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Limitation on the tracking problem due to jω-axis zeros in MIMO feedback control systems

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Abstract: The process of feedback design is necessarily performed in the presence of hard constraints upon properties of the closed-loop system. This paper gives the time domain integral constraints of the feedback control systems tracking error that arise from the presence of stable zeros on or near the jω-axis, which shows that for a linear time invariant system, in order to guarantee the stability of the closed-loop system, the time domain integral constraints must be satisfied by the tracking error. The presence of stable zeros near or on the jω-axis shows that there is trade offs between the tracking error and the settling time. For fixed settling time, this paper gives an effective lower bound on the infinite norm of the tracking error, which shows that when the absolute value of the stable zero on the jω-axis becomes small, the lower bound becomes large. The constraints are illustrated by an example.

Key words: time domain integral constraints; feedback system; response; simulation

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多输入输出反馈系统虚轴的零点对跟踪问题的约束

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摘要：反馈系统的设计会受到闭环系统特性的制约.本文给出了反馈控制系统接近虚轴或虚轴上的稳定零点对跟踪误差的时间域积分约束.这一时间域积分约束是任何一个线性时不变系统为保证其闭环系统的稳定性,其跟踪误差必须满足的.接近虚轴或虚轴上的稳定零的存在表明跟踪误差与调节时间之间存在某种折中.对固定的调节时间,本文给出了在虚轴上存在零点情形下,其跟踪误差的无穷范数下界的一个有效估计.此估计表明,零点的绝对值越小,其无穷范数的下界越大.这些约束由一个例子加以解释.

关键词：时间域积分约束；反馈系统；响应；仿真

1 Introduction

There are always basic limitations on the achievable performance involved with the feedback control of any physical plant. These limitations come from several sources. Bode developed the fundamental work on structural limitations in the control of linear time invariant systems. In [1,2] the waterbed effect for non-minimum-phase plant are given, which shows that if the system gain is pushed down on one frequency range, it pops up somewhere else. [2] also derives the area formula, which applies minimum and non-minimum phase plants. [3,4] extend the corresponding work to multivariable systems and to discrete time systems. [5] shows that the performance and robust stability properties are limited by the presence of RHP poles and zeros for SISO system. [6] explores the time-domain integral constraints to show that slow stable poles place constraints on the settling time of the closed-loop systems. [7] is based on unit step response and shows that fundamental limitations arise from the presence of stable zeros near or on the jω-axis. [8] treats multivariable systems by using singular values and the theory of subharmonic functions; Refinements of these results have also been presented in [9]. [2] converts the multivariable probl-

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em into a scalar one by pre- and post-multiplying the sensitivity function by vectors or by use of determinants[5]. A similar idea appears in the work of [10] which uses directions associated with poles and zeros of the system resulting in a directional study of trade-offs. [3] develops integral constraints on sensitivity vectors for multivariable feedback systems due to either unstable poles or non-minimum-phase zeros of the plant; the use of these integral constraints give the inherent trade-offs in sensitivity reduction and the cost of decoupling.

The aim of the present paper is to continue the research of [7], and extend the corresponding result to multivariable feedback systems. This paper shows the effect of stable zeros near or on the $j\omega$-axis for MIMO feedback control systems tracking problem. Time domain integral constraints of the feedback control system tracking error is developed, which shows stable zeros near or on the $j\omega$-axis and implies a lower bound on the achievable settling time of the feedback control systems. An example is given to explain the results of this paper.

2 Preliminaries

We consider the linear time invariant feedback control systems shown in Fig.1. The symbols in Fig.1 have the following meaning. $P(s)$ is the $n \times m$ matrix of proper rational plant transfer function; $C(s)$ is the $m \times n$ matrix of proper rational controller transfer function ; $u(t)$, $e(t)$, and $y(t)$ are, respectively, the m-tuple vectors of reference, error signal, and plant output.

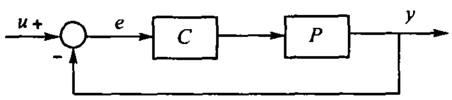

Fig. 1 Feedback control systems

Suppose the plant and the controller are described by coprime fractional representations (over the ring of proper stable matrices of transfer functions)[11]

$$P = \widetilde{D}_P^{-1}\widetilde{N}_P = N_P D_P^{-1}, \tag{1}$$

$$C = N_C D_C^{-1} = \widetilde{D}_C^{-1}\widetilde{N}_C. \tag{2}$$

Further, we assume that C is chosen so that the closed loop is internally stable (i.e. the matrix $\widetilde{N}_C N_P + \widetilde{D}_C D_P$ and $\widetilde{N}_P N_C + \widetilde{D}_P D_C$ are unimodular). Then the sensitivity function and the complementary sensitivity function are defined, respectively, by

$$S(s) = D_C(\widetilde{D}_P D_C + \widetilde{N}_P N_C)^{-1}\widetilde{D}_P \tag{3}$$

and

$$T(s) = I - S(s) = (I + PC)^{-1}PC = P(I + CP)^{-1}C = N_P(\widetilde{D}_C D_P + \widetilde{N}_C N_P)^{-1}\widetilde{N}_C.$$

The L_∞^m norm defined by

$$\| f \|_\infty^m = | f(t) |_{\mathrm{esssup}\, t \geq 0}, \tag{4}$$

where $| f(t) |$ is the m-tuple vector Euclidean norm. When the plant P is stable, the set of all compensators that stabilize P is given in [11] by

$$S(P) = \{C : C = Q(I - PQ)^{-1}\}, \tag{5}$$

where Q is a proper, causal stable transfer function matrix.

3 Time-domain constraints of the feedback systems for zeros on or near $j\omega$-axis

Lemma 1 Suppose the plant P has zeros at $-\sigma \pm j\omega_0$ $(\sigma \geq 0, \omega_0 \geq 0)$, then there exists a right coprime factorization (N_P, D_P) such that $N_P(-\sigma \pm j\omega_0) = 0$. Further there exists nonzero normalized complex vector ξ such that $\xi' N_P(-\sigma + j\omega_0) = 0$ and $\xi^{\mathrm{T}} N_P(-\sigma - j\omega_0) = 0$.

Proof Suppose the plant P has rank r . The first part can be found in the proof of Theorem 4.1.49 of [11], where

$$N_P = M^{-1}\mathrm{diag}\,(a_1(s), a_2(s), \cdots, a_r(s), 0, \cdots, 0), \tag{6}$$

M is a unimodular matrix, $a_i (i = 1, \cdots, r)$ are the numerators of the diagonal entries of the Smith-McMillan form for the plant P satisfying $a_i \mid a_{i+1}$ (over the ring of polynomials). Hence $a_r = 0$ contains all of the zeros of the plant $P(s)$. Now let

$$\xi'(s) = (0, \cdots, 1, \cdots, 0)M, \tag{7}$$

then

$$\xi'(s)N_P(s) = (0, \cdots, a_r(s), \cdots, 0). \tag{8}$$

Hence

$$\xi = (\xi(-\sigma + j\omega_0))/(|\xi(-\sigma + j\omega_0)|)$$

satisfy the requirements.

Theorem 1 Consider the feedback control system shown in Fig. 1. Suppose the following two conditions hold: i) The plant $P(s)$ has at least one pair of zeros on or near the jω-axis at $-\sigma \pm j\omega_0 (\sigma \geqslant 0, \omega_0 \geqslant 0)$; ii) All poles of the closed-loop system have real parts less than $-\alpha(\alpha > 0, \alpha > \sigma)$.

Under these conditions the following integral constraints hold on the error signal $e(t)$ of the tracking problem shown in Fig. 1

$$\xi' \int_0^{\infty} e^{(\sigma - j\omega_0)t} e(t)\,dt = \xi' U(-\sigma + j\omega_0), \quad (9)$$

$$\xi^T \int_0^{\infty} e^{(\sigma + j\omega_0)t} e(t)\,dt = \xi^T U(-\sigma - j\omega_0). \quad (10)$$

Proof The Laplace transform of the tracking error

$$e(t) = u(t) - y(t).$$

Satisfies

$$E(s) = (I - T(s))U(s). \quad (11)$$

According to the internal theorem, in order to tracking the signal $u(t)$, the tracking error $e(t)$ satisfies

$$\lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s(1 - T(s))U(s) = 0.$$

So $s = 0$ is not a pole of (11). Hence, by Assumption ii), $s = 0$ lies inside the region of convergence of the transform

$$\int_0^{\infty} e^{-st} e(t)\,dt = (1 - T(s))U(s). \quad (12)$$

Because of $T(-\sigma \pm j\omega_0) = 0$, set $s = -\sigma + j\omega_0$ and left multiplying by ξ', by Lemma 1 Eq. (12) gives

$$\xi' \int_0^{\infty} e^{(\sigma - j\omega_0)t} e(t)\,dt = \xi' U(-\sigma + j\omega_0). \quad (13)$$

Set $s = -\sigma - j\omega_0$ and left multiplying by ξ^T, by Lemma 1 Eq. (12) gives

$$\xi^T \int_0^{\infty} e^{(\sigma + j\omega_0)t} e(t)\,dt = \xi^T U(-\sigma - j\omega_0). \quad (14)$$

4 Lower bounds on $\| e \|_{\infty}^{m}$

Definition 1 Define the settling time of the system to be

$$t_s = \inf\{\tau : \| e(t) \| \leqslant 0.02, \ \forall t > \tau\}. \quad (15)$$

In order to simplify the expression, let $\sigma = 0$ under this condition, we have the following:

Corollary 1 Consider the feedback control system shown in Fig. 1. Assume that the plant has zeros at $\sigma = \pm j\omega_0$, and the settling time is t_s, then the tracking error's infinite norm has the following lower bound

$$\| e \|_{\infty}^{m} \geqslant \sqrt{\max\left\{\frac{|\xi' U(j\omega_0)|}{t_s}, \frac{|\xi^* U(-j\omega_0)|}{t_s}\right\}}. \quad (16)$$

Proof From Definition 1 and Eqs. (13) and (14), using Schwarz inequality we immediately get the required inequalities.

Because the actual inputs are unit step, unit ramp, unit accelerate and their linear combinations, their corresponding Laplace transforms are $1/s, 1/s^2, 1/s^3$, and their linear combinations. Hence the lower bounds show that for fixed t_s, as ω_0 becomes small, the lower bounds on $\| e \|_{\infty}^{m}$ become arbitrarily large. Which means that under this condition, it would be very difficult to get a desired settling time.

5 Example

Consider the plant

$$P(s) = \begin{pmatrix} \dfrac{100s^2 + 1}{(s+1)^2} & \dfrac{s+3}{(s+1)(s+2)} \\ \dfrac{100s^2 + 1}{(s+1)(s+2)} & \dfrac{s+4}{(s+2)^2} \end{pmatrix},$$

$P(s)$ has zeros at $\pm 0.1j$, the corresponding normalized complex vector ξ in Lemma 1 is given by $\xi' = (-0.55 - 0.07j, 0.83 + 0.07j)$. In order to find a compensator C that stabilize P and also track a step reference signal, according to (5), we have

$$S(s) = (I + PC)^{-1} = I - PQ.$$

Let $S(0) = 0$, we have

$$Q(s) = \begin{pmatrix} 4 & -6 \\ -2 & 4 \end{pmatrix}; C(s) = Q(I - PQ)^{-1}.$$

The simulation response of $e_1(t)$ and $e_2(t)$ are shown in Fig. 2 and Fig. 3. where

$$E_1(s) = (201s^3 + 404s^2 - s)/(s(s+1)^2(s+2)),$$

$$E_2(s) = (201s^3 + 403s^2)/(s(s+1)(s+2)^2).$$

By (16), the lower bound satisfies

$$\| e \|_{\infty} \geqslant 1.25.$$

Numerical integrations yield

$$(-0.55 - 0.07j, 0.83 + 0.07j) \int_0^{t_s} e^{-0.1j} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} dt =$$

$$(-0.55 - 0.07j, 0.83 + 0.07j) \cdot \begin{pmatrix} \int_0^8 e^{-0.1j}(206e^{-t} - 204te^{-t} - 5e^{-2t})dt \\ \int_0^8 e^{-0.1j}(-202e^{-t} + 403e^{-2t} + 2te^{-2t})dt \end{pmatrix} \approx$$

$$-0.11 - 2.8j \approx (-0.55 - 0.07j, 0.83 + 0.07j)\begin{pmatrix} -10j \\ -10j \end{pmatrix}.$$

Hence (13) is satisfied.

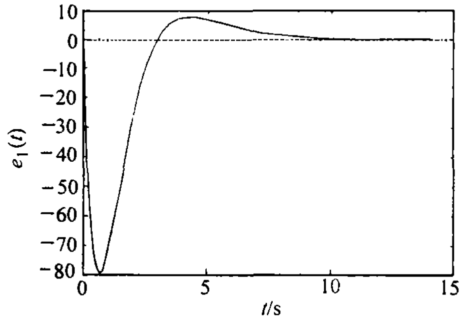

Fig. 2 Impulse response of $E_1(s)$

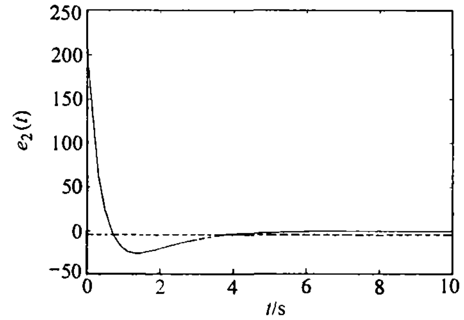

Fig. 3 Impulse response of $E_2(s)$

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