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# Predictive control for polytopic uncertain linear systems with guaranteed constraints satisfaction

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Abstract: Based on the invariant set theory, invariance constraint predictive control (IC-PC) first proposed by Chiscil et al, is extended and generalized to a framework of model predictive control for constrained linear systems with polytopic uncertainty. The crucial point is to reformulate online optimization problem corresponding to nominal model with an appropriate additional robust and feasible constraint. It is shown that in this configuration feasibility of online optimization problem as well as satisfaction of constraints for the real plant can be guaranteed in all time steps if the optimization problem is feasible at the initial stage. Moreover, a sufficient condition of robust stability is given for closed-loop uncertain system, which provides a guide to the choice of cost function to guarantee robust stability.

Key words: predictive control; invariant set; constrained systems; polytopic uncertainty CLC number: TP273 Document code: A

## 多面体不确定线性系统具有约束满足保证的预测控制

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摘要:基于不变集理论,拓展了 Chiscil 等人提出的约束不变预测控制方法(IC-PC),提出了一种适用于带约束 多面体不确定线性系统的预测控制的框架.其关键在于为针对标称系统设计的在线优化问题附加适当的额外的鲁 棒可行约束.若优化问题在初始阶段可行,则此约束可保证在线优化问题始终可行,从而保证了实际系统中约束条 件的始终满足.同时提出了闭环系统鲁棒稳定的一个充分条件,可为成本函数的选择提供指导以保证预测控制器 的鲁棒镇定.

关键词:预测控制;不变集;约束系统;多面体不确定性

### 1 Introduction

It is well accepted to date that model predictive control (MPC) is the most effective way to address the complex constrained multivariable control problem. Robust predictive control is often used to name MPC algorithms that can be successfully applied to uncertain system. The existing robust predictive control methods can be generally classified into two categories: one makes use of only nominal plant but seeks an ad hoc way to ensure some robust properties, while the other takes into account the uncertainty explicitly in receding horizon implementation of MPC, which is often referred to as minmax robust predictive control. Both methods have some drawbacks. The former can not guarantee robust constraints satisfaction for uncertain system in all time steps, while the latter tends to cause the propagation of the effect of uncertainty over the predictive horizon and make online computation prohibitive when uncertainty is very complex or long horizon is indispensable.

In this note, we propose a frame of robust feasible model predictive control (RFMPC) for constrained polytopic linear systems, which is able to overcome the aforementioned drawbacks to some extent. It formulates online optimization problem corresponding to nominal plant with an additional feasibility constraint to guarantee robust constraints satisfaction. The idea is based on, ca-

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lled invariance constraint predictive control (IC-PC), proposed in [1]. Using invariant set theory, we extend this pioneer work to a more general form by further enlarging domain. It is equally important to be aware that robust feasibility does not involve robust stability for IC-PC, noting that simulation of a numerical example in [1] showed closed-loop system could sometimes demonstrate limit recycle behavior. Neither analysis criterion nor synthesis technique was given in [1] for robust stability. Here we formulate a sufficient condition for robust stability of RFMPC, which can be used as a guide to the selection of cost function to provide stability guarantee.

## 2 Problem formulation

Consider the following discrete time linear uncertain system with state and input constraints:

$$x(k + 1) = A(k)x(k) + B(k)u(k),$$
 (where

$$[A(k) \quad B(k)] \in \Omega,$$
  

$$\Omega = \operatorname{Co} \{ [A_1 \quad B_1], \cdots, [A_l \quad B_l] \}, \qquad (2)$$
  

$$x(k) \in X \subset \mathbb{R}^n, \ u(k) \in U \subset \mathbb{R}^m. \qquad (3)$$

Co in (2) denotes the convex hull. Without loss of generality, we assume that X is close and U is compact, each containing its own origin as an interior point. Moreover, we assume that they are all convex to avoid technicalities. Hereafter we will always use X, U in (3) to denote state constraint set and input constraint set respectively.

Model predictive control (MPC) strategy is often deployed to regulate the states and control inputs to zeros for systems  $(1) \sim (3)$ . In order to derive computational efficient algorithm, only the nominal model based MPC method is discussed in this paper.

Define nominal plant for uncertain systems (1) and (2):

$$\begin{aligned} x(k+1) &= A_0 x(k) + B_0 u(k), \\ [A_0 & B_0] \in \Omega. \end{aligned}$$
 (4)

MPC control action is determined at each time step by solving the following optimization problem over finite horizon with regard to nominal model (4):

$$\min_{u_{\{0,N-1\}}(k)} \left[ \sum_{i=0}^{N-1} L(x(k+i+k), u(k+i+k)) + F(x(k+N+k)) \right],$$
(5)

s.t.

$$\begin{cases} x(k+i+1+k) = A_0 x(k+i+k) + B_0 u(k+i+k), \\ x(k+k) = x(k), \\ x(k+i+k) \in X, \ u(k+i+k) \in U, \\ i = 0, 1, \cdots, N-1, \\ x(k+N+k) \in T_0 \subseteq X. \end{cases}$$
(6)

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In (5), the stage cost  $L(\cdot, \cdot)$  defined on  $X \times U$  is continuous, non-negative, time-invariant and  $F(\cdot)$  defined on X possesses these properties too.  $T_0$  is named as terminal constraint set in MPC literature which is often carefully selected to provide stability guarantee in conjunction with  $F(\cdot)^{[2]}$ . The decision variable in (5) and (6) is the control sequence, i.e.,

 $u_{[0,N-1]}(k) =$ 

1)

 $[u^{T}(k + k) \quad u^{T}(k + 1 + k) \quad \cdots \quad u^{T}(k + N - 1 + k]^{T}$ . When the optimal control sequence is computed, only the first control action is applied to the real system. In the next time step, the computation is repeated from the new state and over a shifted horizon, leading to a moving horizon policy. Now there arise such questions as whether the MPC problem (5) and (6) is always feasible and whether the MPC controller based on the above optimization problem can stabilize all plants within the polytopic uncertain models. These questions will be addressed in the next section.

# 3 Robust feasible predictive control algorithm

#### 3.1 Preliminary about invariant set

Invariant set theory plays a fundamental role in the design of controller for constrained systems since the constraints can be satisfied all the time if and only if the initial state and state evolution remain inside some invariant set. For the convenience of further discussion, we first introduce some definitions and the relevant propositions in invariant set theory. We recommend readers to [3] and references therein for detail. Except of particular specification, definitions in the following are all with regard to the uncertain system dynamics and constraints in  $(1) \sim (3)$ .

**Definition 1** (Robust one step controllable set O(E))

$$O(E) \triangleq \{x(k) \in X \mid \exists u(k) \in U : x(k+1) \in E\}.$$

 Next, we give some basic properties of robust on step controllable set and the conception of controllable invariant set together with the geometric condition for invariance.

**Proposition 1**  $O(E_1) \subseteq O(E_2)$  if  $E_1 \subseteq E_2 \subseteq X$ .

**Proposition 2** Let  $O_1(E)$ ,  $O_2(E)$  be robust one step controllable set of *E* corresponding to system dynamics  $\Sigma_1$ ,  $\Sigma_2$  respectively. If  $\Sigma_1 \subseteq \Sigma_2$ , then

$$O_1(E) \supseteq O_2(E).$$

**Remark 1** As for system dynamics like (1), (2),  $\Sigma_1 \subset \Sigma_2$  is equivalent to  $\Omega_1 \subset \Omega_2$ .

**Definition 2**(Robust controllable invariant set) The set  $E \subseteq X$  is a robust controllable invariant set if and only if  $\forall x(k) \in E, \exists u(k) \in U$  such that  $x(k+1) \in E$ .

**Proposition 3** (Geometric condition for invariance) *E* is robust controllable invariant if and only if  $E \subseteq O(E)$ .

Proposition 1,2 is quite simple to prove, so the proof is omitted. Detailed proof of proposition 3 can be found in [4].

# 3.2 Robust feasible predictive control algorithm

In this section, we first give a framework of the robust feasible predictive control problem for constrained polytopic uncertain linear system, then proceed to discuss its robust feasibility with the above preliminary.

Robust feasible predictive control problem. At each sample time k, solve

$$\min_{\substack{u_{[0,N-1]}(k)}} \left[ \sum_{i=0}^{N-1} L(x(k+i|k), u(k+i|k)) + F(x(k+N|k)) \right],$$
(7)

s.t.

$$x(k+i+1|k) = A_0 x(k+i|k) + B_0 u(k+i|k),$$
  
 
$$x(k+k) = x(k),$$

$$(8.1)$$

$$x(k+i|k) \in X, \ u(k+i|k) \in U, \ i=0,1,\cdots,N-1,$$

$$(8.2)$$

$$A_{jx}(k|k) + B_{ju}(k|k) \in X_{R}, \ j = 1, 2, \cdots, l,$$
 (8.3)

$$x(k+N+k) \in T_0 \subseteq X.$$
(8.4)

The control action is applied to the real plant in traditional receding horizon manner.

Problem posed above is the same as the problems (5)

and (6), but with additional constraint (8.3). We refer to  $X_R$  as robust feasible constraint set and assume that it is convex. Let  $X_F(X_R, T_0, N)$  denote the set of states for which problem (7) and (8) has a feasible solution, in other words,  $X_F(X_R, T_0, N)$  is the feasible domain of problem (7) and (8). For the consistency of notation, we denote by  $X_F(T_0, N)$  the feasible domain of problem (5) and (6).

**Definition 3**(Robust strong feasibility) MPC optimization problem is robust strong feasible if and only if the feasibility at the current time step implies the feasibility at the next time step.

As far as strong feasibility of problem (7) and (8) is concerned, the following theorem holds.

**Theorem 1**(Robust feasibility) If  $X_R$  is convex and robust controllable invariant, satisfying  $X_R \subseteq X_F(T_0, N-1)$ , then (7) and (8) are robust strong feasible with the feasible domain

$$X_F(X_R, T_0, N) = O(X_R).$$

**Proof** Assume that problem (7) and (8) is feasible at the current time step k, i.e.,

$$x(k) \in X_F(X_R, T_0, N).$$

Satisfaction of (8.3) means  $x(k) \in O(X_R)$ . Fulfillment of (8.1), (8.2) and (8.4) means that  $x(k) \in X_F(T_0, N)$ . So it can be claimed that

 $X_F(X_R, T_0, N) = O(X_R) \cap X_F(T_0, N).$  (9) Let  $\begin{bmatrix} A_0 & B_0 \end{bmatrix} = \Omega_0$  and use  $O_0(\cdot)$  to denote one step controllable set under nominal system dynamics (4). It is clear that

$$X_F(T_0, N) = O_0(X_F(T_0, N-1)).$$
(10)

By Proposition 1,  $X_F(T_0, N-1) \supseteq X_R$  implies that

$$O_0(X_F(T_0, N-1)) \supseteq O_0(X_R).$$
(11)

Furthermore, it follows from Proposition 2 that

$$O_0(X_R) \supset O(X_R). \tag{12}$$

From  $(10) \sim (12)$ , we get

$$X_F(T_0,N)\supseteq O(X_R).$$

Thus  $X_F(X_R, T_0, N) = O(X_R)$  via (9). By this we have proven the formula of feasible domain. Moreover, we can infer that  $x(k + 1) \in X_R$  from constraint (8.3), system dynamics (1), (2) and the convexity of  $X_R$ . On the other hand, invariance of  $X_R$  implies that

$$X_R \subseteq O(X_R)$$

$$x(k+1) \in X_R \subset O(X_R) = X_F(X_R, T_0, N)$$

i.e., problem (7) and (8) is still feasible at next time step k + 1. Consequently, problem (7) and (8) is strong feasible. Q.E.D.

In the following, we shall use RFMPC to refer MPC approaches based on (7) and (8) with the choices of  $X_R$  and  $T_0$  satisfying the condition in robust feasible theorem.

#### 3.3 Design of RFMPC

Kernel point in the design of RFMPC is to construct  $X_R$  and  $T_0$  to satisfy

 $X_R \subset X_F(T_0, N-1), X_R \subset O(X_g),$ 

for convexity of  $X_R$  is easier to fufill. In this section, we shall give a scheme similar to the approach of [1] to determine  $T_0$  and  $X_R$ , but as being demonstrated later, our method can encompass the way in [1] and bring a much larger feasible domain.

First,  $T_0$  can be selected as the maximal invariant set of nominal system under a constant linear state feedback control law as proposed in [1]. The choice of  $X_R$  may be generally a little more complicated. Before addressing this question, we give another conception involving robust invariant set for the ease of the following discussion.

**Definition 4**(*j*-step robust stabilizable set) Let  $T \subseteq X$  be robust controllable invariant with system dynamics and constraints (1) ~ (3), its *j*-step robust stabilizable set  $S_i(X, T)$  is difined as follows:

$$S_{j}(X,T) = \{x \in X \mid \exists u \in U: (A_{n}x + B_{n}u) \in S_{j-1}(X,T), \\ n = 1, 2, \cdots, l\},\$$

$$S_0(X,T) = T.$$

The following two propositions are based on the invariance of T.

**Proposition 4** The *j*-step robust stabilizable set contains all the previous set, i,e.,

$$S_i(X,T) \supset S_{i-1}(X,T), \ j \ge 1$$

**Proposition 5** The *j*-step robust stabilizable set  $S_i(X, T)$  is robust controllable invariant.

Let  $\hat{S}_j(\cdot, \cdot)$  denote *j*-step stabilizable set corresponding to nominal dynamics (4) and controllable invariant set  $T_0$ , then it is apparent that

$$X_F(T_0, N-1) = \hat{S}_{N-1}(X, T_0).$$

In the following, we shall denote by T the maximal invariant set<sup>[3]</sup> of uncertain system (1) ~ (3) applying the same linear state feedback law as  $T_0$ . Then  $X_R$  can be chosen in the following two ways:

1) Let  $X_R = S_j^*(X, T)$ , where  $j^*$  is the maximal positive integer *j* satisfying

$$S_j(X,T) \subseteq \hat{S}_{N-1}(X,T_0)$$

2) Let  $X_R = S_j(\hat{S}_{N-1}(X, T_0), T)$ , where *j* can be chosen as large as possible. Especially when  $S_{\infty}(\hat{S}_{N-1}(X, T_0), T)$  can be exactly determined, we can choose it as  $X_R$  to get the largest feasible domain by this choice.

From the definition and properties of stabilizable set, it is obvious that the above two choices of  $X_R$  can both satisfy the sufficient condition in robust feasible theorem, thus guarantee the robust feasibility of RFMPC and robust constraint fulfillment of the real plant. The first choice covers the algorithm presented in [1].

#### 3.4 Robust stability analysis of RFMPC

In the preceding section, we give the sufficient condition for robust feasibility of RFMPC, but this method can not provide any guarantee on robust stability spontaneously. In this section, we limit  $L(\cdot, \cdot)$  and  $F(\cdot)$  to quadratic functions, i.e.,

$$L(x, u) = x^{T}Q_{1}x + u^{T}Ru, F(x) = x^{T}Q_{2}x,$$
(13)

where

 $Q_1 = Q_1^T > 0$ ,  $R = R^T > 0$ ,  $Q_2 = Q_2^T > 0$ . This paper uses '>0' on a matrix to mean that the ma-

trix is positive definite. Here we also demand that all the sets included in RFMPC problem are convex polyhedrons which can be represented by standard linear inequalities. In the following, we shall discuss robust stability of RFMPC in this case.

Cost function (7) corresponding to nominal model can be written as

$$J(x(k), u_{[0,N-1]}(k)) =$$

$$\frac{1}{2} u_{[0,N-1]}^{\mathsf{T}}(k) H u_{[0,N-1]}(k) +$$

$$x^{\mathsf{T}}(k) Y u_{[0,N-1]}(k) + \frac{1}{2} x^{\mathsf{T}}(k) P x(k). \quad (14)$$

RFMPC problem can be rewritten as

1

$$V(x(k)) = \min_{u_{[0,N-1]}(k)} J(x(k), u_{[0,N-1]}(k)),$$
(15)

s.t. 
$$Cx(k) + Du_{[0,N-1]}(k) + E \leq 0.$$
 (16)

E in the left-hand side of (16) is a vector and vector Ineq. (16) applies element-by-element. For (15) and (16), we further suppose that

$$V(0) = 0,$$
  
arg min<sub>u<sub>[0,N-1]</sub><sup>(k)</sup></sub>  $J(0, u[0,N-1](k)) = 0.$  (17)

From robust feasible theorem, we know that RFMPC problem will be always feasible if it is feasible at the initial stage. According to mathematical program theory, if problem (15) and (16) has a feasible solution, it must have the unique optimal solution since H is positive definite. This paper denotes by  $u_{[0,N-1]}^*(k)$  the optimal solution at the current time step k. Based on the convexity of uncertainty model and the linearity of constraints, the following theorem establishes the sufficient robust stability condition for RFMPC.

Theorem 2(Robust stability) Let

$$z(k) = \begin{bmatrix} x(k)^{\mathrm{T}} & u_{[0,N-1]}^{*}(k)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$
  

$$z^{(i)}(k+1) = \begin{bmatrix} x^{(i)}(k+1) & u_{[0,N-1]}^{*}(k+1) \end{bmatrix},$$
  
here  $u_{i}^{*}$  =  $z(k)$  is the extinct solution to (15) or

where  $u_{[0,N-1]}^{*}(k)$  is the optimal solution to (15) and (16) at x(k);

$$x^{(i)}(k+1) = A_i x(k) + B_i u^*(k \mid k),$$
  
$$i = 1, 2, \cdots, l,$$

 $u_{[0,\tilde{N}-1]}^{*(i)}(k+1)$  is the optimal solution to (15) and (16) at  $x^{(i)}(k+1)$ . If there exists  $\tilde{P} \in \mathbb{R}^{n \times n}$ , such that  $\tilde{P} = \tilde{P}^{T}$ ,

$$M = \begin{bmatrix} \bar{P} & Y \\ Y^{T} & H \end{bmatrix} > 0,$$
  
$$\frac{1}{2}z(k)^{T}Mz(k) - \frac{1}{2}z^{(i)}(k+1)^{T}Mz^{(i)}(k+1) >$$
  
$$\varepsilon \parallel x(k) \parallel^{2}, \ i = 1, 2, \cdots, l,$$

where  $\epsilon$  is a positive constant real variable, then RFMPC controller is robust stabilizing for the constrained uncertain systems (1) ~ (3).

The preceding theorem can be proven using the standard Lyapunov arguments and the convexity of the quadratic cost function. For lack of space, we leave it to readers.

**Remark 2** The robust stability condition here can be further converted to a more conservative condition in terms of LMIs using KT condition and S-procedure. We recommend readers to [5] for details.

## 4 An illustrative example

Consider the constrained polytopic uncertain system  $(1) \sim (3)$  with l = 2,

$$A_{1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, A_{2} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1.8 \end{bmatrix}$$
$$B_{1} = B_{2} = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix},$$

and

$$U = \{ u \mid -2 \leq u \leq 2 \}, X = \mathbb{R}^2.$$

This is the same model as used in [1]. In addition, let

$$N = 3, A_0 = (A_1 + A_2)/2$$

$$B_0 = (B_1 + B_1)/2,$$

in RFMPC problem (7), (8) and choose in (13)

$$Q_1 = \text{diag}(5000, 1), R = 1,$$

$$Q_2 = \text{diag}(1, 1).$$

The linear state feedback control law K essential to the terminal set  $T_0$  is chosen as

K = [-35.4755 - 16.8591],

which is the optimal feedback gain of the unconstrained, infinite horizon LQR problem with weights  $Q_1$  and R for the nominal system. It can be verified that K can locally stabilize the constrained uncertain system  $(1) \sim (3)$  by the method presented in [6].  $T_0$  is selected as the maximal admissible set for the nominal system under linear state feedback control law  $K^{[7]}$ .  $X_R$  selected by IC-PC is only  $S_3(X, T)$ . By choice 1 presented in 3.3,  $X_R$  can be extended up to  $S_5(X, T)$ . Although for the boundlessness of X here,  $S_{\infty}(\hat{S}_2(X, T_0), T)$  can not be determined by finite steps, we can still choose  $X_R$  as  $S_6(\hat{S}_2(X, T_0), T)$  to circumvent  $X_R$  in choice 1. These different choices of  $X_R$  are reported in Fig. 1 and Fig. 2 compares the corresponding feasible domains attained. The robust stability of RFMPC is affirmed by the robust stability theorem in Section 3.4 and the approach presented in [5]. Fig.3 shows the state evolution of RFM-PC with the second choice of  $X_R$  for an initial state

$$x(0) = [-0.1 \quad 0.1]^{\mathrm{T}},$$

which is outside of the feasible domain of IC-PC, and for the realization of

$$A(k) = (0.5 + 0.5\sin 0.1k)A_1 +$$

$$(0.5 - 0.5 \sin 0.1k)A_2.$$

Fig.4 displays input response.







Fig. 3 State evolution of RFMPC

# 5 Conclusion

In this paper, a framework for a robust feasible MPC algorithm is presented with a sufficient condition derived for guaranteeing robust strong feasibility. Compared to other similar MPC methods corresponding to nominal model, feasible domain attained by RFMPC can be shown further enlarged without much more online computation demand. In addition, a sufficient condition is given for robust stability of closed-loop system. Current research is devoted to developing a systematic design scheme to guarantee robust strong feasibility and robust stability simultaneously.

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