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Extension and Proof of the Continuous-Time Dynamic Replication Theorem *

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Abstract: This paper extends the continuous-time dynamic replication theorem for incomplete Markets, which is proposed by Bertsimas, Kogan and Lo (1997)[1]. Then this extended dynamic replication theorem is proved using the theory of the stochastic optimal control.

Key words: dynamic replication; incomplete markets; continuous-time; stochastic optimal control

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连续时间动态复制定理的推广与证明

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摘要: 推广了由 Bertsimas, Kogan and Lo (1997)[1]提出的非完全市场中的连续时间动态复制定理, 然后, 我们运用随机最优控制理论证明了这个定理.

关键词: 动态复制; 非完全市场; 连续时间; 随机最优控制

1 Introduction

In financial risk management, pricing and hedging derivative security is an important problem (see [2, 3]). Perfect hedging is impossible in incomplete markets. To evaluate contingent claims in incomplete markets, researchers have proposed many new concepts. Schweizer (1992, 1995)[4,5] solved the dynamic replication problem with a mean-squared-error loss function under the probability measure of the original price process. Recently, Bertsimas, Kogan and Lo (1997)[1] proposed an interesting term ε-arbitrage to replicate and price options in incomplete markets. Although Schweizer considered the general stochastic processes, Bertsimas, Kogan and Lo focused only on Markov price processes and used various principles to characterize the optimal replication strategy. The Markov assumption allows them to obtain thorough results. However, Bertsimas, Kogan and Lo's result is derived under the assumption that riskless interest rate is zero, and they did not prove the theorem. In this paper, we extend their result to the case that riskless interest rate is not zero. Finally, this extended dynamic replication theorem is proved using the theory of the stochastic optimal control.

2 Problem

Consider an asset with price P_t at time t, where $0 \leqslant t \leqslant T$. Let $F(P_t, Z_t)$ denote the payoff of a European derivative security at maturity date T. It is a function of P_T and some other variables Z_T.

As suggested by Merton (1973)[6], the derivation of the Black-Scholes formula is to find a dynamic hedging strategy-purchase and sale of the stock and the riskless asset on $[0, T]$. The strategy is supposed to be self-financing and to come as close as possible to the payoff $F(P_T, Z_T)$ at T. To formulate the dynamic replication problem more precisely, we begin with the following assumptions:

H1) Markets are frictionless, i.e., there are no taxes, transaction costs, short sale restrictions and borrowing restrictions.

H2) The riskless borrowing and lending rate is r.

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H3) There exists a finite-dimensional vector Z_t of state variables whose components are not perfectly correlated with the prices of any traded securities, and $[P_t, Z_t]'$ is a vector Markov process.

Consider a portfolio consisting of the stock and the riskless bond with initial value V_0 at time 0. Let θ_t, B_t and V_t denote respectively the number of the shares of the stock held in the portfolio, the dollar value of the bonds, and the market value of the portfolio at time t, where $t \in [0, T]$. We have

$$V_t = \theta_t P_t + B_t. \tag{1}$$

In addition, we impose a condition that after time 0, the portfolio is self-financing, i.e., all long positions in one asset are completely financed by short positions in the other asset so that the portfolio experiences no cash inflows or outflows. In continuous time, this implies that

$$\begin{aligned} \mathrm{d}V_t = {} & [rV_t + (\mu_0(t, P_t, Z_t) - r)\theta_t P_t]\mathrm{d}t + \\ & \theta_t \sigma_0(t, P_t, Z_t) P_t \mathrm{d}W_{0t}, \end{aligned} \tag{2}$$

where $r, \mu_0(t, P_t, Z_t), \sigma_0(t, P_t, Z_t)$, are called the interest rate, the appreciation rate and the volatility, respectively; W_{0t} are Wiener processes.

We seek a self-financing hedging strategy $\{\theta_t\}$, $t \in [0, T]$, such that the terminal value V_T of the portfolio is as close as possible to the option's payoff $F(P_t, Z_t)$. While there are many criteria to measure the "closeness", and each given rise to a different dynamic replication problem, we choose a mean-squared-error loss function. Hence our version of the dynamic replication problem is (see [7,8])

$$\min_{\theta_t} E^v[V_T - F(P_T, Z_T)]^2 \tag{3}$$

subject to the self-financing condition (2), the dynamics of $[P_t, Z_t]'$ and the initial wealth V_0. The expectation E^v is taken with respect to a probability measure v that represents the randomness of the difference $V_T - F(P_T, Z_T)$, conditional on information at time 0.

3 Statement of the theorem

For the continuous-time case, let $[P_t, Z_t]'$ follow a vector Markov diffusion process

$$\mathrm{d}P_t = \mu_0(t, P_t, Z_t) P_t \mathrm{d}t + \sigma_0(t, P_t, Z_t) P_t \mathrm{d}W_{0t}, \tag{4}$$

$$\begin{aligned} \mathrm{d}Z_{jt} = {} & \mu_j(t, P_t, Z_t) Z_{jt}\mathrm{d}t + \sigma_j(t, P_t, Z_t) Z_{jt}\mathrm{d}W_{jt}, \\ & j = 1, 2, \cdots, N, \end{aligned} \tag{5}$$

where w_{jt}, $j = 0, 1, \cdots, N$ are Wiener processes with mutual variation $\mathrm{d}W_{jt}\mathrm{d}W_{kt} = \rho_{jk}(t, P_t, Z_t)\mathrm{d}t$.

The continuous-time Bellman principle is the Hamilton-Jacobi-Bellman equation (see [9, 10]), and this yields the following results:

Theorem Under Assumptions H1) ~ H3) and (2), if the value function $J(t, V_t, P_t, Z_t)$ is quadratic with respect to V_t, i.e., there are functions $a(t, P_t, Z_t)$, $b(t, P_t, Z_t)$ and $c(t, P_t, Z_t)$ such that

$$\begin{aligned} J(t, V_t, P_t, Z_t) = {} & a(t, P_t, Z_t)[V_t - b(t, P_t, Z_t)]^2 + \\ & c(t, P_t, Z_t), \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{6}$$

then the solution of the dynamic replication problem (3) is characterized by the following conditions:

i) For $t \in [0, T]$ the functions $a(t, P_t, Z_t)$, $b(t, P_t, Z_t)$ and $c(t, P_t, Z_t)$ satisfy the following system of partial differential equations

$$\begin{aligned} & \frac{\partial a}{\partial t} + \sum_{j=0}^{N} \mu_j Z_j \frac{\partial a}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N} \sigma_i \sigma_j Z_i Z_j \rho_{ij} \frac{\partial^2 a}{\partial Z_i \partial Z_j} = \\ & \left((\frac{\mu_0 - r}{\sigma_0})^2 - 2r \right) a + \frac{2(\mu_0 - r)}{\sigma_0} \sum_{j=0}^{N} \sigma_j Z_j \rho_{0j} \frac{\partial a}{\partial Z_j} + \\ & \frac{1}{a} \sum_{i,j=0}^{N} \sigma_i \sigma_j Z_i Z_j \rho_{0i} \rho_{0j} \frac{\partial a}{\partial Z_i} \frac{\partial a}{\partial Z_j}, \end{aligned} \tag{7}$$

$$\begin{aligned} & \frac{\partial b}{\partial t} + \sum_{j=0}^{N} \mu_j Z_j \frac{\partial b}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N} \sigma_i \sigma_j Z_i Z_j \rho_{ij} \frac{\partial^2 b}{\partial Z_i \partial Z_j} = \\ & rb + \frac{\mu_0 - r}{\sigma_0} \sum_{j=0}^{N} \sigma_j Z_j \rho_{0j} \frac{\partial b}{\partial Z_j} + \\ & \frac{1}{a} \sum_{i,j=0}^{N} \sigma_i \sigma_j Z_i Z_j (\rho_{0i}\rho_{0j} - \rho_{ij}) \frac{\partial a}{\partial Z_i} \frac{\partial b}{\partial Z_j}, \end{aligned} \tag{8}$$

$$\begin{aligned} & \frac{\partial c}{\partial t} + \sum_{j=0}^{N} \mu_j Z_j \frac{\partial c}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N} \sigma_i \sigma_j Z_i Z_j \rho_{ij} \frac{\partial^2 c}{\partial Z_i \partial Z_j} = \\ & a \sum_{i,j=0}^{N} \sigma_i \sigma_j Z_i Z_j (\rho_{0i}\rho_{0j} - \rho_{ij}) \frac{\partial b}{\partial Z_i} \frac{\partial b}{\partial Z_j} \end{aligned} \tag{9}$$

with boundary conditions

$$\begin{cases} a(T, P_T, Z_T) = 1, \ b(T, P_T, Z_T) = F(P_T, Z_T), \\ c(T, P_T, Z_T) = 0, \end{cases} \tag{10}$$

where Z_i denotes the i-th component of Z_t and $Z_0 \equiv P_t$.

ii) The optimal control $\theta^*(t, V_t, P_t, Z_t)$ is linear in V_t and is given by

$$\begin{aligned} & \theta^*(t, V_t, P_t, Z_t) = \\ & \sum_{j=0}^{N} \frac{\sigma_j Z_j}{\sigma_0 Z_0} \rho_{0j} \frac{\partial b}{\partial Z_j} - \frac{V_t - b}{a} \sum_{j=0}^{N} \frac{\sigma_j Z_j}{\sigma_0 Z_0} \rho_{0j} \frac{\partial a}{\partial Z_j} - \end{aligned}$$

$$\frac{\mu_0 - r}{\sigma_0^2 Z_0}(V_t - b). \tag{11}$$

4 Proof of the theorem

Proof First we define value function of the stochastic optimal control problems (2), (4), (5) and (3) as follows

$$J(t, V_t, P_t, Z_t) = \min_{\theta(\cdot)} E^v[V_T - F(P_T, Z_T)]^2. \tag{12}$$

Applying the stochastic optimal control theory to value function $J(t, V_t, P_t, Z_t)$, we obtain the following partial differential equation

$$\frac{\partial J(t,V_t,P_t,Z_t)}{\partial t} + \min_{\theta(\cdot)}\Big\{\sum_{j=0}^{N}\mu_j Z_j \frac{\partial J(t,V_t,P_t,Z_t)}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 J(t,V_t,P_t,Z_t)}{\partial Z_i \partial Z_j} + [rV_t + (\mu_0 - r)\theta(t,V_t,P_t,Z_t)Z_0]\frac{\partial J(t,V_t,P_t,Z_t)}{\partial V_t} + \frac{1}{2}(\theta(t,V_t,P_t,Z_t)\sigma_0 Z_0)^2\frac{\partial^2 J(t,V_t,P_t,Z_t)}{\partial V_t^2} + \theta(t,V_t,P_t,Z_t)\sum_{j=0}^{N}\sigma_0\sigma_j\rho_{0j}Z_0 Z_j\frac{\partial^2 J(t,V_t,P_t,Z_t)}{\partial V_t \partial Z_j}\Big\} = 0 \tag{13}$$

with boundary condition

$$J(T, V_T, P_T, Z_T) = [V_T - F(P_T, Z_T)]^2. \tag{14}$$

Let

$$J(t,V_t,P_t,Z_t) = a(t,P_t,Z_t)[V_t - b(t,P_t,Z_t)]^2 + c(t,P_t,Z_t),\ 0 \leqslant t \leqslant T. \tag{15}$$

Using equations (7) and (10), we can check that function $a(\cdot)$ is positive. Therefore the first-order condition is sufficient for the minimum in (13). We have

$$\theta^* = -\frac{(\mu_0 - r)\frac{\partial J}{\partial V_t}}{\sigma_0^2 Z_0 \frac{\partial^2 J}{\partial V_t^2}} - \frac{1}{\sigma_0 Z_0 \frac{\partial^2 J}{\partial V_t^2}}\sum_{j=0}^{N}\sigma_j Z_j\rho_{0j}\frac{\partial^2 J}{\partial V_t \partial Z_j}. \tag{16}$$

Substitute (15) into (16) to obtain

$$\theta^* = -\frac{(\mu_0 - r)}{\sigma_0^2 Z_0}(V_t - b) - \frac{1}{\sigma_0 Z_0 a}\sum_{j=0}^{N}\sigma_j Z_j\rho_{0j}\Big[\frac{\partial a}{\partial Z_j}(V_t - b) - a\frac{\partial b}{\partial Z_j}\Big]. \tag{17}$$

Rearranging the terms we can get

$$\theta^* = \sum_{j=0}^{N}\frac{\sigma_j Z_j}{\sigma_0 Z_0}\rho_{0j}\frac{\partial b}{\partial Z_j} - \frac{V_t - b}{a}\cdot\sum_{j=0}^{N}\frac{\sigma_j Z_j}{\sigma_0 Z_0}\rho_{0j}\frac{\partial a}{\partial Z_j} - \frac{\mu_0 - r}{\sigma_0^2 Z_0}(V_t - b), \tag{18}$$

which indicates that (11) holds.

Apply the optimal strategy (16) to (13), one obtains

$$\frac{\partial J}{\partial t} + \sum_{j=0}^{N}\mu_j Z_j\frac{\partial J}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 J}{\partial Z_i\partial Z_j} + rV_t\frac{\partial J}{\partial V_t} - \frac{1}{2}\frac{(\mu_0 - r)^2(\frac{\partial J}{\partial V_t})^2}{\sigma_0^2\frac{\partial^2 J}{\partial V_t^2}} - \frac{1}{2}\frac{1}{\frac{\partial J}{\partial V_t^2}}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{0i}\rho_{0j}\frac{\partial^2 J}{\partial V_t\partial Z_i}\frac{\partial^2 J}{\partial V_t\partial Z_j} - \frac{(\mu_0 - r)\frac{\partial J}{\partial V_t}}{\sigma_0\frac{\partial^2 J}{\partial V_t^2}}\sum_{j=0}^{N}\sigma_j Z_j\rho_{0j}\frac{\partial^2 J}{\partial V_t\partial Z_j}. \tag{19}$$

Substituting (15) into (19) and rearranging the terms to obtain

$$\Big[\frac{\partial a}{\partial t} + \sum_{j=0}^{N}\mu_j Z_j\frac{\partial a}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 a}{\partial Z_i\partial Z_j} + 2ar - \Big(\frac{\mu_0 - r}{\sigma_0}\Big)^2 a - \frac{2(\mu_0 - r)}{\sigma_0}\sum_{j=0}^{N}\sigma_j Z_j\rho_{0j}\frac{\partial a}{\partial Z_j} - \frac{1}{a}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{0i}\rho_{0j}\frac{\partial a}{\partial Z_i}\frac{\partial a}{\partial Z_j}\Big](V_t - b)^2 + 2a\Big[-\frac{\partial b}{\partial t} - \sum_{j=0}^{N}\mu_j Z_j\frac{\partial b}{\partial Z_j} + rb - \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 b}{\partial Z_i\partial Z_j} - \frac{1}{a}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial a}{\partial Z_i}\frac{\partial b}{\partial Z_j} + \frac{1}{a}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{0i}\rho_{0j}\frac{\partial b}{\partial Z_i}\frac{\partial b}{\partial Z_j} - \frac{\mu_0 - r}{\sigma_0}\sum_{j=0}^{N}\sigma_j Z_j\rho_{0j}\frac{\partial b}{\partial Z_j}\Big](V_t - b) + \frac{\partial c}{\partial t} + \sum_{j=0}^{N}\mu_j Z_j\frac{\partial c}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 c}{\partial Z_i\partial Z_j} + a\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial b}{\partial Z_i}\frac{\partial b}{\partial Z_j} - a\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{0i}\rho_{0j}\frac{\partial b}{\partial Z_i}\frac{\partial b}{\partial Z_j} = 0. \tag{20}$$

Since (20) holds for all $t \in [0, T]$, we have

$$\frac{\partial a}{\partial t} + \sum_{j=0}^{N}\mu_j Z_j\frac{\partial a}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j\rho_{ij}\frac{\partial^2 a}{\partial Z_i\partial Z_j} +$$

$$2ar - \left(\frac{\mu_0 - r}{\sigma_0}\right)^2 a - \frac{2(\mu_0 - r)}{\sigma_0}\sum_{j=0}^{N}\sigma_j Z_j \rho_{0j}\frac{\partial a}{\partial Z_j} -$$

$$\frac{1}{a}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j \rho_{0i}\rho_{0j}\frac{\partial a}{\partial Z_i}\frac{\partial a}{\partial Z_j} = 0, \tag{21}$$

$$-\frac{\partial b}{\partial t} - \sum_{j=0}^{N}\mu_j Z_j\frac{\partial b}{\partial Z_j} - \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j \rho_{ij}\frac{\partial^2 b}{\partial Z_i \partial Z_j} -$$

$$\frac{1}{a}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j \rho_{ij}\frac{\partial a}{\partial Z_i}\frac{\partial b}{\partial Z_j} + rb +$$

$$\frac{1}{a}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j \rho_{0i}\rho_{0j}\frac{\partial a}{\partial Z_i}\frac{\partial b}{\partial Z_j} +$$

$$\frac{\mu_0 - r}{\sigma_0}\sum_{j=0}^{N}\sigma_j Z_j \rho_{0j}\frac{\partial b}{\partial Z_j} = 0, \tag{22}$$

$$\frac{\partial c}{\partial t} + \sum_{j=0}^{N}\mu_j Z_j\frac{\partial c}{\partial Z_j} + \frac{1}{2}\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j \rho_{ij}\frac{\partial^2 c}{\partial Z_i \partial Z_j} +$$

$$a\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j \rho_{ij}\frac{\partial b}{\partial Z_i}\frac{\partial b}{\partial Z_j} -$$

$$a\sum_{i,j=0}^{N}\sigma_i\sigma_j Z_i Z_j \rho_{0i}\rho_{0j}\frac{\partial b}{\partial Z_i}\frac{\partial b}{\partial Z_j} = 0. \tag{23}$$

The results of (7), (8) and (9) can be obtained by rearranging terms in (21), (22) and (23).

By boundary condition (14) we can obtain (10) immediately.

5 Conclusion

In this paper, we extend a method for replication derivative securities in dynamically incomplete market. Using the theory of the stochastic optimal control, we construct a self-financing dynamic portfolio strategy that best approximates an arbitrary payoff function in a mean-squared sense. When riskless interest rate is zero, our optimal hedging strategy coincides with the results of Bertsimas, Kogan and Lo, which are special case of our research.

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