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General Linear Quadratic Optimal Control for Periodically Time-Varying Linear Systems *

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Abstract: This paper deals with the general linear quadratic optimal control (LQOC) problem for periodically time-varying (PTV) linear systems, i.e. the LQOC when the state equation is non-homogenous and the quadratic functional criterion contains linear terms. A series of necessary and sufficient conditions for the solvability of the LQOC problem are given, the optimal control is constructed and the optimal criterion value is given.

Key words: periodically time-varying linear systems; linear quadratic optimal control; Riccati matrix differential equations; Hamiltonian differential equations; linear matrix differential inequality

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周期时变线性系统的一般线性二次型最优控制

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摘要:讨论周期时变线性系统的一般线性二次型最优控制问题,即状态方程为非齐次方程且二次型性能指标包含线性项的一般情况.给出了该问题可解的一系列充分必要条件,同时给出了最优控制的解析构造以及最优性能指标值.

关键词: 周期时变线性系统; 线性二次型最优控制; Riccati 矩阵微分方程; Hamiltonian 微分方程; 线性矩阵微分不等式

1 Introduction

In recent decades, periodically time-varying linear system has absorbed many researchers' attention. Some basic problems, including linear quadratic optimal control[1], optimal filtering[2] etc., have been studied. In this paper, we deal with the general linear quadratic optimal control (LQOC) problem, i.e. the LQOC when the state equation is non-homogenous and the quadratic functional criterion contains linear terms. A series of necessary and sufficient conditions for the solvability of the LQOC problem are given, the optimal control is constructed and the optimal criterion value is given. The obtained results are the generalization of those in [1].

2 General linear quadratic optimal control for PTV linear systems

Consider the following PTV linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + B(t)u + f(t),\ x(0) = a,\ t \geqslant 0 \tag{1}$$

and quadratic functional criterion

$$\begin{cases} J[u] = \int_0^\infty F(t, x(t), u(t))\mathrm{d}t, \\ F(t,x,u) = F_q(t,x,u) + F_l(t,x,u), \\ F_q(t,x,u) = \frac{1}{2}(x^* G(t)x + 2x^* g(t)u + u^* \Gamma(t)u), \\ F_l(t,x,u) = x^* k(t) + u^* d(t), \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and control variables, respectively; matrix functions $A(t)$, $B(t)$, $G(t) = G^*(t)$, $g(t)$, $\Gamma(t) = \Gamma^*(t)$ have appropriate dimensions and their entries are real periodic with period T, bounded and measurable functions; the vectors fun-

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ction $f(t), k(t), d(t)$ have appropriate dimensions and are Lebesque square integrable, i.e. $f(\cdot) \in L_2[0, \infty), k(\cdot) \in L_2[0, \infty), d(\cdot) \in L_2[0, \infty)$. The asterisk $*$ denotes the conjugate transpose operator (in the real valued case, it stands for transposition); $|\cdot|$ is the Euclidean norm; $\|\cdot\|_2$ is the norm of Lebesque square integrable function-vector space $L_2[0, \infty)$. The LQOC problem is to find (if exists) a control $u(t)$ such that the criterion $J[u]$ achieves its minimal value over the set $\{x(t), u(t)\}$ of the processes defined by (1).

Definition 1 Matrix function $A(\cdot)$ is called to be Hurwitz stable, if there exist numbers $c > 0$ and $\varepsilon > 0$ such that for any solution $x(t)$ of the equation $\mathrm{d}x/\mathrm{d}t = A(t)x$ the inequality

$$|x(t)| \leqslant c \exp(-\varepsilon(t-s)) |x(s)|$$

holds for any $t \geqslant s$. The pair $(A(\cdot), B(\cdot))$ is called to be L_2-stabilizable if for any vector a and any function $u(t) \in L_2[0, \infty)$ the solution $x(t)$ to equation (1), in which $f(t) = 0$, satisfies $x(\cdot) \in L_2[0, \infty)$. Some equivalent definitions about stabilizability can be found in [1,3].

Associated with the LQOC problem (1),(2) we can derive the following canonical differential equation

$$J \frac{\mathrm{d}z}{\mathrm{d}t} = H(t)z, \tag{3}$$

where

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad H(t) = \begin{bmatrix} H_{11}(t) & H_{21}{}^*(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix},$$

$$H(t) = H^*(t) = H(t+T).$$

I_n is an $n \times n$ unit matrix, each partition $H_{ij}(t)$ is an $n \times n$ matrix. By the well-known Floquet-Lyapunov theorem[4], (3) has a fundamental matrix $Z(t) = F(t)\mathrm{e}^{tK}$, where $F(t) = F(t+T), F(0) = I_{2n}, K$ is constant matrix (in general, $F(t), K$ may be complex matrices). The equation (3) is said to meet frequency condition[1] if

$$\det[Z(T) - \mathrm{e}^{\mathrm{i}\omega} I_{2n}] \neq 0 \ (\forall \omega \in [0, 2\pi)), \ \mathrm{i} = \sqrt{-1}. \tag{4}$$

If the condition (4) holds, the matrix $Z(T)$ has n-eigenvalues inside and outside the open unit circle, respectively. Let $L^{(+)}, L^{(-)}$ be the eigen-subspaces of C^{2n}, corresponding respectively to the eigenvalues of $Z(T)$ inside and outside the open unit circle. Thus, $C^{2n} = L^{(+)} \oplus L^{(-)}$. Construct matrices S^+, S^- such that their columns compose the bases of $L^{(+)}$ and $L^{(-)}$, respectively. Forming matrix $S = [S^+ \quad S^-]$, we have

$$Z(t)S = F(t)S(S^{-1}\mathrm{e}^{tK}S) =: F_1(t)\begin{bmatrix} \mathrm{e}^{tN_1} & 0 \\ 0 & \mathrm{e}^{-tN_2} \end{bmatrix},$$

where $F_1(t) = F(t)S$, both N_1 and N_2 are $n \times n$ Hurwitz constant matrices. Hence, for every nonsingular $n \times n$ matrix C, the columns $z_i(t) = \mathrm{col}[x_i(t), \psi_i(t)]$ $(i = 1, 2, \cdots, n)$ of matrix

$$\begin{bmatrix} X(t) \\ \Psi(t) \end{bmatrix} =: F_1(t)\begin{bmatrix} \mathrm{e}^{tN_1}C \\ 0 \end{bmatrix} =$$

$$F_1(t)\begin{bmatrix} C\mathrm{e}^{tC^{-1}N_1C} \\ 0 \end{bmatrix} =: \begin{bmatrix} P_1(t)\mathrm{e}^{tN_0} \\ P_2(t)\mathrm{e}^{tN_0} \end{bmatrix} \tag{5}$$

compose n linearly independent solutions with property $|z_i(t)| \to 0$ as $t \to 0$. If the following condition

$$\det X(t) \neq 0 \ (\forall t) \tag{6}$$

is satisfied we say the equation (3) is nonoscillatory. One can show that nonoscillatory condition (6) is well-defined, i.e. it is independent of the choice of matrix C. Now we consider the relation between (3) and the following Riccati equation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = [I_n \quad -R]H(t)\begin{bmatrix} I_n \\ -R \end{bmatrix}. \tag{7}$$

Lemma 1 If (3) satisfies the frequency condition (4) (thus, matrices $X(t), \Psi(t)$ can be defined as in (5)) and nonoscillatory condition (6), the matrix $R = -\Psi X^{-1} = -P_2P_1^{-1}$ is uniquely defined by $H(t)$ and is the unique T-periodic symmetric matrix solution of (7) such that $H_{21} - H_{22}R$ is a Hurwitz matrix. Inversely, if (7) admits a T-periodic symmetric matrix solution R such that $H_{21} - H_{22}R$ is Hurwitz, (3) satisfies the conditions (4) and (6).

Proof Suppose (3) satisfies the frequency condition (4) and nonoscillatory condition (6). Let $X(t), \Psi(t)$ be defined as in (5). One can verify that $R = -\Psi X^{-1}$ is the solution with the above stated property. In fact, multiplying the equation

$$J\begin{bmatrix} \dot{X} \\ \dot{\Psi} \end{bmatrix} = H\begin{bmatrix} X \\ \Psi \end{bmatrix} \tag{8}$$

left by $[I_n \quad \Psi X^{-1}]$ and right by X^{-1} we have that

$$\Psi X^{-1}\dot{X}X^{-1} - \dot{\Psi}X^{-1} = [I_n \quad \Psi X^{-1}]H\begin{bmatrix} I_n \\ \Psi X^{-1} \end{bmatrix}. \tag{9}$$

One easily verifies the left side of the identity (9) as $-d(\Psi X^{-1})/\mathrm{d}t$. Let $R = -\Psi X^{-1}$, we conclude from (9) that (7) admits the solution $R = -\Psi X^{-1}$. The solution is T-periodic and its symmetry can be inferred from the property that $Z^*(t)JZ(t) \equiv 0$ (see [4]). Finally, from (8) we obtain that $\dot{X} = (H_{21} - H_{22}R)X$. It has the fundamental solution $X = P_1 e^{tN_0}$. Thus, $H_{21} - H_{22}R$ is a Hurwitz matrix. Conversely, if (7) admits a solution R with the property stated above, the equation $\dot{X} = (H_{21} - H_{22}R)X$ has a solution of the form $X(t) = P_1(t)e^{tN_0}$ in which $P_1(t)$ is a T-periodic nonsingular matrix and N_0 is a constant Hurwitz matrix. Let $\Psi = -RX$. Thus, one easily verifies that the columns of $\mathrm{col}[X \quad \Psi]$ are n linearly independent solutions of (3). It indicates that (3) meets the conditions (4) and (6). The proof is complete.

The following theorem gives a complete solution of the LQOC problem described by (1), (2).

Theorem 1 Let the pair $(A(\cdot), B(\cdot))$ be L_2-stabilizable and $\Gamma(t) \geqslant \gamma I_m > 0$ ($\forall t$). Then the following statements are equivalent to each other (for the sake of simply writing, time t is often omitted):

Ⅰ) For arbitrary initial states $x(0) = a$ the LQOC problem (1), (2) is solvable, i.e., there exists an optimal control $u(t)$ (and in fact it is even unique).

Ⅱ) The canonical equation (3) with the matrix $H(t)$ defined as follows

$$H(t) = \begin{bmatrix} -G + g\Gamma g^* & A^* - g\Gamma^{-1}B^* \\ A - B\Gamma^{-1}g^* & B\Gamma^{-1}B^* \end{bmatrix} \qquad (10)$$

satisfies the frequency condition (4) and nonoscillatory condition (6).

Ⅲ) The Riccati differential equation (7) with the matrix $H(t)$ defined in (10) admits an absolutely continuous matrix solution $R(t) = R^*(t) = R(t+T)$ such that $\bar{B} =: A + Br^*$ is a Hurwitz matrix function, where

$$r = -[RB + g]\Gamma^{-1}. \qquad (11)$$

Ⅳ) There is a quadratic form

$$V(t,x) = x^* R(t)x + 2s^*(t)x + \sigma(t).$$

Here $R(t)$ is T-periodic, symmetric and absolutely continuous matrix function, both $s(t)$ and $\sigma(t)$ are absolutely continuous and Lebesque square integrable functions over $[0, \infty)$ such that the identity

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} + 2F = |\Gamma^{1/2}(u - r^*x - \rho)|^2 (\forall t, x, u) \qquad (12)$$

holds and $\bar{B} =: A + Br^*$ is Hurwitz, where $r(t)$ is defined as in (11) and $\rho(t)$ as follows

$$\rho = -\Gamma^{-1}[B^*s + d]. \qquad (13)$$

Ⅴ) There exists a number $\delta > 0$ such that, for any process (x, u) satisfying the relation $\dot{x} = A(t)x + B(t)u, x(0) = 0$ and being Lebesque square integrable, i.e., $(x(\cdot), u(\cdot)) \in L_2[0, \infty)$, the following inequality holds

$$\int_0^\infty F_q \mathrm{d}t \geqslant \delta \int_0^\infty (|x|^2 + |u|^2)\mathrm{d}t.$$

Ⅵ) There exists a number $\delta > 0$ such that, for any Lebesque square integrable complex vector-value functions $x(t), u(t)$ and any real number ω satisfying condition $0 \leqslant \omega < 2\pi$ and relation

$$\dot{x} = A(t)x + B(t)u, \; x(T) = e^{i\omega}x(0),$$

the following inequality holds

$$\int_0^T F_q \mathrm{d}t \geqslant \delta \int_0^T (|x|^2 + |u|^2)\mathrm{d}t.$$

Ⅶ) There exists a quadratic form $V_0(t,x) = x^* R_0(t)x$ with T-periodic, symmetric and absolutely continuous function $R_0(t)$ and a real number $\delta > 0$ such that the inequality

$$\frac{\mathrm{d}V_0}{\mathrm{d}t} + 2F_q \geqslant \delta(|x|^2 + |u|^2) \; (\forall t, x, u)$$

holds, where $\frac{\mathrm{d}V_0}{\mathrm{d}t}$ is the Lyapunov derivative along the trajectory of the system $\dot{x} = A(t)x + B(t)u$. Or equivalently, the following inequality admits a T-periodic, symmetric and absolutely continuous function $R_0(t)$

$$\begin{bmatrix} \dot{R}_0 + R_0A + A^*R_0 + G & R_0B + g \\ R^*R_0 + g^* & \Gamma \end{bmatrix} \geqslant \delta \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}.$$

Suppose that one of the conditions Ⅰ) ~ Ⅶ) holds. Then the matrix $R(t)$ in Ⅲ) and Ⅳ) is defined uniquely and has the following formula $R(t) = -\Psi(t)X^{-1}(t)$, where $X(t), \Psi(t)$ are given in (5) for the equation (3) with coefficient matrix (10). The optimal control is the state feedback

$$u(t) = r^*(t)x(t) + \rho(t)$$

and the optimal criterion value is

$$\min_u J = \frac{1}{2}a^* R(0)a + s^*(0)a + \frac{1}{2}\sigma(0),$$

where $r(t)$ is defined in (11) and $\rho(t)$ in (13). Functions $s(t)$ and $\sigma(t)$ are absolutely continuous and Lebesque square integrable on $[0, \infty)$ and are defined as follows

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\bar{B}^* s - Rf - k - rd,\ \bar{B} = A + Br^*,$$
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \rho^* \Gamma\rho - 2s^* f. \tag{14}$$

Proof Ⅰ)$\Rightarrow$Ⅱ). If Condition Ⅰ) holds, according to the well known Pontryagin's maximum principle one can conclude that the following equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\partial \mathscr{H}}{\partial \psi}\right)^*,\ \frac{\mathrm{d}\psi}{\mathrm{d}t} = \left(\frac{\partial \mathscr{H}}{\partial x}\right)^*,\ \left(\frac{\partial \mathscr{H}}{\partial u}\right) = 0 \tag{15}$$

have solutions $x(t), \psi(t) \in L_2[0,\infty)$ for any $x(0) = a$, where $\mathscr{H}$ is the Hamiltonian

$$\mathscr{H}(t,x,u,\psi) = \psi^*(A(t)x + B(t)u + f(t)) - F(t,x,u).$$

Noticing that $\Gamma(t) \geqslant \gamma I_m > 0$, by the third equation in (15) we have that

$$u = \Gamma^{-1}(B^*\psi - g^*x - d).$$

Substituting it into the first and second equation of (15) we derive

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -G + g\Gamma^{-1}g^* & A^* - g\Gamma^{-1}B^* \\ A - B\Gamma^{-1}g^* & B\Gamma^{-1}B^* \end{bmatrix}\begin{bmatrix} x \\ \psi \end{bmatrix} + \begin{bmatrix} g\Gamma^{-1}d - k \\ -B\Gamma^{-1}d + f \end{bmatrix}. \tag{16}$$

Thus, the equation (16) has a solution $x(t), \psi(t) \in L_2[0,\infty)$ for arbitrary initial state $x(0) = a$. On the other hand, (16) admits the solution stated above for arbitrary $x(0) = a$ if and only if the equation (3) with the matrix $H(t)$ defined in (10) has solutions with the same property for arbitrary $x(0) = a$. It indicates that if for arbitrary initial states $x(0) = a$, the LQOC problem (1),(2) is solvable, so is the homogeneous LQOC problem (the case $f \equiv 0, k \equiv 0, d \equiv 0$). Hence, by the analogous line of deduction as in [1] (see Lemmas 1 ~ 9 in [1]) one can verify Ⅰ)$\Rightarrow$Ⅱ). The equivalence of the conditions Ⅱ) and Ⅲ) has been stated in Lemma 1. Ⅲ)$\Rightarrow$Ⅳ) is easily verified by defining quadratic form as in Ⅳ), where $R(t)$ is the solution given in Ⅲ) and the functions $s(t), \sigma(t)$ are defined in (14). The equivalence between the conditions Ⅱ), Ⅲ) and Ⅴ) ~ Ⅶ) can be obtained from Theorem 2 in [1] since the non-homogenous LQOC problem (1),(2) and the homogeneous LQOC problem have the same solvability. Finally, from Ⅳ) we derive that

$$V(\infty) - V(0) + 2J = \int_0^\infty |\Gamma^{1/2}(u - rx - \rho)|^2 \mathrm{d}t.$$

It indicates that the optimal control is $u = r^*x + \rho$ and the optimal criterion value is $\min_u J = \frac{1}{2}V(0)$.

3 Conclusion

This paper gives a very general solution for the linear quadratic optimal control problem where the state equation is non-homogenous and the quadratic functional criterion contains linear terms. A series of necessary and sufficient conditions for the solvability of the problem are given. These conditions are represented in terms of Hamiltonian equation, Riccati matrix differential equation, the linear matrix differential inequality or the others. The optimal control is constructed and the optimal criterion value is given as well. The general result of LQOC for time-invariant linear system is easily derived from Theorem 1 and herein is omitted due to the limit of the paper's length.

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