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Memoryless H_∞ controller of linear systems with delay in state

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Abstract: Using the method of linear matrix inequality(LMI), this paper gives sufficient condition to solve the state feedback H_∞ control problem for linear systems with delay in state. The design of the controller ensures the closed-loop systems be of index one and asymptotically stable. The controller can be obtained by solving LMIs. Finally a simple example is given to illustrate the validity of the given result.

Key words: LMI; H_∞ control; time-delay system
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状态滞后线性系统的 H_∞ 状态反馈控制器

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摘要：利用 LMI(线性矩阵不等式)的方法，首先给出了一般的状态滞后自治系统内稳定且具有 H_∞ 范数界的一个充分条件．并由此得到了一般的状态滞后系统 H_∞ 问题有解的一个充分条件，通过解 LMI 可以获得控制器的解．最后举例说明方法的正确性．

关键词：线性矩阵不等式；H_∞ 控制；时滞系统

1 Introduction

H_∞ control theory, as an important branch of robust control theory, has received a considerable amount of attention in recent years[1~3]. The H_∞ control problem for linear time-delay systems has been is being studied[4~6]. Yet, owing to the complex of time-delay systems, the study of H_∞ control problem for time-delay systems is still in developing stage. Most literatures deal with H_∞ control problem for some special kind of time-delay systems[6,7]. In this paper, by using the method of LMI, attention is focused on the H_∞ state feedback control for the most general linear time-delay systems. The sufficient condition for the existence of H_∞ state feedback controller, and the design of the corresponding controller are given.

2 System description

We consider the H_∞ control problem for the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_\tau x(t-\tau) + Bw(t) + Eu(t), \\ z(t) = Cx(t) + C_\tau x(t-\tau) + Dw(t) + Fu(t), \end{cases} \tag{1}$$

and the state feedback:

$$u(t) = Kx(t). \tag{2}$$

The aim is to design the state feedback controller (2) for system (1) such that the closed-loop system satisfies

1) The closed-loop system is stable;

2) $\| T_{zw}(s) \|_\infty \leqslant \gamma$.

Where $x(t) \in \mathbb{R}^n$ is the state variable, $w(t) \in \mathbb{R}^q$ is the exogenous input and $u(t) \in \mathbb{R}^p$ is the control input, $z(t) \in \mathbb{R}^m$ is the controlled output, $\tau > 0$ delay constant, $A, A_\tau, B, E, C, C_\tau, D, F$ are known constant real matrices with appropriate dimensions, only K is unknown matrix, $T_{zw}(s)$ is the transfer function from $w(t)$ to $z(t)$, $\gamma > 0$ is given.

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For the simplicity, we first gives the condition that makes the system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_\tau x(t-\tau) + Bw(t), \\ z(t) = Cx(t) + C_\tau x(t-\tau) + Dw(t), \end{cases} \tag{3}$$

have the property (*) above.

To this end, the following lemmas are needed.

Lemma 1[7] If there exist positive-definite matrices $P > 0, Q > 0$ satisfying the following inequality:

$$PA + A^{\mathrm{T}}P + PA_\tau Q^{-1}A_\tau^{\mathrm{T}}P + Q < 0, \tag{4}$$

the system $\dot{x}(t) = Ax(t) + A_\tau x(t-\tau)$ is zero-solution asymptotically stable.

The transfer function from $w(t)$ to $z(t)$ of system (3) is $T_{zw}(s)$:

$$T_{zw}(s) = (C + C_\tau \mathrm{e}^{-s\tau})(sI - A - A_\tau \mathrm{e}^{-s\tau})^{-1}B + D. \tag{5}$$

From Eq. (5), we note that there is one term which is the product of two transcendental function, which makes the problem more difficult than the problem of literature[5~7] where $C_\tau = 0$ or $C_\tau = 0, D = 0$. There are few literatures[8,9] which consider H_∞ control of the general system, but the results can not be applied easily to design the state feedback controller. In the paper, a sufficient condition for H_∞ state feedback control of the general system is given using the method of LMI. Let

$$\begin{cases} \hat{E} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} B \\ D \end{bmatrix}, \ \hat{A} = \begin{bmatrix} A & 0 \\ 0 & -I_m \end{bmatrix}, \\ \hat{A}_\tau = \begin{bmatrix} A_\tau & 0 \\ C_\tau & 0 \end{bmatrix}, \ \hat{C} = [C \quad I_m], \end{cases} \tag{6}$$

then, the transfer function (5) of system (3) is equivalent to

$$T_{zw}(s) = \hat{C}(s\hat{E} - (\hat{A} + \hat{A}_\tau \mathrm{e}^{-s\tau}))^{-1}\hat{B}, \tag{7}$$

so the H_∞-norm of the transfer function (5) is equivalent to the H_∞-norm of the transfer function (7). Next, we revise Theorem 1 of the literature [7] and give the sufficient condition such that $\| T_{zw}(s) \|_\infty \leqslant \gamma$.

Lemma 2 For the given constant $\gamma > 0$, if there exists matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}$, where

$$P_{11} = P_{11}^{\mathrm{T}}, P_{11} \in \mathbb{R}^{n\times n}, \ P_{22} \in \mathbb{R}^{m\times m},$$

and positive-definite matrix $Q \in \mathbb{R}^{(n+m)\times(n+m)}$ satisfying

$$\hat{A}^{\mathrm{T}}P^{\mathrm{T}} + P\hat{A} + P\hat{A}_\tau Q^{-1}\hat{A}_\tau^{\mathrm{T}}P^{\mathrm{T}} + Q + \hat{C}^{\mathrm{T}}\hat{C} + \gamma^{-2}P\hat{B}\hat{B}^{\mathrm{T}}P^{\mathrm{T}} < 0, \tag{8}$$

the transfer function of system (3) satisfies $\| T_{zw}(s) \|_\infty \leqslant \gamma$.

Remark 1 Lemma 1 and Lemma 2 provide respectively a sufficient condition which guarantees system (3) with the property (* 1) and (* 2). If $C_\tau = 0, D = 0$, Lemma 2 is in fact Theorem 1 of Ref. [7].

Remark 2 We have the similar result for the system with multiple delays in state.

3 Main result

For the design of the controller (2), we need to revise the condition (8) of Lemma 2. Particularly, we take the following forms for the matrices Q, P of Ineq. (8):

$$P = \mathrm{diag}\,(P_1, P_2), \ Q = \mathrm{diag}\,(Q_1, Q_2), \tag{9}$$

where matrices $P_i, Q_i \ (i = 1,2)$ have appropriate dimensions, and $P_1 > 0, \ Q_1 > 0, Q_2 > 0, \det(P_2) \neq 0$, then Ineq. (8) is changed into:

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^{\mathrm{T}} & W_{22} \end{bmatrix} < 0, \tag{10}$$

$$\begin{cases} W_{11} = A^{\mathrm{T}}P_1 + P_1A + P_1A_\tau Q_1^{-1}A_\tau^{\mathrm{T}}P_1 + \\ \qquad \gamma^{-2}P_1BB^{\mathrm{T}}P_1 + Q_1 + C^{\mathrm{T}}C, \\ W_{22} = -2P_2 + P_2C_\tau Q_1^{-1}C_\tau^{\mathrm{T}}P_2 + \\ \qquad \gamma^{-2}P_2DD^{\mathrm{T}}P_2 + Q_2 + I_m, \\ W_{12} = P_1A_\tau Q_1^{-1}C_\tau^{\mathrm{T}}P_2 + \gamma^{-2}P_1BD^{\mathrm{T}}P_2 + C^{\mathrm{T}}. \end{cases} \tag{11}$$

If there exist positive-definite matrices $P_1 > 0, Q_1 > 0, Q_2 > 0$, and nonsingular matrix P_2 satisfying the Ineq. (10), then there exist matrices P and Q satisfying Ineq. (8). Ineq. (10) holds if and only if:

$$\begin{bmatrix} \overline{W}_{11} & \overline{W}_{12} \\ \overline{W}_{12}^{\mathrm{T}} & \overline{W}_{22} \end{bmatrix} \triangleq P^{-1}WP^{-\mathrm{T}} < 0, \tag{12}$$

where

$$\begin{cases} \overline{W}_{11} = \overline{P}_1A^{\mathrm{T}} + A\overline{P}_1 + A_\tau\overline{Q}_1A_\tau^{\mathrm{T}} + \\ \qquad \overline{P}_1\overline{Q}_1^{-1}\overline{P}_1 + \overline{P}_1C^{\mathrm{T}}C\overline{P}_1, \\ \overline{W}_{22} = X + C_\tau\overline{Q}_1C_\tau^{\mathrm{T}} + \gamma^{-2}DD^{\mathrm{T}}, \\ \overline{W}_{12} = Y + \overline{P}_1C^{\mathrm{T}}\overline{P}_2^{\mathrm{T}}, \end{cases} \tag{13}$$

$$\begin{cases} \overline{P}_1 = P_1^{-1}, \ \overline{P}_2 = P_2^{-1}, \\ \overline{Q}_1 = Q_1^{-1}, \ \overline{Q}_2 = Q_2, \\ X = -\overline{P}_2 - \overline{P}_2^{\mathrm{T}} + \overline{P}_2\overline{P}_2^{\mathrm{T}} + \overline{P}_2\overline{Q}_2\overline{P}_2^{\mathrm{T}}, \\ Y = A_\tau\overline{Q}_1C_\tau^{\mathrm{T}} + \gamma^{-2}BD^{\mathrm{T}}, \end{cases} \tag{14}$$

so we only need to prove that there exist matrices $\overline{P}_1 > 0, \overline{Q}_1 > 0, \ \overline{Q}_2 > 0$ and nonsingular matrix $\overline{P}_2$ satisfying Ineq. (12). The following theorem gives the sufficient condition which guarantees system (3) having the property (*).

Theorem 1 For the given constant $\gamma > 0$, if there exist positive-definite matrices $\bar{P}_1 > 0, \bar{Q}_1 > 0$, negative-definite matrix $X < 0$, and nonsingular matrix $\bar{P}_2$ satisfying the following LMI, then system (3) have the property (*).

$$\begin{bmatrix} \tilde{A} & \bar{P}_1 & 3\bar{P}_1C^T & 2Y \\ \bar{P}_1 & -Q_1 & & \\ 3C\bar{P}_1 & & -3I & \\ 2Y^T & & & 2\widetilde{W}_{22} \end{bmatrix} < 0, \tag{15a}$$

$$\begin{bmatrix} -I & -\bar{P}_2^T & & \\ -\bar{P}_2 & \widetilde{W}_{22} & & \\ & & -X-\bar{P}_2-\bar{P}_2^T & -\bar{P}_2 \\ & & -\bar{P}_2^T & -I \end{bmatrix} < 0, \tag{15b}$$

where

$$\begin{cases} \tilde{A} = \bar{P}_1A^T + A\bar{P}_1 + A_r\bar{Q}_1A_r^T + \gamma^{-2}BB^T, \\ \widetilde{W}_{22} = X + C_r\bar{Q}_1C_r^T + \gamma^{-2}DD^T, \\ Y = A_r\bar{Q}_1C_r^T + \gamma^{-2}BD^T. \end{cases} \tag{16}$$

Using Theorem 1, the sufficient condition is established for H_∞-control synthesis by state feedback.

Theorem 2 For the given constant $\gamma > 0$, if there exist positive-definite matrices $\bar{P}_1 > 0, \bar{Q}_1 > 0$, negative-definite matrix $X < 0$, nonsingular matrix $\bar{P}_2$ and matrix M satisfying LMIs: (15b) and

$$\begin{bmatrix} \overset{\triangle}{A} & \bar{P}_1 & 3(\bar{P}_1C^T+M^TF^T) & 2Y \\ \bar{P}_1 & -\bar{Q}_1 & & \\ 3(C\bar{P}_1+FM) & & -3I & \\ 2Y^T & & & 2\widetilde{W}_{22} \end{bmatrix} < 0, \tag{17}$$

where $\overset{\triangle}{A} = \bar{P}_1A^T + A\bar{P}_1 + A_r\bar{Q}_1A_r^T + \gamma^{-2}BB^T + EM + M^TE^T$, $\widetilde{W}_{22}$ and Y are the same as Eq.(16), then the closed-loop system formed by system (1) and the state feedback control

$$u(t) = Kx(t),\ K = M\bar{P}_1^{-1} \tag{18}$$

has the property (*).

Remark 3 We have the similar result for the system with multiple delays in state.

Remark 4 If the delays of the state equation and the output equation are different, we still have the similar result.

Remark 5 Because of the special forms of matrices P and Q in Eq.(9), the results have certain conservation. (How to reduce this kind of conservation needs further research).

Example We design the state controller (2) of system (1), and the matrices of system (1):

$$A = \begin{bmatrix} -3 & 0.5 \\ 0 & 0.1 \end{bmatrix},\ A_r = \begin{bmatrix} 0 & 0.5 \\ -0.4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix},\ C = [0.5\ \ 0.2],\ D = -0.5,$$

$$E = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ C_r = [-0.5\ \ 0.5],\ F = -1,$$

when $\gamma = 1$, using the LMI toolbox of Matlab, we get directly the controller: $u(t) = [0.3053\ \ -1.5472]x(t)$.

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