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# Linearization and Stability of Generalized One-Dimensional Delay Discrete Logistic Systems *

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**Abstract**: Linearization and stability of all solutions of the generalized one-dimensional delay discrete logistic system $x_{n+1} - \mu_n x_n + \sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{\alpha_i} = 0$ are investigated. Some sufficient conditions for the stability of this equation are derived.

**Key words**: generalized delay discrete logistic system; linearization; stability

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# 一维时滞离散广义 Logistic 系统的线性化和稳定性

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**摘要**: Logistic 系统是一经典的离散非线性系统,它的混沌,以及混沌的各种控制和其它渐近性质已经有了大量的研究,本文主要研究了一类更为广泛的一维时滞离散 Logistic 系统 $x_{n+1} - \mu_n x_n + \sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{\alpha_i} = 0$ 的线性化和稳定性,得到了系统所有解稳定的一些充分条件.

**关键词**: 时滞离散广义 Logistic 系统; 线性化; 稳定性

## 1 Introduction

In recent years, there have been a lot of studies on the dynamical properties such as chaos and synchronization as well as other asymptotic behavior of the one-dimensional logistic system

$$x_{n+1} = \mu x_n(1 - x_n) \tag{1.1}$$

and its continuous counterpart

$$\frac{dx}{dt} = x(t)[\mu(1 - x(t)) - 1],$$

where $\mu \in \mathbb{R}$. Some results obtained in [1 ~ 6] are collected in the book of Gyori and Ladas[1].

In this paper, we consider the generalized one-dimensional delay logistic system

$$x_{n+1} - \mu_n x_n + \sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{\alpha_i} = 0, \tag{E}$$

where $\mu_{i,n}, \mu_n$, are positive real parameters, $\alpha_i$ are real numbers, $\sigma_i$ are nonnegative integers, $i = 1, 2, \cdots$, and $u$ is a positive integer, $i, j \in N_0 = \{0, 1, 2, \cdots\}$.

When $\alpha_1 = 1, \mu_n = \mu, \mu_{1,n} = -\mu, u = 1, \sigma_1 = 0$, system (E) becomes (1.1). Let

$\sigma = \max\{\sigma_i \mid 1 \leqslant i \leqslant u\}$ and $N_0 = \{0, 1, 2, \cdots\}$.

Our aim in this paper is to establish some linearization results as well as some stability criteria for all the solutions of system (E).

Moreover, we need some concepts and notation. First, for a given function $\varphi(i)$ defined on $\Omega = \{-\sigma, -\sigma + 1, \cdots, -1\}$, it is easy to construct a unique solution of system (E) by induction a sequence $\{x_i\}$, which equals $\varphi(i)$ on $\Omega$ and satisfies Eq. (E) for $i = 0, 1, 2, \cdots$. Indeed, we can write Eq. (E) in the form

$$x_{n+1} = -\sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{\alpha_i} + \mu_n x_n$$

and then use it to successively calculate

$$x_1, x_2, x_3, \cdots.$$

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Such a sequence is unique and is said to be a solution of Eq. (E) subject to the initial condition

$$x_i = \varphi(i),\ i \in \Omega.$$

**Definition 1** A solution $\{x_n\}$ of (1.1) is said to be eventually positive (negative) if $x_n > 0$ ($< 0$) for all large enough $n$; it is said to be oscillatory if it is neither eventually positive nor eventually negative.

**Definition 2** Eq. (E) is said to be $\gamma$-exponentially stable if, for any $\lambda > 0$ and $\omega > 1$, $|x_0| < \gamma$, such that

$$|x_n| \leqslant \omega |x_0| e^{-\lambda n},\ n \in \Omega.$$

It is clear that Eq. (E) can be regarded as a discrete analog of the functional differential equation

$$x'(t) = \lambda x(t)((1 - x(t-\sigma)) - 1. \tag{E$_0$}$$

Therefore, qualitative properties of Eq. (E) may provide useful information for this delay differential equation.

## 2 Linearization results of equation (E)

For $s \in N_0$ such that

$$s > n,\ \text{for all } n \in N_0, \tag{2.1}$$

let

$$C_s = \mu_n^{n+1-s} x_s + \left(\frac{1}{2}\sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i}\right)^{s-n} + \left[-\left(\sum_{i=1}^{u} \mu_{i,n} + \mu_n\right) x_{n-\sigma}\right]^{s-n} - 2. \tag{2.2}$$

The following result can be established based on (2.2):

**Theoerm 1** Assume that (2.1) holds and that $\{x_n\}$ is an eventually positive solution of Eq. (E) with $0 < \mu_{i,n}, \mu_n \leqslant 1$. Then,

i) $C_n$ is a monotone decreasing sequence in $n$, that is,

$$C_{n+1} \leqslant C_n. \tag{2.3}$$

ii) $C_n$ is eventually positive, and $C_n \leqslant x_n$.

iii) Eq. (E) reduces to the linear inequality

$$C_{n+1} - C_n \leqslant -\mu_n x_{n-\sigma}.$$

Proof i) From (2.2), we obtain

$$C_{n+1} = x_{n+1} + \frac{1}{2}\sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i} - \left(\sum_{i=1}^{u} \mu_{i,n} + \mu_n\right) x_{n-\sigma} - 2, \tag{2.4}$$

$$C_n = \mu_n x_n + 1 + 1 - 2 = \mu_n x_n. \tag{2.5}$$

Since $x_n > 0$, we have

$$C_{n+1} - C_n = \frac{1}{2}\sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i} + x_{n+1} - \mu_n x_n - \left(\sum_{i=1}^{u} \mu_{i,n} + \mu_n\right) x_{n-\sigma} - 2 \leqslant$$

$$\sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i} + x_{n+1} - \mu_n x_n - \left(\sum_{i=1}^{u} \mu_{i,n} + \mu_n\right) x_{n-\sigma} - 2 \leqslant$$

$$\sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i} - \sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i} - \left(\sum_{i=1}^{u} \mu_{i,n} + \mu_n\right) x_{n-\sigma} - 2 \leqslant$$

$$-\mu_n x_{n-\sigma} \leqslant 0,$$

that is, $C_{n+1} - C_n \leqslant -\mu_n x_{n-\sigma} \leqslant 0$.

ii) From (2.5), we immediately know that $C_n$ is eventually positive. Note also that $0 < \mu_n < 1$. So we have

$$C_n = \mu_n x_n \leqslant x_n. \tag{2.6}$$

iii) Using the above $C_{n+1}$ and $C_n$, we get the following linear inequality:

$$C_{n+1} - C_n + \mu_n x_{n-\sigma} \leqslant 0. \tag{2.7}$$

The proof is thus completed.

## 3 Stability of equation (E)

**Theorem 2** Suppose that $\mu_{i,n} \leqslant \mu_{i,0}, \mu_n \leqslant \mu_0$, $i = 1,2,\cdots,u$ and

$$\mu_0 + \sum_{i=1}^{u} \mu_{i,0}\gamma^{a_i - 1} < 1. \tag{3.1}$$

If $\max\{|\psi_{-\sigma+1}|,\cdots,|\psi_0|\} \leqslant \gamma$, then the solution $\{x_n\}_{-\sigma}^{\infty}$ of Eq. (E), determined by $x_i = \psi_i, \sigma + 1 \leqslant i \leqslant 0$, satisfies

$$|x_n| < \gamma e^{-n\lambda},\ n \in \Omega,$$

where $\lambda$ is some positive number.

Proof Assume that $a_i \geqslant 0, i = 1,2,\cdots,u$, and let $\lambda$ be a positive number such that

$$e^{\lambda}\left(\mu_0 + \sum \mu_{i,0}\gamma^{a_i - 1}\right) < 1. \tag{3.2}$$

First, note that Eq. (E) can be reduced to

$$x_{n+1} = \mu_n x_n - \sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i} \tag{3.3}$$

and that $\max\{|\psi_{-\sigma+1}|,\cdots |\psi_0| \leqslant \gamma$. Next, note that

$$|x_1| e^{\lambda} \leqslant e^{\lambda}\left(\mu_0 |x_0| + \sum_{i=1}^{u} \mu_{i,0} |x_{-\sigma_i}^{a_i}|\right) \leqslant$$

$$\gamma e^{\lambda}\left(\mu_0 + \sum_{i=1}^{u} \mu_{i,0}\gamma^{a_i - 1}\right) < \gamma.$$

Thus, $|x_1| < \gamma e^{-\lambda}$.

Assume, by induction, that

$$|x_n| e^{n\lambda} < \gamma,\ n = 1,2,\cdots,m.$$

Then,

$$|x_{m+1}| e^{(m+1)\lambda} \leqslant$$

$e^{\lambda} e^{m\lambda}\left( | \mu_{m-1} x_{m-1} | + \left| \sum_{i=1}^{u} \mu_{i,m-1} x_{m-1-\sigma_i}^{\alpha_i} \right| \right) \leqslant$

$e^{\lambda} e^{(m-1)\lambda}\left( \mu_{m-1} | x_{m-1} | + \sum_{i=1}^{u} \mu_{i,m-1} | x_{m-1-\sigma_i} |^{\alpha_i} \right) \leqslant$

$e^{\lambda} (\mu_{m-1} | e^{(m-1)\lambda} x_{m-1} | +$

$\sum_{i=1}^{u} \mu_{i,m-1} | e^{(m-1)\lambda} x_{m-1-\sigma_i} || x_{m-1-\sigma_i} |^{\alpha_i - 1}) \leqslant$

$\gamma e^{\lambda} (\mu_0 + \sum_{i=1}^{u} \mu_{i,0} \gamma^{\alpha_i - 1}) \leqslant \gamma,$

as required. The proof is thus completed.

**Example** Consider the equation

$$x_{n+1} - \frac{5}{e^2 + n} x_n + \frac{1}{e^2(e^4 + n)} x_{n-1}^2 + \frac{1}{e^2}\left( \frac{5}{e^2 + n} - \frac{1}{e} - \frac{1}{e^n(e^4 + n)} \right) x_{n-2} = 0, \; n \in \Omega, \quad (3.4)$$

where $\mu_n = \frac{1}{e^2 + n}, \mu_{1,n} = \frac{1}{e^2(e^4 + n)}, \mu_{2,n} = \frac{1}{e^2}\left( \frac{5}{e^2 + n} - \frac{1}{e} - \frac{1}{e^n(e^4 + n)} \right), \sigma_1 = 1, \sigma_2 = 2,$ $\alpha_1 = 2, \alpha_2 = 1$, since $\mu_n \leqslant \frac{1}{e^2}, \mu_{1,n} \leqslant \frac{1}{e^6}, \mu_{2,n} \leqslant \frac{5}{e^4}$, and

$$\sum_{i=1}^{2} \mu_{0,i} \gamma^{\alpha_i - 1} + \mu_0 = \frac{1}{e^6}\gamma + \frac{5}{e^4} + \frac{1}{e^2} < 1 \quad (3.5)$$

holds for $0 < \gamma < e^6(1 - \frac{5}{e^4} - \frac{1}{e^2})$. Thus, Theorem 2 asserts that if $\max \{ | \psi_{-\sigma+1} |, \cdots, | \psi_0 | \} \leqslant \gamma < e^6$, then the solution $\{x_n\}$ of (3.4), determined by $x_i = \psi_i, -\sigma + 1 \leqslant i \leqslant 0$, will also satisfy $\lim_{n \to \infty} x_n = 0$. In fact, $x_n = \frac{1}{e^n}$ is one such solution of (3.4).

The following stability result follows from Theorem 2.

**Theorem 3** Assume that one of the following two conditiions is satisfied:

i) There exists a positive constant $\alpha_0$ such that

$$\mu_n \geqslant \alpha_0 \text{ for } n \geqslant \sigma; \quad (3.6)$$

ii)

$$\sum_{i=\sigma}^{\infty} \mu_i = \infty. \quad (3.7)$$

Then, every positive solution of Eq. (E) tends to zero as $n \to \infty$.

Proof It suffices to show that every eventually positive solution $\{x_n\}$ of Eq. (E) tends to zero as $n \to \infty$.

By Theorem 1, $\{C_n\}$ is eventually decreasing and positive. Hence,

$$\lim_{n \to \infty} C_n = \eta \in \mathbb{R}^+, \quad (3.8)$$

where $\mathbb{R}^+ = [0, \infty)$. Also by Theorem 1, it is easy to see that

$$C_{n+1} - C_n \leqslant - \mu_n x_{n-\sigma}. \quad (3.9)$$

Taking $n$ sufficiently large, and summing both sides of (3.9) from $n_1 \to \infty$, we get

$$\lim_{n \to \infty} \sum_{i=n_1}^{n} (C_{i+1} - C_i) \leqslant - \sum_{i=n_1}^{\infty} \mu_i x_{i-\sigma}. \quad (3.10)$$

Applying (3.8), we have

$$\eta - C_{n_1} \leqslant - \sum_{i=n_1}^{\infty} \mu_i x_{i-\sigma}. \quad (3.11)$$

Now, if (3.6) holds, then (3.11) implies that

$$\sum_{i=n_1}^{\infty} \mu_i x_{i-\sigma} < \infty.$$

Since $x_{i-\sigma}$ is a positive solution of Eq. (E), for $i \geqslant n_1$, by (3.11), we have

$$\lim_{n \to \infty} x_n = 0.$$

The proof is completed when (3.7) holds.

Next we assume that (3.7) indeed holds. It follows from (3.11) that

$$\liminf_{n \to \infty} x_n = 0.$$

Also by (2.5), $C_n = \mu_n x_n$, and in view of (3.8), $\eta = 0$.

We now claim that $\{x_n\}$ is a bounded sequence. Otherwise, there would exist a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that

$$x_{n_r} = \max\{x_{n-\sigma} \mid n \leqslant n_r + \sigma \text{ for } r = 1,2,\cdots\} \text{ and } \lim_{n \to \infty} x_{n_r} = \infty.$$

By this, (2.5), and (3.7), we have

$$C_{n_r} = \mu_{n_r} x_{n_r} \to \infty \text{ as } r \to \infty,$$

which contradicts $\eta = 0$, leading to the claimed result.

Let $\lambda = \limsup_{n \to \infty} x_n$ and let $\{x_{n_s}\}$ be a subsequence of $\{x_n\}$ such that $\lim_{s \to \infty} x_{n_s} = \lambda$. Then, for sufficiently small $\varepsilon > 0$ and sufficiently large $s$, it follows from (2.5) that

$$C_{n_s} = \mu_{n_s} x_{n_s}.$$

Taking a limit as $s \to \infty$ and using the fact that $\eta = 0$, we obtain $0 = \lambda \lim_{s \to \infty} \mu_{n_s}$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lambda = 0$. The proof is thus completed.

## 4 Conclusion

In this paper, we have investigated the linearization and stability of all solutions of the generalized one-dimensional delay discrete logistic system $x_{n+1} - \mu_n x_n + \sum_{i=1}^{u} \mu_{i,n} x_{n-\sigma_i}^{a_i} = 0$, and obtain some sufficient conditions for this equation to be stable.

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