Article ID: 1000 - 8152(2002)03 - 0471 - 04

On Covariance Upper Bound Control Problem with Finite-Precision Consideration *

XU Weihua, WU Jun and CHU Jian

(National Laboratory of Industrial Control Technology, Institute of Advanced Process Control, Zhejiang University · Hangzhou, 310027, P. R. China)

Abstract: A covariance upper bound control problem is formulated for digital finite-precision controllers with synchronous sampling and fixed-point arithmetic. The finite-precision covariance upper bound controller design problem is reduced to the problem of solving some matrix inequalities such that the closed-loop system is asymptotically stable.

Key words: finite-precision controller; covariance upper bound control; linear matrix inequality (LMI); stabilization; digital control theory

Document code: A

有限精度的协方差上界控制问题

徐巍华 吴 俊 褚 健

(浙江大学工业控制技术国家重点实验室,先进控制研究所·杭州,310027)

摘要: 研究一个采用定点算法的同步采样有限精度数字控制器的协方差上界控制问题,并把这个有限精度协方差上界控制器设计问题简化为解一组矩阵不等式,使闭环系统是渐近稳定的.

关键词: 有限精度控制器; 协方差上界控制; 线性矩阵不等式; 镇定; 数字控制理论

1 Introduction

Finite-precision controller design has been an important issue in modern control theory and engineering with the recent advances in fixed-point implementation of digital controller. Improved control performance and increased levels of integration are especially important in many application areas, such as consumer electronic products, automotive and electromechanical control systems. The fixed-point arithmetic offers the advantages of speed, memory space, cost and simplicity over floating-point arithmetic[1]. However, due to finite-word-length (FWL) effects, a performance degradation of the closed-loop system usually occurs since the infinite-precision controller is implemented using a fixed-point processor.

In recent years many results have been reported in the literature dealing with the issues of FWL controller implementation. It is well known that there exists an optimal realization of a given controller[2,3], so that the synthesis in these optimal coordinates will minimize a proposed coefficient sensitivity measure or the noise gain from round-off effects. A controller designed without regard to controller synthesis can then be implemented in an optimal realization for this given controller. It is also known that such controllers are not optimal overall. That is, the design and synthesis problems are not independent problems.

Some improvements are made in [4], where the finite-word-length covariance control has been studied. All dynamic controllers which assign state covariances to the closed-loop system are characterized in the presence of quantization error in the control computer and in the A/D and D/A devices. It also presents a general design idea, that is, the control problems are reduced to a problem in linear algebra.

It is well known that upper bounds on performances might be more useful than the more difficult task of assigning exact performance[4]. In many cases, the "sma-

* Foundation item: supported by National Natural Science Foundation of China (60174026) and National Outstanding Youth Science Foundation of China (60025308).

Received date: 2000 - 12 - 04; Revised date: 2001 - 09 - 24.

ller" the output covariance, the better the system performance. This provides motivation to consider a performance specification given in terms of a bound on the output covariance. Mathematically, this allows for the use of inequality constraints in lieu of equality constraints on covariance matrices[5]. In this paper, we consider the control design problem when the controller synthesized in a digital computer with synchronous sampling and fixed-point arithmetic. It is shown that the problem is reduced to a problem of solving some matrix inequalities.

2 Problem formulation

Consider the linear time-invariant system and the controller (with the assumption of infinite precision implementation)

$$\begin{bmatrix} x_p(k+1) \\ y_p(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_p & D_p & B_p \\ C_p & D_y & 0 \\ M_p & D_z & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ w_p(k) \\ u(k) \end{bmatrix}, \quad (1)$$

$$\begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(k) \\ z(k) \end{bmatrix}, \quad (2)$$

where x_p and x_c are the plant and the controller states, respectively, y_p is output of interest, w_p is the finite energy disturbance, z and u are the measured output and the control input, respectively.

However, in most application cases, the controller is synthesized in a digital computer with finite wordlength and fixed-point arithmetic, and we must take the quantization errors into consideration. It is well known that the effects of the quantization error in the control computer depend on the realization of the controller[2,3]. To this end, we shall study the control design problem in a transformed set of controller parameters $A_c = T_c^{-1}\tilde{A}_c T_c$, $B_c = T_c^{-1}\tilde{B}_c$, $C_c = \tilde{C}_c T_c$, $D_c = \tilde{D}_c$ with $\det T_c \neq 0$ and write the controller dynamics

$$\begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(k) + e_x(k) \\ z(k) + e_z(k) \end{bmatrix}, \quad (3)$$

where $e_x(k)$ is the quantization error introduced by the controller state computation $x_c(k)$ in the control computer (with wordlength β_x), and $e_z(k)$ is the quantization error introduced by the A/D converter (with wordlength β_z). The plant is described by

$$\begin{bmatrix} x_p(k+1) \\ y_p(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_p & D_p & B_p \\ C_p & D_y & 0 \\ M_p & D_z & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ w_p(k) \\ u(k) + e_u(k) \end{bmatrix}, \quad (4)$$

where $e_u(k)$ is the quantization error introduced by the D/A converter (with wordlength β_u). Under sufficient excitation conditions we can approximate the quantization errors $e_x(k), e_z(k)$ and $e_u(k)$ to be zero-mean white noise processes

$$W_x = \mathrm{diag}[\cdots\ q_i\ \ \cdots],\ W_z = q_z I,\ W_u = q_u I,$$

respectively, where $q_i = (1/12)2^{-2\beta_i}$, $q_z = (1/12)2^{-2\beta_z}$, $q_u = (1/12)2^{-2\beta_u}$, and β_i is the length of the fractional part of the word storing the i-th controller state variable[4].

Define the matrix

$$W = \begin{bmatrix} q_u I & 0 & 0 & 0 \\ 0 & W_p & 0 & 0 \\ 0 & 0 & W_z + V & 0 \\ 0 & 0 & 0 & T_c W_x T_c^{\mathrm{T}} \end{bmatrix},$$

where W_p and V are the covariance of the plant noise $w_p(k)$ and the measurement noise $D_z w_p(k)$, respectively. The closed-loop system is described by

$$\begin{bmatrix} x_p(k+1) \\ x_c(k+1) \end{bmatrix} = \begin{bmatrix} A_p + B_p D_c M_p & B_p C_c \\ B_c M_p & A_c \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_c(k) \end{bmatrix} + \begin{bmatrix} B_p & D_p & B_p D_c & B_p C_c \\ 0 & 0 & B_c & A_c \end{bmatrix} \begin{bmatrix} e_u \\ w_p \\ e_z + D_z w_p \\ e_x \end{bmatrix}, \quad (5)$$

$$y_p(k) = [C_p \quad 0] \begin{bmatrix} x_p(k) \\ x_c(k) \end{bmatrix} + [0\ D_y\ 0\ 0] \cdot \begin{bmatrix} e_u \\ w_p \\ e_z + D_z w_p \\ e_x \end{bmatrix}, \quad (6)$$

or simply

$$x(k+1) = (A + BGM)x(k) + (D + BGE)w(k), \quad (7)$$

$$y(k) = Cx(k) + Fw(k) \quad (8)$$

with

$$A = \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} B_p & 0 \\ 0 & I \end{bmatrix},\ C = [C_p \quad 0],$$

$$D = \begin{bmatrix} B_p & D_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = [0 \ D_y \ 0 \ 0],$$

$$M = \begin{bmatrix} M_p & 0 \\ 0 & I \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad T = \begin{bmatrix} I & 0 \\ 0 & T_c \end{bmatrix},$$

$$\tilde{G} = \begin{bmatrix} D_c & \tilde{C}_c \\ \tilde{B}_c & \tilde{A}_c \end{bmatrix}, \quad G = T^{-1}\tilde{G}T = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix},$$

and

$$x = [x_p^{\mathrm{T}} \quad x_c^{\mathrm{T}}]^{\mathrm{T}},$$

$$w = [e_u^{\mathrm{T}} \quad w_p^{\mathrm{T}} \quad e_z^{\mathrm{T}} + D_z^{\mathrm{T}} w_p^{\mathrm{T}} \quad e_x^{\mathrm{T}}], \quad y = y_p.$$

The FWL covariance upper bound control problem can be stated as follows.

Determine if there exists a controller which stabilizes the system and yields an output covariance bounded above by a given matrix Ω. Find all such controllers when they exist.

3 Main results

Lemma 1 Let a symmetric matrix Ω be given. Consider the linear time-invariant discrete-time system

$$x(k+1) = A_{cl}x(k) + B_{cl}w(k),$$
$$y(k) = C_{cl}x(k) + D_{cl}w(k),$$

where w is the stochastic white noise process with covariance W. Then the following statements are equivalent:

1) There exists a stabilizing feedback gain G such that

$$\lim_{k\to\infty} \varepsilon[y(k)y^{\mathrm{T}}(k)] < \Omega;$$

2) There exists a matrix $X > 0$ such that

$$X > A_{cl}XA_{cl} + B_{cl}WB_{cl}^{\mathrm{T}}, \quad C_{cl}XC_{cl} + D_{cl}WD_{cl}^{\mathrm{T}} < \Omega.$$

Proof See Lemma 6.1.2 of [4].

Lemma 2(see Corollary 2.3.6 of [4]) Let matrices A, B, Q and R be given. Suppose $Q = Q^{\mathrm{T}}$, $R = R^{\mathrm{T}} > 0$ and $BB^{\mathrm{T}} > 0$. Then the following statements are equivalent:

1) There exists a matrix X such that

$$(A + BX)R(A + BX)^{\mathrm{T}} < Q; \qquad (*)$$

2) $Q > 0$ and $B^{\perp}(Q - ARA^{\mathrm{T}})B^{\perp\mathrm{T}} > 0$ or $BB^{\mathrm{T}} > 0$.

If the above statements hold, then all matrices X satisfying $(*)$ are given by

$$X = -(B^{\mathrm{T}}Q^{-1}B)^{-1}B^{\mathrm{T}}Q^{-1}A + (B^{\mathrm{T}}Q^{-1}B)^{-1/2}L\Psi,$$

where L is an arbitrary matrix such that $\| L \| < 1$ and

$$\Psi \triangleq R^{-1} - A^{\mathrm{T}}Q^{-1}A + A^{\mathrm{T}}Q^{-1}B(B^{\mathrm{T}}Q^{-1}B)^{-1}B^{\mathrm{T}}Q^{-1}A.$$

Thus the finite-precision covariance upper bound problem reduces to the following theorem:

Theorem 1 Let a symmetric matrix Ω be given and consider the system (6), then the following statements are equivalent:

1) There exists a stabilizing feedback gain G such that $\lim_{k\to\infty} \varepsilon[y(k)y^{\mathrm{T}}(k)] < \Omega$;

2) There exists a matrix X such that

$$B^{\perp}(X - AXA^{\mathrm{T}} - DWD^{\mathrm{T}})B^{\perp\mathrm{T}} > 0, \tag{9}$$

$$CXC^{\mathrm{T}} + FWF^{\mathrm{T}} < \Omega, \tag{10}$$

$$AXA^{\mathrm{T}} + DWD^{\mathrm{T}} - AXM^{\mathrm{T}}(MXM^{\mathrm{T}} + EWE^{\mathrm{T}})^{-1}MXA^{\mathrm{T}} < X. \tag{11}$$

In this case, all such state feedback gains are given by

$$G = -(B^{\mathrm{T}}Q^{-1}B)^{-1}B^{\mathrm{T}}Q^{-1}AXM^{\mathrm{T}}R^{-1} + (B^{\mathrm{T}}Q^{-1}B)^{-1/2}L\Psi^{1/2}, \tag{12}$$

where L is an arbitrary matrix such that $\| L \| < 1$ and

$$Q \triangleq X - AXA^{\mathrm{T}} - DWD^{\mathrm{T}} + AXM^{\mathrm{T}}R^{-1}MXA^{\mathrm{T}} > 0,$$

$$\Psi \triangleq R^{-1} - \Phi^{\mathrm{T}}(Q^{-1} - Q^{-1}B(B^{\mathrm{T}}Q^{-1}B)^{-1}B^{\mathrm{T}}Q^{-1})\Phi > 0,$$

$$R \triangleq MXM^{\mathrm{T}} + EWE^{\mathrm{T}}, \quad \Phi \triangleq AXM^{\mathrm{T}}R^{-1}.$$

Proof The Lyapunov inequality in this case is given by

$$X > (A + BGM)X(A + BGM)^{\mathrm{T}} + (D + BGE)W(D + BGE)^{\mathrm{T}}.$$

Noting that $EWD^{\mathrm{T}} = 0$, the covariance inequality can be expanded as

$$X > AXA^{\mathrm{T}} + DWD^{\mathrm{T}} + BGMXA^{\mathrm{T}} + AXM^{\mathrm{T}}G^{\mathrm{T}}B^{\mathrm{T}} + BGRG^{\mathrm{T}}B^{\mathrm{T}},$$

where $R \triangleq MXM^{\mathrm{T}} + EWE^{\mathrm{T}}$.

Since $X > 0$ and there is no redundant sensor ($MM^{\mathrm{T}} > 0$), we have $R > 0$. Hence, we can complete the square with respect to BG as follows:

$$(BG + AXM^{\mathrm{T}}R^{-1})R(BG + AXM^{\mathrm{T}}R^{-1})^{\mathrm{T}} < Q,$$

$$Q = X - AXA^{\mathrm{T}} - DWD^{\mathrm{T}} + AXM^{\mathrm{T}}R^{-1}MXA^{\mathrm{T}}.$$

Using Lemma 2, the above inequality is solvable for G.

Now the FWL covariance bounding control problem has been converted to an algebraic problem of finding matrices X and G satisfying the condition (12), which is given by the intersection of the three sets defined by two LMIs (9),(10) and a Riccati-like inequality (11). Although none of them are immediately verifiable, they may be useful to develop computational algorithms or algebraically verifiable tests to determine if a given system is stabilizable with a finite-precision digital controller. As shown in [4], the results obtained from this upper bound approach are computable via convex programming or some other algorithms.

4 Conclusions

In this paper, we have shown that the FWL covariance upper bound control problem can be reduced to some matrix inequalities problem. The matrix inequality approach to the control problem essentially derives from [6] and [7], the covariance upper bound control problem with finite wordlength and fixed-point arithmetic considerations, however, is first discussed in this paper. For discrete-time systems, there are lots of similar problems to be solved, such as LQG control problem, H_∞ control problem, H_2 control problem, subject to synchronous or skewed sampling between measurement and control, and subject to finite precision computing in the A/D, D/A devices and in the controller state noise with variance related with the wordlength.

References

[1] Masten M K and Panahi I. Digital signal processors for modern control systems [J]. Control Engineering Practice, 1997, 5(4):449 - 458

[2] Williamson D. Digital Control and Implementation: Finite Wordlength Considerations [M]. New Jersey: Prentice Hall, 1997

[3] Gevers M and Li G. Parametrizations in Control, Estimation and Filtering Problems [M]. New York: Springer-Verlag, 1993

[4] Skelton R E, Iwasaki T A and Grigoriadis K M. A Unified Algebraic Approach to Linear Control Design [M]. London: Taylor & Francis, 1998

[5] Corless M, Zhu G and Skelton R E. Robustness of covariance controllers [A]. Proc. IEEE Conference on Decision and Control [C], Tampa, Florida, American, 1989, 2667 - 2672

[6] Iwasaki T and Skelton R E. Parametrization of all stabilizing controllers via quadratic Lyapunov functions [J]. J. on Optimization Theory and Applications, 1995, 85(2):291 - 307

[7] Iwasaki T. A unified matrix inequality approach to linear control design [D]. West Lafayatte: Purdue University, IN 47907, USA, 1993

本文作者简介

徐巍华 1976 年生.现为浙江大学先进控制研究所博士研究生.研究方向为先进控制中数字控制器的有限字长问题,线性控制器设计理论及应用. Email: whxu@iipc.zju.edu.cn

吴 俊 1967 年生.分别于 1989 年和 1994 年在华中理工大学工业自动化专业获学士和博士学位,1994 年至 1996 年在浙江大学工业控制技术研究所做博士后研究工作,现为浙江大学先进控制研究所副研究员.主要研究兴趣是鲁棒控制和计算机控制.

褚 健 见本刊 2002 年第 1 期第 108 页.