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Optimal financing and dividend control of a corporation with transaction costs

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Abstract: In the financial markets corporations have to pay for the fixed and proportional transaction costs when distributing dividends and issuing external equity. But no discussions have been found on the optimal financing and dividends policy influenced by both the fixed and proportional transaction costs. To address this inadequacy, an optimal control problem is discussed using stochastic impulse control theory to determine the optimal policy. First the associated Hamilton-Jacobi-Bellman (HJB) equation is given, then its continuously differentiable solution is constructed. From the solution and generalized Itô Lemma, the optimal financing and dividends policy is derived. Finally the economic interpretations are presented to illustrate the applications of the results, and comparisons are made with existing literatures.

Key words: transaction cost; stochastic impulse control; HJB equation; quasi-variational inequalities; value function
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考虑交易费的融资与分红最优控制模型

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摘要:在实际金融市场中股份公司在红利分配和再融资过程中都需要支付固定交易费和比例交易费.而如何确定交易费对公司财务决策的影响还没有进行过讨论.本文利用随机脉冲控制理论研究了在收取固定和比例交易费的市场环境下,公司如何制定其最优的财务策略.首先给出了最优控制问题对应的Hamilton-Jacobi-Bellman方程,接着构造出了它的连续可微解.利用解的性质和推广的Itô公式,构造出了最优的再融资及分红策略.最后对模型的应用做了经济学上的解释,并与已有模型做了比较.

关键词:交易费;随机脉冲控制;HJB方程;拟变分不等式;值函数

1 Introduction

In corporate finance a basic problem is to find the optimal financing and dividend pay-out policy for a corporation, whose objective is to maximize the net expecting all the discounted dividends to be distributed to its shareholders. Several authors[1~3] considered a continuous time model in which the liquid capital of a firm evolved like an arithmetic Brownian motion, and their problem was to find the balance between high return and the probability of going bankruptcy. The problem was addressed with singular stochastic control method if transaction costs were not considered. It was shown that the firm pays no dividend until its liquid assets reach a certain level and then pays for everything in excess of this level as dividends. But when fixed and proportional costs existed, it became an impulse control problem, and the optimal policy was to put the assets down to a certain level whenever it reached the other upper level and the difference between the two levels would be dividends[4]. Another case was considered by Sethi and Taksar[5] when the firm can issue external equity. In [5] they proved there exists a two-fold continuously differentiable value function in the case where the coefficients are not constant, but only proportional transaction costs of new equity issuing were considered in their model.

This paper reformulates the Sethi-Taksar model by considering fixed and proportional costs when the firm pays dividends or issues new equity. Since there is no debt in this model and the company can issue external equity, bankruptcy is not possible in this case, which means the time horizon will be infinite. The value function and its related optimal policy are given explicitly in a stochastic impulse control framework.

2 Model

Consider a probability space (Ω, F, F_t, P) with a standard Brownian motion $W(t)$ adapted to the filtration

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F_t. The value of the liquid assets of the company at time t is denoted by $X(t)$ when there is no jump to the state of assets. The dynamics of $X(t)$ is given by

$$\mathrm{d}X(t) = a(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t).$$

Here the return function is denoted by $a(\cdot)$, so $a(x)$ represents the mean return when $X(t) = x$. The assumptions of $a(\cdot)$ and $\sigma(\cdot)$ are made the same as in Sethi and Taksar[5]. We assume $a(\cdot)$ to be concave and differentiable with $a(0) = 0, a'(0) \geqslant \lambda$ and $a'(\infty) \leqslant \lambda$, where λ is the cost of capital, which is the stockholder's required rate of return. The function $\sigma(\cdot)$ is an increasing, differentiable function satisfying $\sigma(0) = 0, \sigma'(0) > 0$ and $\sigma(x) < Mx$ for some $M > 0$. For the fact that transaction costs exist, paying dividends and issuing new equity will not be on the same moment. Thus we can describe the two actions by one control variable. A policy consists of a sequence of stopping times $\{\tau_0, \tau_1, \cdots\}$ and a sequence of random variables $\{y_0, y_1, \cdots\}$ such that

$$P(0 = \tau_0 < \tau_1 < \cdots \to \infty) = 1,$$
$$y_i \in F_{\tau_i} \text{ for } i = 0,1,\cdots,$$

where τ_i is the time when the controller enforces a jump to the state of the assets, with y_i the size of the jump. Then the assets value satisfies

$$\begin{cases} \mathrm{d}X(t) = a(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) - \mathrm{d}Y(t), \\ X_0 = x - y_0, \end{cases} \tag{1}$$

where

$$Y(t) = \sum_{\tau_i \leqslant t} y_i \tag{2}$$

and $X_{0_-} = x$ is the size of the firm's initial assets. When y_i is positive, the action means the company is paying dividends at the moment τ_i, and y_i is negative if external equity is issued. The policy $\{(\tau_i, y_i)\}$ is said to be admissible if

$$P_x(X(t) \geqslant 0 \text{ for all } t \geqslant 0) = 1, \tag{3}$$

$$E_x\Big(\sum_{i=0}^{\infty} |y_i| \mathrm{e}^{-\lambda\tau_i}\Big) < \infty \text{ for every } x \in \mathbb{R}. \tag{4}$$

In this article there exist fixed and proportional transaction costs which are associated with each dividend pay-out and equity issuing; for instance, some costs like taxes and commissions. The transaction cost scheme is as follows. Shareholders can only get $ry_i - K$ when the firm pays y_i as dividends, and they have to pay $c|y_i| + Q$ out to meet the amount of $|y_i|$ as new equity to the firm, where $0 < r < 1, c > 1$, and $K, Q > 0$. Set

$$f(y) = \begin{cases} ry - K, & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ cy - Q, & \text{if } y < 0 \end{cases} \tag{5}$$

and the index function for any admissible policy $\pi(\cdot) = \{(\tau_i, y_i)\}$ is defined by

$$J_x(\pi(\cdot)) = E_x \sum_{i\geqslant 0} \mathrm{e}^{-\lambda\tau_i} f(y_i). \tag{6}$$

Then, we define the value function as

$$v(x) = \sup_{\pi(\cdot)} J_x(\pi).$$

3 Hamilton-Jacobi-Bellman equation

Theorem 1 Suppose the value function $v(\cdot)$ is continuous. Then, for every $x \in \mathbb{R}$, in the sense of viscosity solutions as in Yong and Zhou[6] it satisfies the following HJB equation

$$\max\Big\{\frac{1}{2}\sigma^2(x)v''(x) + a(x)v'(x) - \lambda v(x),$$
$$\sup_{\delta\leqslant x}\{v(x-\delta) + f(\delta)\} - v(x)\Big\} = 0. \tag{7}$$

Proof The theorem is a direct corollary of Proposition 4.4 in Tang and Yong[7,pp167].

Heuristically from HJB (7), we consider the following quasi-variational inequalities:

$$\frac{1}{2}\sigma^2(x)v''(x) + a(x)v'(x) - \lambda v(x) \leqslant 0 \text{ for every } x \in \mathbb{R}, \tag{8}$$

$$v(x) \geqslant v(y) + f(x-y) \text{ for every } x \in \mathbb{R} \text{ and } y \geqslant 0. \tag{9}$$

Theorem 2 Suppose $v(\cdot): (-\infty, \infty) \to \mathbb{R}$ is continuous differentiable with bounded derivatives, and the second derivatives are continuous at all but a finite number of points. If $v(\cdot)$ is a solution of the QVI (8), (9), then for any admissible policy $\pi(\cdot)$, $v(x) \geqslant J_x(\pi(\cdot))$.

Proof It is well known that Itô formula remains valid even when $v(\cdot)$ is not twice continuously differentiable, provided that it has an absolutely continuous derivative v' and v'' is chosen as any density of v'. The proof can be found in Rogers and David[8]. For any admissible policy $\{(\tau_i, y_i)\}$, X_t is a semimartingale, so we get

$$\mathrm{e}^{-\lambda t}v(X_t) =$$
$$v(X_0) + \int_0^t (-\mathrm{e}^{-\lambda s}\lambda v(X_s))\mathrm{d}s +$$
$$\int_0^t \mathrm{e}^{-\lambda s}v'(X_{s-})\mathrm{d}X_s + \frac{1}{2}\int_0^t \mathrm{e}^{-\lambda s}v''(X_s)\sigma^2(X_s)\mathrm{d}s +$$
$$\sum_{0<s\leqslant t} \mathrm{e}^{-\lambda s}\{v(X_s) - v(X_{s-}) - (X_s - X_{s-})v'(X_{s-})\} =$$
$$v(X_0) + \int_0^t \mathrm{e}^{-\lambda s}Lv(X_s)\mathrm{d}s + \int_0^t \mathrm{e}^{-\lambda s}\sigma(X_s)v'(X_s)\mathrm{d}W_s +$$
$$\sum_{0<\tau_i\leqslant t} \mathrm{e}^{-\lambda\tau_i}[v(X_{\tau_i}) - v(X_{\tau_i-1})],$$

where $Lv(x) = \frac{1}{2}\sigma^2(x)v''(x) + a(x)v'(x) -$

$\lambda v(x)$. Taking expectation on both sides and letting $t \to \infty$, we get

$$v(x) \geqslant v(x) - E_x v(X_0) - E_x\left[\int_0^\infty e^{-\lambda s} Lv(X_s)\mathrm{d}s\right] + E_x\left[\sum_{i\geqslant 1} e^{-\lambda\tau_i}(v(X_{\tau_i-}) v(X_{\tau_i}))\right] = E_x\left[\sum_{i\geqslant 0} e^{-\lambda\tau_i}(v(X_{\tau_i-}) - v(X_{\tau_i}))\right] \geqslant E_x\left[\sum_{i\geqslant 0} e^{-\lambda\tau_i} f(X_{\tau_i-} - X_{\tau_i})\right] = E_x\left[\sum_{i\geqslant 0} e^{-\lambda} f(y_i)\right]. \tag{10}$$

Thus the proof is complete.

4 Explicit solution of the quasi-variational inequalities (QVI)

To find a solution to (8), (9), heuristically we assume there exist two barrier points I and S as defined in what follows:

$$I = \sup\{x \mid v(x) = \sup_{\varepsilon>0}\{v(x+\varepsilon) - c\varepsilon - Q\}\},$$

$$S = \inf\{x \mid v(x) = \sup_{\delta>0}\{v(x-\delta) + r\delta - K\}\}.$$

Then we define

$$A = S - \delta^*,\ B = I + \varepsilon^*,$$

where δ^*, ε^* satisfies

$$v(S) = v(S-\delta^*) + r\delta^* - K,$$

$$v(I) = v(I+\varepsilon^*) - c\varepsilon^* - Q.$$

Next we consider the following free boundary problem:

$$\frac{1}{2}\sigma^2(x)v''(x) + a(x)v'(x) - \lambda v(x) = 0,\ I \leqslant x \leqslant S, \tag{11}$$

$$v(x) = v(B) - c(B-x) - Q,\ x < I, \tag{12}$$

$$v(x) = v(A) + r(x-A) - K,\ x > S \tag{13}$$

subject to the boundary conditions

$$v'(A) = v'(S) = r, \tag{14}$$

$$v'(I) = v'(B) = c, \tag{15}$$

$$v(S) = v(A) + r(S-A) - K, \tag{16}$$

$$v(I) = v(B) - c(B-I) - Q. \tag{17}$$

As will be proved later, finding a solution to the free boundary problem (11) ~ (17) is in essence equivalent to find a solution to (8), (9).

Suppose a solution $v(x)$ to (11) ~ (17) is found. Differentiating (11) we see that $w(x) = v'(x)$ satisfies the following equations

$$\frac{1}{2}\sigma^2(x)w''(x) + (a(x) + \sigma(x)\sigma'(x))w'(x) + (a'(x) - \lambda)w(x) = 0,\ I \leqslant x \leqslant S, \tag{18}$$

$$w(A) = w(S) = r, \tag{19}$$

$$w(I) = w(B) = c, \tag{20}$$

$$\int_A^S (r - w(x))\mathrm{d}x = K, \tag{21}$$

$$\int_I^B (w(x-c)\mathrm{d}x = Q. \tag{22}$$

Next we solve the free boundary problem (18) ~ (22) for $w(x)$ and construct $v(x)$ via $w(x)$.

Theorem 3 There exists a continuously differentiable function $w(x)$ on $[I, S]$ satisfying (18) ~ (22).

Here the shooting method in ordinary differential equations is applied to prove this theorem. That means, first we turn the boundary value problem into an initial value problem with some unknown coefficients, then we show there indeed exist such coefficients to meet the boundary conditions. Now the proof of the theorem is made in several stages.

Let $x^* > 0$ be such that $a'(x^*) = \lambda$ and denote $\Phi = \{x: 0 < x \leqslant x^*\}$ and $\Psi = \{x: x > x^*\}$. Obviously $a'(x) \geqslant \lambda$ when $x \in \Phi$ and $a'(x) < \lambda$ when $x \in \Psi$.

Proposition 1 If $w(x)$ is a positive solution to (18), then $w(x)$ can not have a local minimum in $\Phi \setminus \{x^*\}$ or a local maximum in Ψ.

Proof Suppose w attains a local minimum at $x \in \Phi \setminus \{x^*\}$. Then $w'(x) = 0$ and $w''(x) \geqslant 0$. Since $(a'(x) - \lambda) > 0$ when $x \in \Phi \setminus \{x^*\}$ and $w(x) > 0$ due to our assumption, we see that (18) can not be satisfied. The proof of the second part of this proposition is similar.

Next we choose m, n such that $m > c$ and $0 < n < r$, then focus on the following free boundary problem:

$$\frac{1}{2}\sigma^2(x)w''(x) + (a(x) + \sigma(x)\sigma'(x))w'(x) + (a'(x) - \lambda)w(x) = 0,\ L \leqslant x \leqslant H, \tag{23}$$

$$w(L) = m,\ w'(L) = 0, \tag{24}$$

$$w(H) = n,\ w'(H) = 0, \tag{25}$$

where L and H are unknown boundary points.

Proposition 2 If $w(x)$ is a solution to (23) ~ (25) on $(0, \varepsilon]$, $\varepsilon > 0$, such that $w(0+) > 1$, then $\lim_{x\to 0} w(x) = \infty$.

Proposition 3 There exists a twice continuously differentiable function $w(x)$, $0 < L \leqslant x \leqslant H$, such that $w'(x) \leqslant 0$ on $[L, H]$ and (23) ~ (25) hold.

The proof of Proposition 2 and Proposition 3 can be found in Sethi and Taksar[5, pp161]. Here the boundary values are different from those in that paper but the same method can be applied. Proposition 4 shows that when m, n are given, there exist two positive boundary points L and H such that $w(x)$ is monotonically decreasing from m to n on $[L, H]$. Thus we find the points B and A on $[L, H]$ satisfying $w(B) = c$ and $w(A) = r$.

Proposition 4 Suppose $w(x)$ is the solution to

(23) ~ (25) found in Proposition 3, then $w(x)$ can be extended to $[I, L]$ and $[H, S]$ such that $w(I) = c$, $w(S) = r$ where $0 < I < L < H < S$.

Proof From the basic extensibility theorem for ordinary differential equations, we can extend $w(x)$ to the right neighborhood of H. In view of (18) and the definition of Ψ, for every x in the right neighborhood of H, $w'(x) > 0$. If there exists $\tilde{x} > H$ such that $w'(\tilde{x}) \leqslant 0$, we will find a local maximum in the right domain of H, which is impossible due to Proposition 1. It is easy to see that $w(x) \to \infty$ when $x \to \infty$; otherwise (18) can not be satisfied. Therefore, there exists $S > H$ such that $w(S) = r$ and $w(x)$ is strictly convex and monotonically increasing on $[H, S]$. The proof of the other part of this proposition is similar and Proposition 3 shows that I is positive and $w(x)$ is strictly concave and monotonically increasing on $[I, L]$. Thus the proof is complete.

Proof of Theorem 3 As was shown above, there exists a continuous solution $w_{m,n}(x)$ to (23) on $[I, S]$ satisfying the following boundary conditions:

$$w(I) = w(B) = c,\ w(L) = m, \tag{26}$$

$$w(A) = w(S) = r,\ w(H) = n, \tag{27}$$

where we denote $I(m,n), B(m,n), A(m,n), S(m,n)$ as I, B, A, S for convenience. Set

$$F(m,n) = \int_A^S (r - w_{m,n}(x))\mathrm{d}x - K, \tag{28}$$

$$G(m,n) = Q - \int_I^B (w_{m,n}(x) - c)\mathrm{d}x. \tag{29}$$

Using the theorem of continuous dependence of $w_{m,n}(x)$ on m and n, we see that $F(m,n), G(m,n)$ are continuous functions. By virtue of Proposition 5 and (18), we see that $A \to S$ uniformly as $n \to r$. Therefore, for $K > 0$ there exists $\tilde{r}(K) < r$ such that $F(m, \tilde{r}(K)) < 0$ holds uniformly for any $m > c$. Then, from the theorem of continuous dependence on initial values, for $T = 2K/r$, there exists $\delta(T) > 0$ such that $w_{m,\delta(T)}(x) < r/2$ holds uniformly for any $m > c$ on $[H, H + T]$. Then we have

$$F(m, \delta(T)) > rT/2 - K = 0.$$

Now let $n \in [\delta(K), \tilde{r}(K)]$. The argument is similar to prove that there exist $m < \tilde{c}(Q) < P(Q)$ such that $G(P(Q), n) < 0$ and $G(\tilde{c}(Q), n) > 0$ hold uniformly for any $n \in [\delta(K), \tilde{r}(K)]$ with a little difference that the continuous dependence theorem is applied on $1/w(x)$ to prove the existence of $P(Q)$. Then define $D = [\tilde{c}(Q), P(Q)] \times [\delta(K), \tilde{r}(K)]$ and we get

$$F(m, \delta(K)) > 0,\ F(m, \tilde{r}(K)) < 0, \tag{30}$$

$$G(\tilde{c}(Q), n) > 0,\ G(P(Z), n) < 0 \tag{31}$$

are true for every $(m,n) \in D$.

Define

$$M_1 = \max_D G(m,n),\ m_1 = \min_D G(m,n), \tag{32}$$

$$M_2 = \max_D F(m,n),\ m_2 = \min_D F(m,n) \tag{33}$$

and from (30),(31) we have $M_1, M_2 > 0$ and $m_1, m_2 < 0$. Then put

$$\begin{cases} \xi_1' = \mathrm{d}(\{G(m,n) > 0\}, \{m = P(Q)\}), \\ \xi_1'' = \mathrm{d}(\{G(m,n) < 0\}, \{m = \tilde{c}(Q)\}), \end{cases} \tag{34}$$

$$\begin{cases} \xi_5' = \mathrm{d}(\{F(m,n) > 0\}, \{g = \tilde{r}(K)\}), \\ \xi_2'' = \mathrm{d}(\{F(m,n) < 0\}, \{n = \delta(K)\}), \end{cases} \tag{35}$$

$$\varepsilon_1 = \min\{\frac{\xi_1'}{M_1}, \frac{-\xi_1''}{m_1}\},\ \varepsilon_2 = \min\{\frac{\xi_2'}{M_2}, \frac{-\xi_2''}{m_2}\}, \tag{36}$$

where $\mathrm{d}(A, B)$ is the Euclidean distance between the sets A and B.

Define a map $\Gamma: (m,n) \to (\tilde{m}, \tilde{n})$ as follows:

$$\tilde{m} = m + \varepsilon_1 G(m,n), \tag{37}$$

$$\tilde{n} = n + \varepsilon_2 F(m,n). \tag{38}$$

If $G(m,n) \geqslant 0$, then

$$\tilde{c}(Q) \leqslant m \leqslant \tilde{m} \leqslant m + \varepsilon_1 M_1 \leqslant m + M_1 \frac{\xi_1'}{M_1} \leqslant P(Q). \tag{39}$$

If $G(m,n) < 0$, then

$$\tilde{c}(Q) \leqslant m + m_1(\frac{\xi_1''}{m_1}) \leqslant m + \varepsilon_1 m_1 \leqslant \tilde{m} \leqslant m \leqslant P(Q). \tag{40}$$

Now we get $\tilde{c}(Q) \leqslant \tilde{m} \leqslant P(Q)$ and from the same method we see that $\delta(K) \leqslant \tilde{n} \leqslant \tilde{r}(K)$. So Γ is a continuous self-map of the compact and convex set D. By Brouwer fixed-point theorem, there exists a fixed point $(m^*, n^*) \in D$ such that $\Gamma(m^*, n^*) = (m^*, n^*)$. That means

$$F(m^*, n^*) = 0,\ G(m^*, n^*) = 0.$$

Hence we have found a $w_{m^*,n^*}(r)$ which satisfies

$$w(I) = w(B) = c,\ w(L) = m^*,$$

$$w(A) = w(S) = r,\ w(H) = n^*,$$

$$\int_A^S (r - w_{m^*,n^*}(x))\mathrm{d}x = K,$$

$$\int_I^B (w_{m^*,n^*}(x) - c)\mathrm{d}x = Q.$$

This completes the proof of Theorem 3.

5 Verification theorem and construction of the optimal molicy

In this section that the solution $v(x)$ to the free boundary problem (11) ~ (17) will be shown to satisfy the QVI (8),(9). Then we construct an optimal policy via $v(x)$.

Theorem 4 If $w_{m^*,n^*}(x), I, B, L, A, H$ and S are a solution to the free boundary problem (18) ~ (22) found in Theorem 3, then the function is defined as follows

$$v(x)=\begin{cases} v(B)-c(B-x)-Q, & x<I,\\ \dfrac{a(L)m^*}{\lambda}+\int_L^x w_{m^*,n^*}(y)\mathrm{d}y, & I\leqslant x\leqslant S,\\ v(A)+r(x-A)-K, & x>S \end{cases} \tag{41}$$

is a solution to the QVI (8), (9).

Proof Since $v'(x)=w_{m^*,n^*}(x)$ on $[I,S]$ and w satisfies (19), (20), we see that v is subject to $v'(A)=v'(S)=r, v'(I)=v'(B)=c$. As a result, $v'(t)$ is continuous at the points I and S. Then,

$$v(I-)=v(B)-c(B-I)-Q=$$
$$\frac{a(L)m^*}{\lambda}+\int_L^I w_{m^*,n^*}(x)\mathrm{d}x+$$
$$\int_I^B(w_{m^*,n^*}(x)-c)\mathrm{d}x-Q=v(I),$$
$$v(S+)=v(A)+r(x-A)-K=$$
$$\frac{a(L)m^*}{\lambda}+\int_L^S w_{m^*,n^*}(x)\mathrm{d}x+$$
$$\int_A^S(r-w_{m^*,n^*}(x))\mathrm{d}x-K=v(S).$$

Consequently, $v(x)$ is continuously differentiable. From the construction process it follows that there exists a constant C on $[I,S]$ such that

$$\frac{1}{2}\sigma^2(x)v''(x)+a(x)v'(x)-\lambda v(x)=C. \tag{42}$$

Set $x=L$, then substitute $w'_{m^*,n^*}(L)=0$ and $w_{m^*,n^*}(L)=m^*$ into the left side of (42), we have

$$a(L)m^*-\lambda\frac{a(L)m^*}{\lambda}=0=C.$$

Thus the inequality (8) is tight on $[I,B]$.

For any $y<x$,

$$K=\int_A^S[r-w(s)]\mathrm{d}s\geqslant\int_y^x[r-w(s)]\mathrm{d}s$$

since (A,S) is the maximal interval where $w(s)<r$.

If $y>x$,

$$Q=\int_I^B[w(s)-c]\mathrm{d}s\geqslant\int_x^y[w(s)-c]\mathrm{d}s$$

since (I,B) is the maximal interval where $w(s)>c$. Thus the inequality (9) is valid. Next we need to verify the validity of (8) for $x<I$ and $x>S$. If $x<I$, then

$$\frac{1}{2}\sigma^2(x)v''(x)+a(x)v'(x)-\lambda v(x)=$$
$$a(x)c-\lambda(v(B)-c(B-x)-Q)=$$
$$-c\int_x^I(a'(y)-\lambda)\mathrm{d}y+ca(I)-\lambda v(I)=$$
$$-c\int_x^I(a'(y)-\lambda)\mathrm{d}y-\frac{1}{2}\sigma^2(x)v''(I+)<0$$

since $a'(y)\geqslant\lambda$ for all $y\leqslant x^*$ and $v''(I+)>0$.

If $x>S$, then

$$\frac{1}{2}\sigma^2(x)v''(x)+a(x)v'(x)-\lambda v(x)=$$
$$a(x)r-\lambda(v(S)-rS+rx)=$$
$$r\int_S^x(a'(y)-\lambda)\mathrm{d}y+ra(S)-\lambda v(S)=$$
$$r\int_S^x(a'(y)-\lambda)\mathrm{d}y-\frac{1}{2}\sigma^2(x)v''(S-)<0$$

since $a'(y)<\lambda$ for all $y>x^*$ and $v''(S-)>0$. Hence, we have

$$\max\{\frac{1}{2}\sigma^2(x)v''(x)+a(x)v'(x)-\lambda v(x),$$
$$\sup_{\delta\leqslant x}\{v(x-\delta)+f(\delta)\}-v(x)\}\leqslant 0 \tag{43}$$

for every $x\in$ Ξ. For $x\in[I,S]$,

$$\frac{1}{2}\sigma^2(x)v''(x)+a(x)v'(x)-\lambda v(x)=0.$$

If $x<I$, let $\delta=x-B$, then

$$v(B)+c(x-B)-Q=v(x).$$

If $x>S$, let $\delta=x-A$, then

$$v(A)+(x-A)-K=v(x).$$

Therefore, $v(x)$ satisfies the QVI (8), (9) and the proof is complete.

Now define the strategy $\{(\hat{\tau}_i,\hat{y}_i)\}$ in the following way:

$$\hat{y}_0=\begin{cases} x-B, & x\leqslant I,\\ 0, & I<x<S,\\ x-A, & x\geqslant S. \end{cases} \tag{44}$$

$\hat{\tau}_0=0$, and for $i\geqslant 1$,

$$\hat{\tau}_i=\inf\{t>\hat{\tau}_i-1:\hat{X}(t-)=I \text{ or } S\}, \tag{45}$$

$$\hat{y}_i=\begin{cases} I-B, & \text{if } \hat{X}(\hat{\tau}_i-)=I,\\ S-A, & \text{if } \hat{X}(\hat{\tau}_i-)=S, \end{cases} \tag{46}$$

where $\hat{X}(\cdot)$ is the controlled diffusion process defined as follows

$$\begin{cases} \mathrm{d}\hat{X}(t)=a(\hat{X}(t))\mathrm{d}t+\sigma(\hat{X}(t))\mathrm{d}W(t)-\mathrm{d}\hat{Y}(t),\\ \hat{X}_0=x-\hat{y}_0. \end{cases} \tag{47}$$

Theorem 5 Suppose that $v(x)$ is the function (41) and $\hat{\pi}(\cdot)=\{(\hat{\tau}_i,\hat{y}_i),i\geqslant 2\}$ is the policy defined above. Then $\hat{\pi}(\cdot)$ is the optimal control; that is

$$v(x)=J_x(\hat{\pi}(\cdot)). \tag{48}$$

Proof Making the same calculations as in the proof of Theorem 2, we get

$$E_x[\mathrm{e}^{-\lambda t}v(\hat{X}_t)]=$$
$$E_xv(\hat{X}_0)+E_x[\int_0^t\mathrm{e}^{-\lambda s}Lv(\hat{X}_s)\mathrm{d}s]+$$

$$E_x[\sum_{\tau_i \le t} e^{-\lambda\tau_i}(v(\hat{X}_{\tau_i}) - v(\hat{X}_{\tau_i -}))]. \qquad (49)$$

In view of (45) ~ (47), $\hat{X}_s \in [I, S]$. Since $v(x)$ is subject to (11), the integrand in the first integral in the right-hand side of (49) vanishes. Thus (49) can be rewritten as

$$v(x) - E_x[e^{-\lambda t}v(\hat{X}_t)] =$$
$$v(x) - E_x v(\hat{X}_0 + E_x[\sum_{i<\hat{\tau}_i \le t} e^{-\lambda\tau_i}(v(\hat{X}_{\tau_i -})v(\hat{X}_{\hat{\tau}_i}))] =$$
$$E_x[\sum_{\hat{\tau}_i \le t} e^{-\lambda\hat{\tau}_i}(v(\hat{X}_{\tau_i -}) - v(\hat{X}_{\hat{\tau}_i}))] =$$
$$E_x[\sum_{\hat{\tau}_i \le t} e^{-\lambda\hat{\tau}_i} f(\hat{y}_i)].$$

By virtue of (47), $v(\hat{X}_t)$ is bounded by $v(S)$. Therefore, taking the limit as $t \to \infty$, we get

$$v(x) = E_x[\sum_{i \ge 0} e^{-\lambda\hat{\tau}_i} f(\hat{y}_i)].$$

Hence the proof is complete.

Let us conclude this section by providing an economic interpretation of the optimal policy $\{(\hat{\tau}_i, \hat{y}_i)\}$. If the initial value x is below I, we need to issue new equity as much as $B - x$ immediately at $t = 0$. If x is above the upper assets level S, the dividends pay-out amount to $x - A$ should be distributed at time zero. Then, the conclusion is that the firm pays no dividend and brings in external equity when its reserve drops below I with the amount $B - I$. When the reserve exceeds S, the firm pays $S - A$ in dividends, and no external equity is raised. Next we make a comparison with the result obtained by Sethi and Taksar[5]. In our context, the value function $v(x)$ is not concave any more, and it is not twice continuously differentiable at I and S. From the proof of Theorem 3, we see that when fixed transaction costs $Q \to 0$, $m^* \to \tilde{c}(Q) \to c$, then we have $I \to B$. When $Q = 0$, $K = 0$, the model turns out to be a singular stochastic control problem. In this case, the problem in [5] is the limit situation when $Q \to 0$, $K \to 0$ and $r = 1$ in our model.

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