} \gg,\end{aligned}$$

where $\ll \cdot, \cdot \gg$ represents an inner product in $\mathbb{R}^{2d}$ defined by

$$\ll U, V \gg = \sum_{i=1}^{2} \langle u_i, v_i \rangle,\ u_i, v_i \in \mathbb{R}^d,$$

matrix

$$H = \frac{1}{2}\begin{bmatrix} [I_d - \gamma(A+CM)]^{\mathrm{T}}(A+CM) + (A+CM)^{\mathrm{T}}[I_d - \gamma(A+CM)] - 2\alpha I_d & [I - 2\gamma(A+CM)]^{\mathrm{T}}(B+DM) \\ (B+DM)^{\mathrm{T}}[I_d - 2\gamma(A+CM)] & -2\gamma(B+DM)^{\mathrm{T}}(B+DM) - 2\beta I_d \end{bmatrix}$$

and I_d is a d-dimensional identity matrix. Hence, the linear system (7) belongs to the class $D_A(\alpha, \beta, \gamma)$ iff matrix $(-H)$ is nonnegative definite.

3 Stability of one-leg (ρ, σ)-methods

Applying one-leg (ρ, σ)-methods to (4) and (5), then we obtain

$$\begin{aligned}&\rho(E)x_n = \\ &hf(\sigma(E)x_n, \sigma(E)x_{n-m}, \Gamma(\sigma(E)x_n), \Gamma(\sigma(E)x_{n-m})),\end{aligned} \tag{9}$$

$$\begin{aligned}&\rho(E)u_n = \\ &hf(\sigma(E)u_n, \sigma(E)u_{n-m}, \Gamma(\sigma(E)u_n), \Gamma(\sigma(E)u_{n-m})),\end{aligned} \tag{10}$$

where stepsize $h = \frac{\tau}{m}$, m is a positive integer, E denotes the shift operator, and there is no common divisor between $\rho(\xi) = \sum_{i=1}^{k} \alpha_i \xi^i$ and $\sigma(\xi) = \sum_{i=1}^{k} \beta_i \xi^i$ ($\alpha_i, \beta_i, \xi \in \mathbb{R}$), in which α_i and β_i are real constants with the consistency condition: $\rho(1) = 0, \rho'(1) = \sigma(1) = 1$. Moreover, x_n, u_n are approximations to $x(t_n)$ and $u(t_n)$, respectively. In particular, $x_n = \varphi(t_n), u_n = \psi(t_n)$ whenever $-m \leqslant n \leqslant 0$.

For further analysis, we introduce some notational conventions

$$\omega_n = x_n - u_n,\ W_n = (\omega_n^{\mathrm{T}}, \omega_{n+1}^{\mathrm{T}}, \cdots, \omega_{n+k-1}^{\mathrm{T}})^{\mathrm{T}},$$

$$\| \Delta \| = \sqrt{\sum_{i=1}^{k} \| \delta_i \|^2},\ \| \Delta \|_G = \sqrt{\sum_{i,j=1}^{k} g_{ij} \langle \delta_i, \delta_j \rangle},$$

where $\Delta = (\delta_1^{\mathrm{T}}, \delta_2^{\mathrm{T}}, \cdots, \delta_k^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{kn_1}$, $G = (g_{ij}) \in \mathbb{R}^{k\times k}$ is a real symmetric positive-definite matrix. Clearly, $\| \cdot \|$ and $\| \cdot \|_G$ are norms in $\mathbb{R}^{kn_1}$. Moreover, in the following we assume that each matric norm is subject to the corresponding vectoral norm.

In paper [13], for getting the numerical stability results on one-leg and linear (ρ, σ)-methods (for ODEs)

$$\rho(E)x_n = hf(\sigma(E)t_n, \sigma(E)x_n), \tag{11}$$

$$\rho(E)x_n = h\sigma(E)f(t_n, x_n), \tag{12}$$

S.F. Li generalized the concept of G-stability, proposed by G. Dalquist [15], to that of $G(c, p, q)$-stability.

Definition 3.1 Suppose that c, p, q are real constants with $c > 0$ and $pq < 1$, and there exists a $k \times k$ real symmetric positive-definite matrix $G = (g_{ij})$ such that for any real sequence $\{a_i\}_{i=0}^{k}$,

$$A_1^{\mathrm{T}} G A_1 - c A_0^{\mathrm{T}} G A_0 \leqslant$$
$$2\sigma(E) a_0 \rho(E) a_0 - p(\sigma(E) a_0)^2 - q(\rho(E) a_0)^2, \tag{13}$$

where, $A_i = (a_i, a_{i+1}, \cdots, a_{i+k-1})^{\mathrm{T}} (i = 0,1)$. Then (ρ,σ)-method (11) (or (12)) is called $G(c,p,q)$-algebraically stable. In particular, $(1,0,0)$-algebraically stable method is called G-stable.

The above concept will play an important role in the subsequent analysis. Some $G(c,p,q)$-algebraically stable ODEs methods can be found in paper [13]. In paper [5], based on G-stability and strong A-stability, C.M. Huang, et al. obtain the results on numerical stability of one-leg (ρ,σ)-method for DDEs. In this section, we relax the condition into (resp. strong) $G(c,p,q)$-algebraic stability, and obtain the global and the asymptotic stability of one-leg (ρ,σ)-methods for DDAEs.

Theorem 3.1 Suppose that one-leg (ρ,σ)-method (11) for ODEs is $G(c,p,q)$-algebraically stable with $0 < c \leqslant 1$. Then when

$$2h(\alpha + \beta) \leqslant p,\ \beta \geqslant 0,\ hq \geqslant 2\gamma, \tag{14}$$

the corresponding method (9), for problems of the class $D_A(\alpha,\beta,\gamma)$, satisfies

$$\| \omega_{n+k} \| \leqslant$$
$$\sqrt{\frac{\lambda_M^G}{\lambda_m^G}} \sum_{i=0}^{k-1} \| \omega_i \| + \sqrt{\frac{2\beta\tau}{\lambda_m^G}} \max_{-m \leqslant i \leqslant -1} \| \sigma(E) \omega_i \|, \tag{15}$$

where $n \geqslant 0$, λ_M^G and λ_m^G denote the maximum and the minimum eigenvalue of matrix G, respectively.

Proof Assume that $\{e_\mu\}$ is a group of orthonormal basis in $\mathbb{R}^{n_1}$. Write

$$\omega_n = \sum_{\mu=1}^{\infty} \alpha_n^\mu e_\mu,$$
$$A_i^\mu = (\alpha_{n+i}^\mu, \alpha_{n+i+1}^\mu, \cdots, \alpha_{n+i+k-1}^\mu)^{\mathrm{T}},\ i = 0,1.$$

It follows from $G(c,p,q)$- algebraic stability, (6) and (14) that

$$\| W_{n+1} \|_G^2 - c \| W_n \|_G^2 =$$
$$\sum_{\mu=1}^{\infty} [(A_1^\mu)^{\mathrm{T}} G A_1^\mu - c(A_0^\mu)^{\mathrm{T}} G A_0] \leqslant$$
$$2\langle \sigma(E)\omega_n, \rho(E)\omega_n \rangle -$$
$$p \| \sigma(E)\omega_n \|^2 - q \| \rho(E)\omega_n \|^2 \leqslant$$
$$(2\alpha h - p) \| \sigma(E)\omega_n \|^2 +$$
$$2\beta h \| \sigma(E)\omega_{n-m} \|^2 + \left(\frac{2\gamma}{h} - q\right) \| \rho(E)\omega_n \|^2 \leqslant$$
$$(2\alpha h - p) \| \sigma(E)\omega_n \|^2 + 2\beta h \| \sigma(E)\omega_{n-m} \|^2,$$

by which we have

$$\| W_{n+1} \|_G^2 \leqslant$$
$$c \| W_n \|_G^2 + (2\alpha h - p) \| \sigma(E)\omega_n \|^2 +$$
$$2\beta h \| \sigma(E)\omega_{n-m} \|^2. \tag{16}$$

An induction to (16) yields

$$\| W_{n+1} \|_G^2 \leqslant$$
$$c^{n+1} \| W_0 \|^2 + (2\alpha h - p) \sum_{i=1}^{n} c^i \| \sigma(E)\omega_{n-i} \|^2 +$$
$$2\beta h \sum_{i=0}^{n} c^i \| \sigma(E)\omega_{n-m-i} \|^2. \tag{17}$$

Further, with (17), (14) and condition $0 < c \leqslant 1$ we arrive at

$$\| W_{n+1} \|_G^2 \leqslant$$
$$\| W_0 \|_G^2 + (2\alpha h - p) \sum_{i=0}^{n} \| \sigma(E)\omega_{n-i} \|^2 +$$
$$2\beta h \sum_{i=m}^{n+m} \| \sigma(E)\omega_{n-i} \|^2 \leqslant$$
$$\| W_0 \|_G^2 + (2\alpha h - p) \sum_{i=0}^{n} \| \sigma(E)\omega_{n-i} \|^2 +$$
$$2\beta h \sum_{i=0}^{n+m} \| \sigma(E)\omega_{n-i} \|^2 \leqslant$$
$$\| W_0 \|_G^2 + 2\beta h \sum_{i=n+1}^{n+m} \| \sigma(E)\omega_{n-i} \|^2 =$$
$$\| W_0 \|_G^2 + 2\beta h \sum_{i=-m}^{-1} \| \sigma(E)\omega_i \|^2.$$

Hence

$$\| W_{n+1} \|_G^2 \leqslant \| W_0 \|_G^2 + 2\beta\tau \max_{-m \leqslant i \leqslant -1} \| \sigma(E)\omega_i \|^2, \tag{18}$$

by which it follows

$$\lambda_m^G \| \omega_{n+k} \|^2 \leqslant$$
$$\lambda_M^G \sum_{i=0}^{k-1} \| \omega_i \|^2 + 2\beta\tau \max_{-m \leqslant i \leqslant -1} \| \sigma(E)\omega_i \|^2.$$

Therefore, (15) holds.

The inequality (15) shows that the difference between the numerical solutions of DDAEs (1) and (2) is bounded by the initial values of systems and methods. Hence method (9) is globally stable. In the following, for getting the asymptotic stability of the methods, we introduce

Definition 3.2 The method (11) (or (12)) is called strong $G(c,p,q)$- algebraically stable if this method is $G(c,p,q)$- algebraically stable and the root modulus of $\sigma(\xi)$ is less than 1.

Remark 3.1 Since G-stability is equivalent to A-stability (cf.[15]), strong $G(1,0,0)$- algebraic stability is equivalent to strong A-stability (i.e.the method is A-stability and the root modulus of $\sigma(\xi)$ is less than 1).

A combination of Theorem 105B and Theorem 123D in J. C.Butcher [14] yields.

Lemma 3.1 Given matrix $\tilde{A} \in \mathbb{C}^{k\times k}$ and sequence $v_n \in \mathbb{C}^k$. Then the solution sequence of linear difference equation

$$y_n = \tilde{A} y_{n-1} + v_n$$

satisfies

$$\lim_{n\to\infty} \| y_n \| = 0,$$

iff spectrum radius $r(\tilde{A}) < 1$ and $\lim_{n\to\infty} v_n = 0$.

Theorem 3.2 Suppose that one-leg (ρ,σ)-method (11) for ODEs is strong $G(c,p,q)$-algebraically stable with $0 < c \leqslant 1$ and $\beta_k \neq 0$. Then when

$$2h(\alpha+\beta) < p,\ \beta \geqslant 0,\ hq \geqslant \gamma, \tag{19}$$

the corresponding method (9) for the class $D_A(\alpha,\beta,\gamma)$ is asymptotically stable, i.e.

$$\lim_{n\to\infty} \| \omega_n \| = 0.$$

Proof A slight modification to (18) yields

$$\| W_{n+1} \|_G^2 + [p-2h(\alpha+\beta)] \sum_{i=0}^{n} \| \sigma(E)\omega_{n-i} \|^2 \leqslant$$
$$\| W_0 \|_G^2 + 2\beta\tau \max_{-m\leqslant i\leqslant -1} \| \sigma(E)\omega_i \|^2.$$

Hence

$$[p-2h(\alpha+\beta)] \sum_{i=0}^{n} \| \sigma(E)\omega_i \|^2 \leqslant$$
$$\| W_0 \|_G^2 + 2\beta\tau \max_{-m\leqslant i\leqslant -1} \| \sigma(E)\omega_i \|^2. \tag{20}$$

From (20) and $2h(\alpha+\beta) < p$ it follows

$$\lim_{n\to\infty} \| \sigma(E)\omega_n \| = 0. \tag{21}$$

Write

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{\beta_0}{\beta_k} & -\dfrac{\beta_1}{\beta_k} & -\dfrac{\beta_2}{\beta_k} & \cdots & -\dfrac{\beta_{k-1}}{\beta_k} \end{pmatrix} \otimes I_{n_1},$$

$$K_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{\sigma(E)\omega_n}{\beta_k} \end{pmatrix}.$$

Then

$$W_{n+1} = JW_n + K_n. \tag{22}$$

By (21), we know that $\lim_{n\to\infty} K_n = 0$. Moreover, in terms of the strong $G(c,p,q)$- algebraically stability of the methods we infer that the root modulus of $\sigma(\xi)$ is less than 1. Thus, the spectrum radius $r(J) < 1$. Further, by Lemma 3.1 it follows $\lim_{n\to\infty} \| W_{n+1} \| = 0$. Therefore, $\lim_{n\to\infty} \| \omega_n \| = 0$.

With Theorem 3.1,3.2 and Remark 3.1, we further have

Corollary 3.1 Supposed one-leg (ρ,σ)-method for ODEs is A-stable. Then, when

$$0 \leqslant \beta \leqslant -\alpha,\ \gamma \leqslant 0, \tag{23}$$

the corresponding method (9) for the class $D_A(\alpha,\beta,\gamma)$ satisfies (15), which means the global stability.

Corollary 3.2 Supposed one-leg (ρ,σ)-method for ODEs is strong A-stable. Then, when

$$0 \leqslant \beta \leqslant -\alpha,\ \gamma \leqslant 0, \tag{24}$$

the corresponding method (9) for the class $D_A(\alpha,\beta,\gamma)$ is asymptotically stable.

4 Stability of linear (ρ,σ)-methods

solving (4) and (5) by linear (ρ,σ)-methods yields

$$\rho(E)X_n = h\sigma(E)f(X_n, X_{n-m}, \Gamma(X_n), \Gamma(X_{n-m})), \tag{25}$$

$$\rho(E)U_n = h\sigma(E)f(U_n, U_{n-m}, \Gamma(U_n), \Gamma(U_{n-m})), \tag{26}$$

where X_n and U_n stand *for* the approximations $x(t_n)$ and $u(t_n)$, respectively, $X_n = \varphi(t_n)$ and $U_n = \psi(t_n)$ whenever $-m \leqslant n \leqslant 0$, and the other notations are the same as that of (9) and (10).

A similar argumentation to Lemma 4.2.1 of S. F. Li[13] yields the following generalized result.

Lemma 4.1 Suppose that sequence $\{\tilde{X}_n\}$ satisfies (25) (resp.(26)). Then sequence

$$\tilde{x}_n = p(E)\tilde{X}_n + hq(E)f(\tilde{X}_n, \tilde{X}_{n-m}, \Gamma(\tilde{X}_n), \Gamma(\tilde{X}_{n-m})) \tag{27}$$

satisfies (9) (resp. (10)) and

$$\widetilde{X}_n = \sigma(E)\bar{x}_n. \tag{28}$$

Oppositely, suppose that sequence $\{\bar{x}_n\}$ satisfies (9) (resp.(10)). Then sequence $\{\widetilde{X}_n\}$ defined by (28) satisfies (25) (resp. (26)) and (27), where polynomials $p(\xi) = \sum_{j=0}^{k-1} p_j \xi^j$ and $q(\xi) = \sum_{j=0}^{k-1} q_j \xi^j$ fulfil

$$p(\xi)\sigma(\xi) + q(\xi)\rho(\xi) = 1.$$

Theorem 4.1 Suppose that linear (ρ,σ)-method (12) for ODEs is $G(c,p,q)$-algebraically stable and $0 < c \leqslant 1$. Then when (14) holds, the corresponding method (25) for the class $D_A(\alpha,\beta,\gamma)$ satisfies

$$\|\Omega_{n+k}\| \leqslant \|\sigma(E)\| \left(\sqrt{\frac{\lambda_M^G}{\lambda_m^G}} \sum_{i=0}^{k-1} \|\widetilde{\Omega}_i\| + \sqrt{\frac{2\beta\tau}{\lambda_m^G}} \max_{-m\leqslant i\leqslant -1} \|\Omega_i\|\right), \tag{29}$$

where

$$\Omega_i = X_i - U_i,$$

$$\widetilde{\Omega}_i = p(E)\Omega_i + hq(E)[f(X_i, X_{i-m}, \Gamma(X_i), \Gamma(X_{i-m})) - f(U_i, U_{i-m}, \Gamma(U_i), \Gamma(U_{i-m}))].$$

Proof By (27), we can determine sequences $\{x_n\}$ and $\{u_n\}$. From Lemma 4.1 and Theorem 3.1 it follows

$$\|\Omega_{n+k}\| \leqslant \|\sigma(E)\| \, \|\omega_{n+k}\| \leqslant \|\sigma(E)\| \left(\sqrt{\frac{\lambda_M^G}{\lambda_m^G}} \sum_{i=0}^{k-1} \|\omega_i\| + \sqrt{\frac{2\beta\tau}{\lambda_m^G}} \max_{-m\leqslant i\leqslant -1} \|\Omega_i\|\right). \tag{30}$$

whereas

$$\omega_i = p(E)\Omega_i + hq(E)[f(X_i, X_{i-m}, \Gamma(X_i), \Gamma(X_{i-m})) - f(U_i, U_{i-m}, \Gamma(U_i), \Gamma(U_{i-m}))] = \widetilde{\Omega}_i.$$

Accordingly, this completes the proof of Theorem 4.1.

The inequality characterizes the global stability of methods (25).

Theorem 4.2 Suppose that linear (ρ,σ)-method (12) for ODEs is strong $G(c,p,q)$-algebraically stable with $0 < c \leqslant 1$ and $\beta_k \neq 0$. Then, when (19) holds, the corresponding method (25) for the class $D_A(\alpha,\beta,\gamma)$ is asymptotically stable. i.e. $\lim_{n\to\infty} \|\Omega_n\| = 0$.

Proof By Lemma 4.1 and Theorem 3.2 it follows

$$\|\Omega_n\| \leqslant \|\sigma(E)\| \, \|\omega_n\| \to 0, (n \to \infty).$$

Hence this theorem is proved.

With Theorem 4.1, 4.2 and Remark 3.1, we further have

Corollary 4.1 Supposed linear (ρ,σ)-method (12) for ODEs is A-stable. Then, when (23) holds, the corresponding method (25) for the class $D_A(\alpha,\beta,\gamma)$ satisfies (29).

Corollary 4.2 Supposed linear (ρ,σ)-method (12) for ODEs is strong A-stable. Then, when (24) holds, the corresponding method (25) for the class $D_A(\alpha,\beta,\gamma)$ is asymptotically stable.

References

[1] Hairer E and Wanner G. Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems [M]. Berlin: Springer-Verlag, 1993

[2] Jackiewicz Z. Quasilinear multistep methods and variable step predict-corrector methods for neutral delay-differential equations [J]. SIAM J. Numer. Anal., 1986,23(2):423-452

[3] Zenarro M. Asymptotic stability analysis of Runge-Kutta methods for nonlinear systems of delay differential equations [J]. Numer. Math.,1997,77(4):549-563

[4] Zhang C J and Zhou S Z. Nonlinear stability and D-convergence of Runge-Kutta methods for delay differential equations [J]. J. Comput. Appl. Math.,1997,85(2):225-237

[5] Zhang C J and Zhou S Z. The asymptotic stability of the theoretical and numerical solutions for systems of neutral multidelay-differential equations [J]. Science in China, 1998,41(11):1151-1157

[6] Huang C M, Fu H Y, Li S F, et al. Stability and error analysis of one-leg methods for nonlinear delay differential equations [J]. J. Comput. Appl. Math.,1999,103(2):263-279

[7] Xiao A. Error analysis of numerical methods for several classes of nonlinear stiff differential equations [D]. Beijing: China Academy of Engineering Physics, 1999

[8] Petzold L R. Order results for implicit Runge-Kutta methods applied differential-algebraic systems [J]. SIAM J. Numer. Anal.,1986,23(4):837-852

[9] Arnold M. Stability of numerical methods for differential-algebraic equations of higher index [J]. Appl. Numer. Math.,1993,13(1):5-14

[10] Brenan K E and Petzold L R. The numerical solutions of higher index differential-algebraic equations by implicit methods [J]. SIAM J. Numer. Anal., 1989,26(4):976-996

[11] Ascher U M and Petzold L R. The numerical solutions of delay-differential-algebraic equations of retarded and neutral type [J]. SIAM J. Numer. Anal.,1995,32(5):1635-1657

[12] Zhu W and Petzold L R. Asymptotic stability of linear delay-differential-algebraic equations [J]. Appl. Numer. Math.,1997,24(1):247-264

[13] Li S F. Theory of Computational Methods for Stiff Differential

Equtions [M]. Changsha: Hunan Press of Science and Technology, 1997

[14] Butcher J C. The Numerical Analysis of Ordinary Differential Equations [M]. Chichester: John Wiley and Sons, 1987

[15] Dalquist G. *G*-stability is equivalent to *A*-stability [J]. BIT Numerical Methods, 1978, 18(3): 384 – 401

本文作者简介

ZHANG Chengjian Born in 1963. He received the B.Sc. and the M.Sc. degree from Xiangtan University in 1986 and 1992, respectively, and the Ph. D degree from Hunan University in 1998. From 1998 to 2000 he worked at Department of Automatic Control and Engineering as a post-doctor in Huazhong University of Science and Technology (HUST). Now he is acting as a Professor and the Vice-Head of Department of Mathematics in HUST, and a commissioner of the Chinese Speciality Committee for Simulation Algorithms. Dr. Zhang has published more than forty papers in Science in China, Appl. Num. Math., Computers Math. Appl. and the other key journals at home and abroad, and won the second achievement award in science and technology's progress of Chinese Mechanical Industry Ministry in 1997. At present, his research topic is the digital simulation algorithms of stiff delay systems

LIAO Xiaoxin Born in 1938. He graduated from Wuhan University in 1963. From 1978 to 1986 he once acted as Lecturer, Associate Professor and Professor, serially, in Huazhong Normal University and won the honors such as the Middle-Young Specialist with the Great Contribution and Chinese Excellent Teacher. Since 1995 he has been worked in HUST as a Professor and an Advisor of Doctorial student. Moreover, he has published more than 100 papers in several key journals at home and abroad and therefore won the second achievement award in science and technology's progress of Chinese Eductorial Committee for twice. Currently, his research interest is in the nonlinear control and the mathematical theory of neural net.

(Continued form page 826)

[12] Hahn W. Theory and Application of Lyapunov's Direct Method [M]. New York: Prentice-Hall, Englewood Cliffs, 1963, 193 – 227

[13] Chen P, Qin H and Zhu Q. Dynamic output feedback stabilization of partially linear composite system [A]. Chen Hanfu, Cheng Daizhan and Zhang Jifeng. ed. Preprints of the 14th World Congress of IFAC, Vol. E [C], Pergamon: Elsevier Science, 1999, 345 – 350

[14] Carr J Applications of Centre Manifold Theory [M]. New York: Springer-Verlag, 1981, 3 – 30

[15] Teel A and Praly L. Global stabilizability and observability imply semi-global stabilizability by output feedback [J]. Systems Control Letter, 1994, 22(5): 313 – 325

本文作者简介

陈彭年 1948年生.中国计量学院教授.目前研究方向为稳定性理论和非线性系统镇定.

秦化淑 1934年生.1956年毕业于南开大学数学系.1961年在波兰雅盖龙大学获博士学位.现为中国科学院系统科学研究所教授.从事过常微分方程,线性与非线性系统控制及其在导弹制导和机器人控制的应用等方面的研究工作.目前对非线性系统结构性质、镇定,机器人控制、不确定系统控制等有兴趣.