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# 线性–非二次最优控制问题的一种解法

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**摘要**: 讨论了求解状态终端无约束线性–非二次最优控制问题的拟Riccati方程方法, 并据此提出了计算无约束线性–非二次问题之数值解的方法; 然后将这个方法与一种能近似地化有约束问题为无约束问题的惩罚方法结合起来, 给出了一种算法, 可以计算状态终端有约束的线性–非二次最优控制问题之近似解.

**关键词**: 最优控制; 线性–非二次; 特征线; 计算方法

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# A method for solving the linear-nonquadratic optimal control problem

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**Abstract:** The quasi-Riccati equation method is discussed for solving the linear-nonquadratic optimal control problems without terminal state-constrains; and accordingly propose an efficient method for obtaining numerical solutions to the linear-nonquadratic problems without constraints. By combining this method with the penalty method which approximately converts a problem with constraints to one without constraints, we develop an algorithm for approximately computing the solutions to linear-nonquadratic optimal control problems with terminal state-constrains.

**Key words:** optimal control; linear-nonquadratic; characteristic lines; computational method

## 1 引言(Introduction)

有限时区线性–二次最优控制问题是一类在线性系统控制理论中有着基本的重要性因而广为人知的研究对象, 其标准理论可参见文[1]. 此类问题用一个二次指标泛函来衡量控制函数的优劣. 如果所考虑的最优控制问题还附有状态终端约束, 可以运用惩罚方法的思想引入足够大的惩罚因子$\gamma$把原有约束问题CLNQP近似地化为无约束问题$(\mathrm{LNQP})_\gamma$, 解$(\mathrm{LNQP})_\gamma$得到原问题的一足够好近似最优控制(详见本文第5节). 设状态终端约束集是由某非线性不等方程组限定的, 则第5节中的$(\mathrm{LNQP})_\gamma$已不是线性–二次问题, 而是所谓的线性–非二次问题(尤云程和本文作者之一潘立平曾先后对线性–非二次问题作过研究, 见文[2∼4], 文[4]在较文[3]为弱的假定下建立了相应的状态反馈解理论, 它是关于线性–二次问题的标准理论向线性–非二次问题的自然推广). 应当指出: 状态终端约束集的工程意义是预定的状态目标范围, 当系统完全能控时, 虽然人们可在状态目标范围中任意取定一点然后解状态终端约束集为该单点的最优控制问题以获得能使系统状态实现所欲之转移的控制函数, 但当原状态终端约束集不是单点集时这样做的后果往往是使得问题的允许控制函数集合大大缩小从而导致问题的最优值(即指标泛函在最优控制函数处的取值)变大. 对线性–二次最优控制问题已有计算其最优状态反馈函数的优异算法(见文[5]); 而对线性–非二次问题迄今尚未见计算其最优控制函数的高效方法, 本文的主要目的就是在这方面作点有益的探索(若用面向非线性系统最优控制问题的一般方法—动态规划方法进行解算则要依次求HJB方程之解$V$和取决于$V$等的一微分包含之解, 相当困难故不甚可取, 参见专著[6]或专著[7]).

## 2 基本假定与问题(Basic hypotheses and problem)

假定:

**假设 1** $t_0, t_1 \in \mathbb{R}^1, t_0 < t_1, x_0 \in \mathbb{R}^n$;

**假设 2** $A(\cdot) \in C([t_0, t_1], \mathbb{R}^{n\times n}),\ B(\cdot) \in C([t_0, t_1], \mathbb{R}^{n\times m}),\ f(\cdot) \in C([t_0, t_1], \mathbb{R}^n)$;

**假设 3** $g \in C([t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^1),\ \forall\, t \in$

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$[t_0,t_1], g(t,\cdot,\cdot) \in C^2(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^1)$, $G(\cdot) \in C^1(\mathbb{R}^n, \mathbb{R}^1)$;

**假设 4** $\exists L_0 \in \mathbb{R}^1$使$\forall (t,x,u) \in [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^m$, $\min(g(t,x,u), G(x)) \geqslant L_0$;

**假设 5** $\exists \delta_0 > 0$与$M > 0$使$\forall (t,x,u) \in [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^m$, $\delta_0 I_m \leqslant (\frac{\partial^2 g}{\partial u^2})(t,x,u) \leqslant M I_m$;

**假设 6** $Q \in C([t_0,t_1] \times \mathbb{R}^n, \mathbb{R}^1)$, $\forall\, t \in [t_0,t_1], Q(t,\cdot) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$;

**假设 7** $R \in C([t_0,t_1] \times \mathbb{R}^m, \mathbb{R}^1)$, $\forall\, t \in [t_0,t_1], R(t,\cdot) \in C^2(\mathbb{R}^m, \mathbb{R}^1)$, $\exists \delta_0 > 0$与$M > 0$使$\forall\, (t,u) \in [t_0,t_1] \times \mathbb{R}^m$, $\delta_0 I_m \leqslant (\frac{\partial^2 R}{\partial u^2})(t,u) \leqslant M I_m$;

**假设 8** $S(\cdot) \in C([t_0,t_1], \mathbb{R}^{m\times n})$;

**假设 9** $E_j(\cdot) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$且$E_j : \mathbb{R}^n \to \mathbb{R}^1$凸, $j = 1, \cdots, l$.

考虑由线性时变受控系统

$$\begin{cases} \frac{\mathrm{d}x(t,u(\cdot))}{\mathrm{d}t} = A(t)x(t,u(\cdot)) + B(t)u(t) + \\ \qquad f(t),\ \text{a.e.}\ t \in [t_0,t_1], \\ x(t_0,u(\cdot)) = x_0 \end{cases} \tag{1}$$

与指标泛函

$$\begin{cases} I(u(\cdot)) := \\ \int_{t_0}^{t_1} g(t,x(t,u(\cdot)),u(t))\mathrm{d}t + G(x(t_1,u(\cdot))), \\ \forall\, u(\cdot) \in \mathcal{U} := L^2(t_0,t_1;\mathbb{R}^m) \end{cases} \tag{2}$$

构成的线性–非二次最优控制问题(LNQP): 求$\hat{u}(\cdot) \in \mathcal{U}$使得$I(\hat{u}(\cdot)) = \inf\limits_{u(\cdot)\in\mathcal{U}} I(u(\cdot))$.

## 3 拟 Riccati 方程与最优状态反馈映射(Quasi-Riccati equation and optimal state feedback mapping)

易知: 假设3中关于$g$的假定与假设5蕴涵$\forall (t,x) \in [t_0,t_1] \times \mathbb{R}^n$, $(\frac{\partial g}{\partial u})(t,x,\cdot)^{\mathrm{T}}$是映$\mathbb{R}^m$到$\mathbb{R}^m$上的同胚映照且其逆映照$[(\frac{\partial g}{\partial u})(t,x,\cdot)^{\mathrm{T}}]^{-1}$的导映照为$[(\frac{\partial^2 g}{\partial u^2})(t,x,[(\frac{\partial g}{\partial u})(t,x,\cdot)^{\mathrm{T}}]^{-1})]^{-1}$(参见文 [4] 的引理3.1). 引进如下拟Riccati 1阶非线性偏微分方程:

$$\begin{cases} (\frac{\partial K}{\partial t})(t,x) + (\frac{\partial K}{\partial x})(t,x)\{A(t)x + B(t)[(\frac{\partial g}{\partial u})(t,x,\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}K(t,x)) + \\ f(t)\} + A(t)^{\mathrm{T}}K(t,x) + (\frac{\partial g}{\partial x})\big(t,x,[(\frac{\partial g}{\partial u})(t,x,\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}K(t,x))\big)^{\mathrm{T}} = 0, \\ (t,x) \in [t_0,t_1] \times \mathbb{R}^n,\ K(t_1,x) = (\frac{\partial G}{\partial x})(x)^{\mathrm{T}},\ \forall\, x \in \mathbb{R}^n. \end{cases} \tag{3}$$

紧接着要给出的定理表明: 当LNQP的最优控制与方程(3)的整体经典解都存在时(关于这两者之解的存在唯一性后面再谈), 可利用方程(3)之解立即得到LNQP的最优控制的状态反馈表示.

**定理 1** 假定假设1~5成立. 设: $\hat{u}(\cdot)$是LNQP的最优控制, $\hat{K}$ $(\in C^1([t_0,t_1] \times \mathbb{R}^n, \mathbb{R}^n))$是方程(3)的整体经典解. 则

$$\begin{cases} \hat{F} : [t_0,t_1] \times \mathbb{R}^n \to \mathbb{R}^m, \\ \hat{F}(t,x) := \\ [(\frac{\partial g}{\partial u})(t,x,\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,x)), \\ \forall\, (t,x) \in [t_0,t_1] \times \mathbb{R}^n \end{cases} \tag{4}$$

是LNQP的最优状态反馈映射.

**证** 简记

$$\hat{x}(t) := x(t,\hat{u}(\cdot)),\ \forall\, t \in [t_0,t_1]. \tag{5}$$

根据Pontryagin最大值原理(参见文[8]或文[7]), 可知有$\hat{\psi}(\cdot) \in AC([t_0,t_1], \mathbb{R}^n)$使得

$$\begin{cases} \hat{u}(t) = [(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t)), \\ \qquad \forall\, t \in [t_0,t_1], \\ \frac{\mathrm{d}\hat{\psi}(t)}{\mathrm{d}t} = -A(t)^{\mathrm{T}}\hat{\psi}(t) + (\frac{\partial g}{\partial x})(t,\hat{x}(t),\hat{u}(t))^{\mathrm{T}}, \\ \qquad t \in [t_0,t_1], \\ \hat{\psi}(t_1) = -(\frac{\partial G}{\partial x})(\hat{x}(t_1))^{\mathrm{T}}. \end{cases} \tag{6}$$

记

$$\frac{\partial^2 g(t,x,u)}{\partial x \partial u} := \begin{pmatrix} \frac{\partial^2 g(t,x_1,\cdots,x_n,u_1,\cdots,u_m)}{\partial x_1 \partial u_1} & \cdots & \frac{\partial^2 g(t,x_1,\cdots,x_n,u_1,\cdots,u_m)}{\partial x_1 \partial u_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 g(t,x_1,\cdots,x_n,u_1,\cdots,u_m)}{\partial x_n \partial u_1} & \cdots & \frac{\partial^2 g(t,x_1,\cdots,x_n,u_1,\cdots,u_m)}{\partial x_n \partial u_m} \end{pmatrix}, \tag{7}$$

$$\forall\,(t,x,u)=(t,\begin{pmatrix}x_1\\ \cdots\\ x_n\end{pmatrix},\begin{pmatrix}u_1\\ \cdots\\ u_m\end{pmatrix})\in[t_0,t_1]\times\mathbb{R}^n\times\mathbb{R}^m.$$

式(6)和$\hat{x}(\cdot)$的定义(式(5))蕴涵

$$\begin{aligned}&\frac{\mathrm{d}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]}{\mathrm{d}t}=\frac{\mathrm{d}\hat{\psi}(t)}{\mathrm{d}t}+(\frac{\partial\hat{K}}{\partial t})(t,\hat{x}(t))+(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t}=\\&-A(t)^{\mathrm{T}}\hat{\psi}(t)+(\frac{\partial g}{\partial x})(t,\hat{x}(t),\hat{u}(t))^{\mathrm{T}}+(\frac{\partial\hat{K}}{\partial t})(t,\hat{x}(t))+(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))[A(t)\hat{x}(t)+B(t)\hat{u}(t)+f(t)]=\\&-A(t)^{\mathrm{T}}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]+A(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t))+(\frac{\partial g}{\partial x})\big(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))\big)^{\mathrm{T}}+\\&(\frac{\partial\hat{K}}{\partial t})(t,\hat{x}(t))+(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))\big\{A(t)\hat{x}(t)+B(t)[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))+f(t)\big\},\ \forall\,t\in[t_0,t_1].\end{aligned}\tag{8}$$

上式中$\hat{K}$是方程(3)的整体经典解和

$$\hat{\psi}(t_1)=-(\frac{\partial G}{\partial x})(\hat{x}(t_1))^{\mathrm{T}}=-\hat{K}(t_1,\hat{x}(t_1))\tag{9}$$

一起蕴涵

$$\begin{aligned}&\frac{\mathrm{d}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]}{\mathrm{d}t}+A(t)^{\mathrm{T}}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]=\\&(\frac{\partial\hat{K}}{\partial t})(t,\hat{x}(t))+(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))\big\{A(t)\hat{x}(t)+B(t)[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))+f(t)\big\}+\\&A(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t))+(\frac{\partial g}{\partial x})\big(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))\big)^{\mathrm{T}}=\\&(\frac{\partial\hat{K}}{\partial t})(t,\hat{x}(t))+(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))\{A(t)\hat{x}(t)+B(t)[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))+f(t)\}+\\&A(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t))+(\frac{\partial g}{\partial x})\big(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\big)^{\mathrm{T}}+\\&(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))B(t)\big\{[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))-[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\big\}+\\&(\frac{\partial g}{\partial x})\big(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))\big)^{\mathrm{T}}-\\&(\frac{\partial g}{\partial x})\big(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\big)^{\mathrm{T}}=\\&(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))B(t)\big\{[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))-[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\big\}+\\&(\frac{\partial g}{\partial x})\big(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))\big)^{\mathrm{T}}-\\&(\frac{\partial g}{\partial x})\big(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\big)^{\mathrm{T}},\ \forall\,t\in[t_0,t_1]\end{aligned}\tag{10}$$

与

$$\hat{\psi}(t_1)+\hat{K}(t_1,\hat{x}(t_1))=0.\tag{11}$$

式(10)(11)又蕴涵

$$\begin{aligned}0=\ &\frac{\mathrm{d}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]}{\mathrm{d}t}+A(t)^{\mathrm{T}}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]-\{(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))B(t)+\\&\int_0^1(\frac{\partial^2 g}{\partial x\partial u})(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))+\theta\{[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))-\end{aligned}$$

$$[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\})\mathrm{d}\theta\}\{[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))-$$
$$[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\}=$$
$$\frac{\mathrm{d}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]}{\mathrm{d}t}+\{A(t)^{\mathrm{T}}-\{(\frac{\partial \hat{K}}{\partial x})(t,\hat{x}(t))B(t)+$$
$$\int_0^1(\frac{\partial^2 g}{\partial x\partial u})(t,\hat{x}(t),[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))+\theta\{[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))-$$
$$[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\})\mathrm{d}\theta\}\{\int_0^1[(\frac{\partial^2 g}{\partial u^2})(t,\hat{x}(t),$$
$$[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\{\hat{K}(t,\hat{x}(t))-\vartheta[\hat{\psi}(t)+$$
$$\hat{K}(t,\hat{x}(t))]\}))]^{-1}\mathrm{d}\vartheta\}B(t)^{\mathrm{T}}\}[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))],\ \forall\, t\in[t_0,t_1] \tag{12}$$

与式(11), 即$\hat{\psi}(\cdot)+\hat{K}(\cdot,\hat{x}(\cdot))$是线性齐次常微分方程

$$\begin{cases}\dfrac{\mathrm{d}\phi(t)}{\mathrm{d}t}+\{A(t)^{\mathrm{T}}-\{(\dfrac{\partial \hat{K}}{\partial x})(t,\hat{x}(t))B(t)+\displaystyle\int_0^1(\dfrac{\partial^2 g}{\partial x\partial u})(t,\hat{x}(t),[(\dfrac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))+\\ \theta\{[(\dfrac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(B(t)^{\mathrm{T}}\hat{\psi}(t))-[(\dfrac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))\})\mathrm{d}\theta\}\{\displaystyle\int_0^1[(\dfrac{\partial^2 g}{\partial u^2})(t,\hat{x}(t),\\ [(\dfrac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\{\hat{K}(t,\hat{x}(t))-\vartheta[\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))]\}))]^{-1}\mathrm{d}\vartheta\}B(t)^{\mathrm{T}}\}\phi(t)=0,\\ \forall\, t\in[t_0,t_1],\ \phi(t_1)=0\end{cases} \tag{13}$$

的解, 再注意到假设5蕴涵

$$\begin{cases}\forall\,(t,x,y)\in[t_0,t_1]\times\mathbb{R}^n\times\mathbb{R}^n,\\ \left|[(\dfrac{\partial^2 g}{\partial u^2})(t,x,[(\dfrac{\partial g}{\partial u})(t,x,\cdot)^{\mathrm{T}}]^{-1}(y))]^{-1}\right|\leqslant\dfrac{1}{\delta_0}\end{cases} \tag{14}$$

等, 应用Gronwall引理即知

$$\hat{\psi}(t)+\hat{K}(t,\hat{x}(t))=0,\forall\, t\in[t_0,t_1], \tag{15}$$

从而立知有(回顾式(6)的第1式)

$$\hat{u}(t)=$$
$$[(\frac{\partial g}{\partial u})(t,\hat{x}(t),\cdot)^{\mathrm{T}}]^{-1}(-B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)))=$$
$$\hat{F}(t,\hat{x}(t)),\ \forall\, t\in[t_0,t_1]. \tag{16}$$

定理1证毕.

接下来, 记

$$U(t,s):=\bar{X}(t)\bar{X}(s)^{-1},\ \forall\, t,s\in[t_0,t_1], \tag{17}$$

其中的$\bar{X}(\cdot)$是线性齐次矩阵常微分方程

$$\begin{cases}\dfrac{\mathrm{d}X(t)}{\mathrm{d}t}=A(t)X(t),\ t\in[t_0,t_1],\\ X(t_0)=I_n\end{cases} \tag{18}$$

之解, 分别在$\mathcal{U}, AC([t_0,t_1],\mathbb{R}^n)$中任意取定$u(\cdot)$, $x(\cdot)$, 定义算子$\tilde{R}''(u(\cdot)),\tilde{B},\mathit{\Gamma},\mathit{\Gamma}_1$和$\tilde{Q}''(x(\cdot))$如下:

$$\begin{cases}\tilde{R}''(u(\cdot)):\mathcal{U}\to\mathcal{U},\ v(\cdot)\mapsto(\dfrac{\partial^2 R}{\partial u^2})(\cdot,u(\cdot))v(\cdot),\\ \qquad\forall\, v(\cdot)\in\mathcal{U};\\ \tilde{B}:\mathcal{U}\to\mathcal{X}:=L^2(t_0,t_1;\mathbb{R}^n),\\ \qquad v(\cdot)\mapsto B(\cdot)v(\cdot),\forall\, v(\cdot)\in\mathcal{U};\\ \mathit{\Gamma}:\mathcal{X}\to\mathcal{X},\\ \qquad y(\cdot)\mapsto\displaystyle\int_{t_0}^{\cdot}U(\cdot,\tau)y(\tau)\mathrm{d}\tau,\forall\, y(\cdot)\in\mathcal{X};\\ \mathit{\Gamma}_1:\mathcal{X}\to\mathbb{R}^n,\\ \qquad y(\cdot)\mapsto\displaystyle\int_{t_0}^{t_1}U(t_1,\tau)y(\tau)\mathrm{d}\tau,\forall\, y(\cdot)\in\mathcal{X};\\ \tilde{Q}''(x(\cdot)):\mathcal{X}\to\mathcal{X},\ y(\cdot)\mapsto(\dfrac{\partial^2 Q}{\partial x^2})(\cdot,x(\cdot))y(\cdot),\\ \qquad\forall\, y(\cdot)\in\mathcal{X}.\end{cases} \tag{19}$$

对于LNQP之最优控制和拟Riccati方程(3)之解的存在唯一性以及方程(3)之解关于$(A(\cdot),B(\cdot),f(\cdot))$的连续依赖性, 完全类似于本文作者之一潘立平在文[4]中所作的对应讨论, 可证得如下两结果:

**定理 2** 假定假设1,2,6,7和

**假设 10**

$$G(\cdot)\in C^2(\mathbb{R}^n,\mathbb{R}^1)$$

成立. 设

$$\begin{cases} g(t,x,u)=Q(t,x)+R(t,u),\\ \forall\,(t,x,u)\in[t_0,t_1]\times\mathbb{R}^n\times\mathbb{R}^m \end{cases} \tag{20}$$

和$\exists\delta>0$使

$$\begin{cases} \forall\,(u(\cdot),x(\cdot),x_1)\in\mathcal{U}\times AC([t_0,t_1],\mathbb{R}^n)\times\mathbb{R}^n,\\ \varPhi(u(\cdot),x(\cdot),x_1):=\\ \tilde{R}''(u(\cdot))+\tilde{B}^*[\varGamma^*\tilde{Q}''(x(\cdot))\varGamma+\\ \varGamma_1^*(\dfrac{\partial^2G}{\partial y^2})(x_1)\varGamma_1]\tilde{B}\geqslant\delta I_{\mathcal{U}}. \end{cases} \tag{21}$$

则LNQP的最优控制存在唯一且方程(3)有唯一整体经典解.

**定理 3**　仍假定假设1,2,6,7,10和式(20)成立, 并设

$$\forall\,t\in[t_0,t_1],\ Q(t,\cdot):\mathbb{R}^n\to\mathbb{R}^1 \tag{22}$$

为凸函数, $G:\mathbb{R}^n\to\mathbb{R}^1$亦为凸函数; 设$(\Delta A)(\cdot)\in C([t_0,t_1],\mathbb{R}^{n\times n})$, $(\Delta B)(\cdot)\in C([t_0,t_1],\mathbb{R}^{n\times m})$, $(\Delta f)(\cdot)\in C([t_0,t_1],\mathbb{R}^n)$, 记拟Riccati方程

$$\begin{aligned} &(\frac{\partial K}{\partial t})(t,x)+(\frac{\partial K}{\partial x})(t,x)\{[A(t)+(\Delta A)(t)]x+\\ &[B(t)+(\Delta B)(t)][(\frac{\partial R}{\partial u})(t,\cdot)^{\mathrm{T}}]^{-1}(-[B(t)+\\ &(\Delta B)(t)]^{\mathrm{T}}K(t,x))+f(t)+(\Delta f)(t)\}+\\ &[A(t)+(\Delta A)(t)]^{\mathrm{T}}K(t,x)+(\frac{\partial Q}{\partial x})(t,x)^{\mathrm{T}},\\ &(t,x)\in[t_0,t_1]\times\mathbb{R}^n,\\ &K(t_1,x)=(\frac{\partial G}{\partial x})(x)^{\mathrm{T}},\ \forall\,x\in\mathbb{R}^n \end{aligned}$$

的解为$\hat{K}(\cdot)+(\Delta\hat{K})(\cdot)$. 则

$$\lim_{\|\Delta A(\cdot)\|_{C([t_0,t_1],\mathbb{R}^{n\times n})}\to0,\|\Delta B(\cdot)\|_{C([t_0,t_1],\mathbb{R}^{n\times m})}\to0,\|(\Delta f)(\cdot)\|_{C([t_0,t_1],\mathbb{R}^n)}\to0}\|\,\Delta\hat{K}(\cdot)\|_{C([t_0,t_1],\mathbb{R}^{n\times n})}=0. \tag{23}$$

即此时方程(3)之解连续地依赖于系统(1)的参数函数.

由凸分析的基本常识, 当定理3中关于$Q$与$G$的凸性假定成立时,

$$\begin{cases} \forall\,(t,x)\in[t_0,t_1]\times\mathbb{R}^n,\ (\dfrac{\partial^2Q}{\partial x^2})(t,x)\geqslant0,\\ \forall\,x\in\mathbb{R}^n,(\dfrac{\partial^2G}{\partial x^2})(x)\geqslant0, \end{cases} \tag{24}$$

因而此时(再注意到假设7)

$$\begin{cases} \forall\,(u(\cdot),x(\cdot),x_1)\in\mathcal{U}\times AC([t_0,t_1],\mathbb{R}^n)\times\mathbb{R}^n,\\ \varPhi(u(\cdot),x(\cdot),x_1)\geqslant\tilde{R}''(u(\cdot))(\geqslant\delta_0I_{\mathcal{U}}). \end{cases} \tag{25}$$

故定理2,3有如下直接推论:

**推论 1**　仍假定假设1,2,6,7,10成立, 并设式(20)与式(22)成立. 则LNQP的最优控制存在唯一且方程(3)有唯一连续依赖于参数函数($A(\cdot),B(\cdot),f(\cdot)$)的整体经典解.

其次, 假设

**假设 11**　$R(\cdot)\in C([t_0,t_1],\mathbb{R}^{m\times m})$, $\exists\delta_0>0$使$\forall\,t\in[t_0,t_1],R(t)^{\mathrm{T}}=R(t)\geqslant\delta_0I_m$. 记

$$V(t,s):=\bar{Y}(t)\bar{Y}(s)^{-1},\ \forall\,t,s\in[t_0,t_1], \tag{26}$$

其中的$\bar{Y}(\cdot)$是线性齐次矩阵常微分方程

$$\begin{cases} \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t}=[A(t)-B(t)R(t)^{-1}S(t)]Y(t),\\ \qquad t\in[t_0,t_1],\\ Y(t_0)=I_n \end{cases} \tag{27}$$

之解, 任意取定$x(\cdot)\in AC([t_0,t_1],\mathbb{R}^n)$, 定义算子$\varLambda,\varLambda_1$和$\tilde{W}''(x(\cdot))$:

$$\begin{cases} \varLambda:\mathcal{X}\to\mathcal{X},\ y(\cdot)\mapsto\displaystyle\int_{t_0}^{\cdot}V(\cdot,\tau)y(\tau)\mathrm{d}\tau,\\ \quad\forall\,y(\cdot)\in\mathcal{X};\\ \varLambda_1:\mathcal{X}\to\mathbb{R}^n,y(\cdot)\mapsto\displaystyle\int_{t_0}^{t_1}V(t_1,\tau)y(\tau)\mathrm{d}\tau,\\ \quad\forall\,y(\cdot)\in\mathcal{X};\\ \tilde{W}''(x(\cdot)):\\ \mathcal{X}\to\mathcal{X},\ y(\cdot)\mapsto[(\dfrac{\partial^2Q}{\partial x^2})(\cdot,x(\cdot))-\\ S(\cdot)^{\mathrm{T}}R(\cdot)^{-1}S(\cdot)]y(\cdot),\ \forall\,y(\cdot)\in\mathcal{X}. \end{cases} \tag{28}$$

由

$$\begin{cases} \forall\,(t,x,u)\in[t_0,t_1]\times\mathbb{R}^n\times\mathbb{R}^m,\\ 2u^{\mathrm{T}}S(t)x+u^{\mathrm{T}}R(t)u=\\ -x^{\mathrm{T}}S(t)^{\mathrm{T}}R(t)^{-1}S(t)x+\\ [u+R(t)^{-1}S(t)x]^{\mathrm{T}}\times\\ R(t)[u+R(t)^{-1}S(t)x] \end{cases} \tag{29}$$

等即可知定理2还有直接推论:

**推论 2** 假定假设1,2,6,8,10,11成立, 并设

$$\begin{cases} g(t,x,u)=Q(t,x)+2u^{\mathrm{T}}S(t)x+u^{\mathrm{T}}R(t)u, \\ \forall\,(t,x,u)\in[t_0,t_1]\times\mathbb{R}^n\times\mathbb{R}^m \end{cases} \tag{30}$$

和$\exists\delta>0$使

$$\begin{cases} \forall\,(u(\cdot),x(\cdot),x_1)\in\mathcal{U}\times AC([t_0,t_1],\ \mathbb{R}^n)\times\mathbb{R}^n, \\ \varPsi(u(\cdot),x(\cdot),x_1):= \\ \tilde{R}''+\tilde{B}^*[\varLambda^*\tilde{W}''(x(\cdot))\varLambda+ \\ \varLambda_1^*(\dfrac{\partial^2G}{\partial y^2})(x_1)\varLambda_1]\tilde{B}\geqslant\delta I_{\mathcal{U}}, \end{cases} \tag{31}$$

其中的$\tilde{R}''$为算子:

$$\tilde{R}'':\mathcal{U}\to\mathcal{U},v(\cdot)\mapsto R(\cdot)v(\cdot),\ \forall\,v(\cdot)\in\mathcal{U}. \tag{32}$$

则LNQP有唯一最优控制$\hat{u}(\cdot)$且拟Riccati方程

$$\begin{cases} (\dfrac{\partial L}{\partial t})(t,x)+(\dfrac{\partial L}{\partial x})(t,x)\{[A(t)- \\ B(t)R(t)^{-1}S(t)]x- \\ \dfrac{1}{2}B(t)R(t)^{-1}B(t)^{\mathrm{T}}L(t,x)+f(t)\}+[A(t)- \\ B(t)R(t)^{-1}S(t)]^{\mathrm{T}}L(t,x)+ \\ (\dfrac{\partial Q}{\partial x})(t,x)^{\mathrm{T}}-2S(t)^{\mathrm{T}}R(t)^{-1}S(t)x=0, \\ (t,x)\in[t_0,t_1]\times\mathbb{R}^n, \\ L(t_1,x)=(\dfrac{\partial G}{\partial x})(x)^{\mathrm{T}},\ \forall\,x\in\mathbb{R}^n \end{cases} \tag{33}$$

有唯一整体经典解$\hat{L}$, 从而再由定理1等即知$\hat{u}(\cdot)$有状态反馈表示

$$\begin{aligned} &\hat{u}(t)= \\ &-R(t)^{-1}[\frac{1}{2}B(t)^{\mathrm{T}}\hat{L}(t,x(t,\hat{u}(\cdot)))+ \\ &S(t)x(t,\hat{u}(\cdot))],\forall\,t\in[t_0,t_1]. \end{aligned} \tag{34}$$

## 4 拟Riccati方程的特征线解法与最优控制(Characteristic line method of solving quasi-Riccati equation and opitmal control)

由定理 1 容易看出拟 Riccati 方程 (3) 是求取LNQP之最优控制的关键所在. 方程(3)是一高度非线性的方程, 但在实际应用中, 最关心的是

$$\begin{cases} G(x)=\dfrac{\gamma}{2}h(|\,x-x_1|^2-r_1^2),\ h(\cdot)\text{非负}, \\ x_1\in\mathbb{R}^n,\ r_1\in[0,+\infty),\ \gamma\text{是足够大的正数}, \\ g(t,x,u)=\dfrac{1}{2}\{[C(t)x-p(t)]^{\mathrm{T}}Q(t)\cdot \\ \qquad [C(t)x-p(t)]+u^{\mathrm{T}}R(t)u\}, \\ C(\cdot)\in C([t_0,t_1],\mathbb{R}^{l\times n}),\ p(\cdot)\in C([t_0,t_1],\mathbb{R}^l), \\ Q(\cdot)\in C([t_0,t_1],\mathbb{R}^{l\times l}),\ Q(\cdot)^{\mathrm{T}}=Q(\cdot), \\ Q(t)\geqslant0,\ \forall\,t\in[t_0,t_1],\ R(\cdot)\in C([t_0,t_1],\mathbb{R}^{m\times m}), \\ R(\cdot)^{\mathrm{T}}=R(\cdot),\ \exists\delta_0>0,\ R(t)\geqslant\delta_0I_m,\ \forall\,t\in[t_0,t_1] \end{cases} \tag{35}$$

的情形, 这是因为综合反映状态转移效果与调控所耗费能量等的指标泛函$I(u(\cdot))$中的$G$与$g$恰好分别是如式(35)所定义的较特殊函数, 此时方程(3)成为

$$\begin{cases} (\dfrac{\partial K}{\partial t})(t,x)+(\dfrac{\partial K}{\partial x})(t,x)[A(t)x- \\ B(t)R(t)^{-1}B(t)^{\mathrm{T}}K(t,x)+ \\ f(t)]+A(t)^{\mathrm{T}}K(t,x)+ \\ C(t)^{\mathrm{T}}Q(t)[C(t)x-p(t)]=0, \\ (t,x)\in[t_0,t_1]\times\mathbb{R}^n, \\ K(t_1,x)=(\dfrac{\partial G}{\partial x})(x)^{\mathrm{T}}, \\ \forall\,x\in\mathbb{R}^n. \end{cases} \tag{36}$$

根据1阶拟线性偏微分方程组的几何理论(参看文[9])

$$\begin{cases} (\dfrac{\partial K}{\partial t})(t,x)+(\dfrac{\partial K}{\partial x})(t,x)[A(t)x- \\ B(t)R(t)^{-1}B(t)^{\mathrm{T}}K(t,x)+ \\ f(t)]+A(t)^{\mathrm{T}}K(t,x)+ \\ C(t)^{\mathrm{T}}Q(t)[C(t)x-p(t)]=0, \\ (t,x)\in[t_0,t_1]\times\mathbb{R}^n \end{cases} \tag{37}$$

的积分超曲面是由对应的特征方程组

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}=A(t)x(t)- \\ \qquad B(t)R(t)^{-1}B(t)^{\mathrm{T}}k(t)+f(t), \\ \dfrac{\mathrm{d}k(t)}{\mathrm{d}t}=-\{A(t)^{\mathrm{T}}k(t)+ \\ \qquad C(t)^{\mathrm{T}}Q(t)[C(t)x(t)-p(t)]\}, \\ t\in[t_0,t_1] \end{cases} \tag{38}$$

之解所成的特征线$\{(t,x(t),k(t))^{\mathrm{T}}\big|t\in[t_0,t_1]\}$织成的, 式(38)是一仿射线性常微分方程组因而很易求解. 下面即将给出的定理表明: 可利用方程组(38)的解族构造适当的非线性规划问题, 解之, 然后由该问题的解就能得到最优控制.

**定理 4**　设假设1,2, $G(\cdot)\in C^2(\mathbb{R}^n,\mathbb{R}^1)$,

$$\exists L_0\in\mathbb{R}^1, G(x)\geqslant L_0, \forall\, x\in\mathbb{R}^n \tag{39}$$

和

$$\begin{cases} g(t,x,u)=\dfrac{1}{2}\big\{[C(t)x-p(t)]^{\mathrm{T}}Q(t)[C(t)x-\\ \qquad p(t)]+u^{\mathrm{T}}R(t)u\big\},\\ C(\cdot)\in C([t_0,t_1],\mathbb{R}^{l\times n}),\\ p(\cdot)\in C([t_0,t_1],\mathbb{R}^{l}),\\ Q(\cdot)\in C([t_0,t_1],\mathbb{R}^{l\times l}),\ Q(\cdot)^{\mathrm{T}}=Q(\cdot),\\ Q(t)\geqslant 0,\ \forall\, t\in[t_0,t_1],\\ R(\cdot)\in C([t_0,t_1],\mathbb{R}^{m\times m}),\\ R(\cdot)^{\mathrm{T}}=R(\cdot),\ \exists\delta_0>0,\\ R(t)\geqslant\delta_0 I_m,\ \forall\, t\in[t_0,t_1]; \end{cases} \tag{40}$$

$\forall\, z\in\mathbb{R}^n$, 记

$$\begin{cases} \begin{pmatrix}\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}\\ \dfrac{\mathrm{d}k(t)}{\mathrm{d}t}\end{pmatrix}=\\ \begin{pmatrix} A(t) & -B(t)R(t)^{-1}B(t)^{\mathrm{T}}\\ -C(t)^{\mathrm{T}}Q(t)C(t) & -A(t)^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}x(t)\\ k(t)\end{pmatrix}+\\ \begin{pmatrix} f(t)\\ C(t)^{\mathrm{T}}Q(t)p(t)\end{pmatrix},\ t\in[t_0,t_1],\\ \begin{pmatrix}x(t_1)\\ k(t_1)\end{pmatrix}=\begin{pmatrix} z\\ (\dfrac{\partial G}{\partial z})(z)^{\mathrm{T}}\end{pmatrix} \end{cases} \tag{41}$$

之解为 $(x(\cdot,z),k(\cdot,z))^{\mathrm{T}}$. 则: 非线性规划问题(NLPP)求$\hat{z}\in Z(x_0):=\{z\in\mathbb{R}^n|x(t_0,z)=x_0\}$使得

$$I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k(\cdot,\hat{z}))=\inf_{z\in Z(x_0)} I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k(\cdot,z)) \tag{42}$$

有解; 若$\hat{z}$为NLPP的一个解, 那么$-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k(\cdot,\hat{z})$是LNQP的唯一最优控制.

**证**　记

$$Q(t,x):=\frac{1}{2}[C(t)x-p(t)]^{\mathrm{T}}Q(t)[C(t)x-p(t)],$$
$$R(t,u):=\frac{1}{2}u^{\mathrm{T}}R(t)u, \tag{43}$$

易知定理2的假设都成立, 所以对本定理条件下的LNQP其最优控制存在唯一且方程(36)有唯一整体经典解, 继而由定理1知最优控制$\hat{u}(\cdot)$, 最优状态函数$\hat{x}(\cdot)$和拟Riccati方程(36)之解$\hat{K}$满足关系式

$$\hat{u}(t)=-R(t)^{-1}B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t)),\quad \forall\, t\in[t_0,t_1], \tag{44}$$

从而

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t}=A(t)\hat{x}(t)-B(t)R(t)^{-1}B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t))+f(t),\ \forall\, t\in[t_0,t_1], \tag{45}$$

进而有

$$\begin{aligned}&\frac{\mathrm{d}\hat{K}(t,\hat{x}(t))}{\mathrm{d}t}=\\ &(\frac{\partial\hat{K}}{\partial t})(t,\hat{x}(t))+(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t}=\\ &(\frac{\partial\hat{K}}{\partial t})(t,\hat{x}(t))+(\frac{\partial\hat{K}}{\partial x})(t,\hat{x}(t))[A(t)\hat{x}(t)-\\ &B(t)R(t)^{-1}B(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t))+f(t)]=\\ &-\big\{A(t)^{\mathrm{T}}\hat{K}(t,\hat{x}(t))+C(t)^{\mathrm{T}}Q(t)[C(t)\hat{x}(t)-\\ &p(t)]\big\},\ \forall\, t\in[t_0,t_1].\end{aligned} \tag{46}$$

记

$$\hat{k}(t):=\hat{K}(t,\hat{x}(t)),\ \forall\, t\in[t_0,t_1], \tag{47}$$

式(45)(46)与

$$\hat{K}(t_1,x)=(\frac{\partial G}{\partial x})(x)^{\mathrm{T}},\ \forall\, x\in\mathbb{R}^n \tag{48}$$

一起意味着$(\hat{x}(\cdot),\hat{k}(\cdot))^{\mathrm{T}}$是方程组(41)当其参数向量$z=\hat{x}(t_1)$时之解, 从而有

$$I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k(\cdot,\hat{z}))\leqslant I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}\hat{k}(\cdot)); \tag{49}$$

但

$$\hat{u}(t)=-R(t)^{-1}B(t)^{\mathrm{T}}\hat{k}(t),\ \forall\, t\in[t_0,t_1] \tag{50}$$

是最优控制, 又有

$$I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}\hat{k}(\cdot))\leqslant I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k(\cdot,\hat{z})). \tag{51}$$

故

$$I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k(\cdot,\hat{z}))=I(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}\hat{k}(\cdot)), \tag{52}$$

即本定理的结论真. 定理4证毕.

## 5　附有状态终端约束的LNQP(LNQP with terminal state constraint attached)

本节简单地讨论如何利用惩罚方法的思想通过解充分“逼近”状态终端受限线性–非二次问

题的不受限线性–非二次问题来获得原受限问题的足够好(满足预给的精度要求)近似解.

考虑由线性系统(1)、指标泛函(2)和状态终端约束

$$E_j(x(t_1,u(\cdot)))\leqslant 0,\ j=1,\cdots,l \tag{53}$$

构成的线性–非二次最优控制问题(CLNQP): 求$\hat{u}(\cdot)\in\mathcal{U}_{\rm ad}:=\{u(\cdot)\in\mathcal{U}\mid E_j(x(t_1,u(\cdot)))\leqslant 0,\ j=1,\cdots,l\}$使得

$$I(\hat{u}(\cdot)):=\inf_{u(\cdot)\in\mathcal{U}_{\rm ad}}I(u(\cdot)), \tag{54}$$

换言之, 要求$\hat{u}(\cdot)\in\mathcal{U}$使得

$$\begin{cases}I(\hat{u}(\cdot))\leqslant\inf\limits_{u(\cdot)\in\mathcal{U}_{\rm ad}}I(u(\cdot)),\\ E_j(x(t_1,\hat{u}(\cdot)))\leqslant 0,\ j=1,\cdots,l.\end{cases} \tag{55}$$

针对上述附有状态终端约束的最优控制问题, 引入充分大的正数$\gamma$(称为惩罚因子). 记

$$h(r):=\begin{cases}\dfrac{r^{2+\alpha}}{2+\alpha},\ \forall\, r\in[0,+\infty),\\ 0,\qquad\ \forall\, r\in(-\infty,0).\end{cases} \tag{56}$$

其中的$\alpha$为取自$(0,1)$的数, 定义

$$I_\gamma(u(\cdot)):=I(u(\cdot))+\gamma\sum_{j=1}^{l}h\big(E_j(x(t_1,u(\cdot)))\big),\quad \forall\, u(\cdot)\in\mathcal{U}. \tag{57}$$

考察由线性受控系统(1)与指标泛函(57)构成的状态终端不受限线性–非二次最优控制问题(LNQP)$_\gamma$: 求$\hat{u}_\gamma(\cdot)\in\mathcal{U}$使得

$$I_\gamma(\hat{u}_\gamma(\cdot))=\inf_{u(\cdot)\in\mathcal{U}}I_\gamma(u(\cdot)). \tag{58}$$

注意到$\mathcal{U}_{\rm ad}\subseteq\mathcal{U}$, 式(56)和假设4, 易知: 若$\hat{u}_\gamma(\cdot)$为(LNQP)$_\gamma$的最优控制, 则

$$\begin{cases}I(\hat{u}_\gamma(\cdot))\leqslant I_\gamma(\hat{u}_\gamma(\cdot))=\\ \inf\limits_{u(\cdot)\in\mathcal{U}}I_\gamma(u(\cdot))\leqslant\inf\limits_{u(\cdot)\in\mathcal{U}_{\rm ad}}I_\gamma(u(\cdot))=\\ \inf\limits_{u(\cdot)\in\mathcal{U}_{\rm ad}}I(u(\cdot)),\ \forall\,\gamma\in(0,+\infty),\\ E_j(x(t_1,\hat{u}_\gamma(\cdot)))\leqslant\\ \Big\{\dfrac{(2+\alpha)[\inf\limits_{u(\cdot)\in\mathcal{U}_{\rm ad}}I(u(\cdot))-L_0(t_1-t_0+1)]}{\gamma}\Big\}^{\frac{1}{2+\alpha}}\to\\ 0(\gamma\uparrow+\infty),\ j=1,\cdots,l.\end{cases} \tag{59}$$

**定义 1** 对预先给定的非负数$\varepsilon_0$, 称$\mathcal{U}$中满足

$$\begin{cases}I(\hat{u}_{\rm a}(\cdot))\leqslant\varepsilon_0+\inf\limits_{u(\cdot)\in\mathcal{U}_{\rm ad}}I(u(\cdot)),\\ E_j(x(t_1,\hat{u}_{\rm a}(\cdot)))\leqslant\varepsilon_0,\ j=1,\cdots,l\end{cases} \tag{60}$$

的$\hat{u}_{\rm a}(\cdot)$为CLNQP的一个满足$\varepsilon_0$精度要求的近似最优控制.

显见: $\varepsilon_0$愈小, $\hat{u}_{\rm a}(\cdot)$近似满足

$$\begin{cases}I(u(\cdot))\leqslant\inf\limits_{u(\cdot)\in\mathcal{U}_{\rm ad}}I(u(\cdot)),\\ E_j(x(t_1,u(\cdot)))\leqslant 0,\ j=1,\cdots,l\end{cases} \tag{61}$$

的程度愈高($\varepsilon_0=0$时, $\hat{u}_{\rm a}(\cdot)$精确地满足上式, 为CLNQP的精确最优控制), 即从满足对控制作用的代价–效果综合要求的程度看$\hat{u}_{\rm a}(\cdot)$愈好.

如能设法找到一个$\bar{u}(\cdot)\in\mathcal{U}_{\rm ad}$, 则由式(59)便可知: $\forall\,\gamma>\dfrac{(2+\alpha)[I(\bar{u}(\cdot))-L_0(t_1-t_0+1)]}{{\varepsilon_0}^{2+\alpha}}$, $\hat{u}_\gamma(\cdot)$都是CLNQP的满足$\varepsilon_0$精度要求的近似最优控制. 如果是用第4节提供的方法对$\gamma_0:=\dfrac{(2+\alpha)[\varepsilon_0+I(\bar{u}(\cdot))-L_0(t_1-t_0+1)]}{\varepsilon_0^{2+\alpha}}$算得了(LNQP)$_{\gamma_0}$的一个$\varepsilon_0$ near-optimal控制$\hat{u}_0(\cdot)$ (参看文[10]), 所谓$\hat{u}_0(\cdot)$为(LNQP)$_{\gamma_0}$的$\varepsilon_0$ near-optimal控制, 意即它满足

$$I_{\gamma_0}(\hat{u}_0(\cdot))\leqslant\varepsilon_0+\inf_{u(\cdot)\in\mathcal{U}}I_{\gamma_0}(u(\cdot)), \tag{62}$$

则亦易知$\hat{u}_0(\cdot)$即是CLNQP的一个满足$\varepsilon_0$精度要求的近似最优控制.

第4节和本节的以上讨论为在定理4之假设条件和假设9成立的前提下设计求CLNQP之近似解的计算方法提供了充分的理论依据.

**算法 1**

**第1步** 在$\Big[\dfrac{(2+\alpha)[\varepsilon_0+I(\bar{u}(\cdot))-L_0(t_1-t_0+1)]}{\varepsilon_0^{2+\alpha}},+\infty\Big)$中取定一个正数$\gamma_0$;

**第2步** 解线性常微分方程组

$$\begin{cases}\begin{pmatrix}\dfrac{{\rm d}x(t)}{{\rm d}t}\\ \dfrac{{\rm d}k(t)}{{\rm d}t}\end{pmatrix}=\\ \begin{pmatrix}A(t) & -B(t)R(t)^{-1}B(t)^{\rm T}\\ -C(t)^{\rm T}Q(t)C(t) & -A(t)^{\rm T}\end{pmatrix}\begin{pmatrix}x(t)\\ k(t)\end{pmatrix}+\\ \begin{pmatrix}f(t)\\ C(t)^{\rm T}Q(t)p(t)\end{pmatrix},\\ t\in[t_0,t_1],\\ \begin{pmatrix}x(t_1)\\ k(t_1)\end{pmatrix}=\\ \begin{pmatrix}z\\ \big[(\dfrac{\partial G}{\partial z})(z)+\gamma_0\sum\limits_{j=1}^{l}h'(E_j(z))(\dfrac{\partial E_j}{\partial z})(z)\big]^{\rm T}\end{pmatrix}\end{cases} \tag{63}$$

得$\{(x_{\gamma_0}(\cdot,z),k_{\gamma_0}(\cdot,z))^{\mathrm{T}}|z\in\mathbb{R}^n\}$;

**第3步**(搜索)　解非线性规划问题$(\mathrm{NLPP})_{\gamma_0}$: 求$\hat z\in Z_{\gamma_0}(x_0):=\{z\in\mathbb{R}^n|x_{\gamma_0}(t_0,z)=x_0\}$使得

$$I_{\gamma_0}(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k_{\gamma_0}(\cdot,\hat z))=\inf_{z\in Z_{\gamma_0}(x_0)}I_{\gamma_0}(-R(\cdot)^{-1}B(\cdot)^{\mathrm{T}}k_{\gamma_0}(\cdot,z)),\quad (64)$$

得$(\mathrm{NLPP})_{\gamma_0}$的一个解$\hat z$;

**第4步**

$$\hat u_a(t):=-E(t)^{-1}B(t)^{\mathrm{T}}k_{\gamma_0}(t,\hat z),\ \forall\, t\in[t_0,t_1]\quad (65)$$

即为当定理4之所有假设条件和假设9都成立时CLNQP的一个满足$\varepsilon_0$精度要求的近似最优控制函数.

用如下典型算例来说明上述算法的有效性.

**例 1**　特别取

$$\begin{cases}t_0:=0,\ x_0:=10,\ t_1:=1,\ x_1:=0,\ r_1=2,\\ A(t)\equiv B(t)\equiv 1,\ f(t)\equiv 0,\\ g(t,x,u):=\dfrac{1}{2}(x^2+u^2),\\ \forall\,(x,u)\in\mathbb{R}^2, G(\cdot)=0\end{cases}\quad (66)$$

和

$$\gamma_0:=10^4,\ \alpha:=0.5.\quad (67)$$

CLNQP由受控系统

$$\begin{cases}\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}=x(t)+u(t),\ \text{a.e.}\ t\in[0,1],\\ x(0)=10,\end{cases}\quad (68)$$

状态终端约束

$$|\,x(1)\,|\leqslant 2\quad (69)$$

和指标泛函

$$I(u(\cdot)):=\frac{1}{2}\int_0^1[x(t)^2+u(t)^2]\mathrm{d}t,\quad \forall\,u(\cdot)\in L^2(0,1;\mathbb{R}^1)\quad (70)$$

构成. 使用算法1获得其一个近似最优控制(见图1所示).

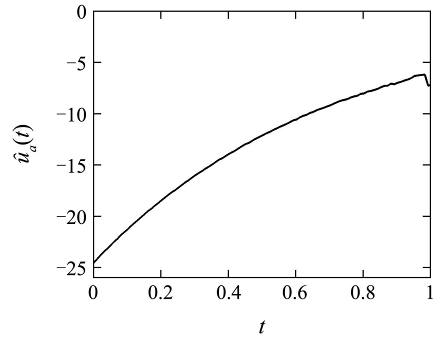


图 1　近似最优控制

Fig. 1　The approximate optimal control

$\hat u_a(t)=$

$-24.4819, -24.1290, -23.7719, -23.4232,$
$-23.1249, -22.7984, -22.4563, -22.1643,$
$-21.8038, -21.5208, -21.2465, -20.9052,$
$-20.6255, -20.3401, -20.0536, -19.7720,$
$-19.4650, -19.2403, -18.9435, -18.7022,$
$-18.4137, -18.1537, -17.8851, -17.6324,$
$-17.4291, -17.1796, -16.9280, -16.6763,$
$-16.4656, -16.2090, -15.9671, -15.7532,$
$-15.5494, -15.3272, -15.1263, -14.9188,$
$-14.6827, -14.4572, -14.3019, -14.0947,$
$-13.8700, -13.7015, -13.5182, -13.2958,$
$-13.0852, -12.9020, -12.7820, -12.5694,$
$-12.3823, -12.2200, -12.0450, -11.8790,$
$-11.7303, -11.5589, -11.4006, -11.2423,$
$-11.1003, -10.9525, -10.7832, -10.6061,$
$-10.5123, -10.3086, -10.1543, -10.0550,$
$-9.8736, -9.7723, -9.6048, -9.4906,$
$-9.3774, -9.2482, -9.1256, -8.9799,$
$-8.8227, -8.6919, -8.6126, -8.5220,$
$-8.3743, -8.2654, -8.1987, -8.0178,$
$-7.9696, -7.7815, -7.7279, -7.6345,$
$-7.5242, -7.3714, -7.2257, -7.2098,$
$-7.0105, -7.0755, -6.9099, -6.8272,$
$-6.7063, -6.5987, -6.4647, -6.2566,$
$-6.2069, -6.1603, -6.1300, -7.1962, -7.1044,$

时间间隔取0.01;

相应的状态函数为

$\hat x_a(t)=$

$10.0000, 9.8552, 9.7124, 9.5718, 9.4333,$
$9.2964, 9.1614, 9.0285, 8.8971, 8.7680, 8.6405,$
$8.5144, 8.3905, 8.2682, 8.1475, 8.0284, 7.9110,$
$7.7954, 7.6810, 7.5683, 7.4570, 7.3474, 7.2394,$
$7.1329, 7.0279, 6.9239, 6.8214, 6.7203, 6.6207,$
$6.5223, 6.4254, 6.3300, 6.2358, 6.1426, 6.0508,$
$5.9600, 5.8704, 5.7823, 5.6956, 5.6095, 5.5247,$
$5.4412, 5.3586, 5.2770, 5.1968, 5.1179, 5.0401,$

$4.9627, 4.8866, 4.8117, 4.7376, 4.6645, 4.5924,$
$4.5210, 4.4506, 4.3811, 4.3125, 4.2446, 4.1775,$
$4.1115, 4.0465, 3.9819, 3.9186, 3.8562, 3.7943,$
$3.7335, 3.6731, 3.6138, 3.5550, 3.4968, 3.4392,$
$3.3824, 3.3264, 3.2714, 3.2172, 3.1633, 3.1097,$
$3.0571, 3.0050, 2.9530, 2.9024, 2.8517, 2.8024,$
$2.7532, 2.7043, 2.6561, 2.6090, 2.5628, 2.5164,$
$2.4714, 2.4254, 2.3805, 2.3361, 2.2924, 2.2493,$
$2.2071, 2.1667, 2.1262, 2.0859, 2.0455, 1.9940,$

其终端值

$$\hat{x}_a(1) = 1.9940, \tag{71}$$

落在终端约束集$\{z \mid |z| \leqslant 2\}$之中, 实现了所欲之状态转移. 近似最优状态轨线见图2所示.

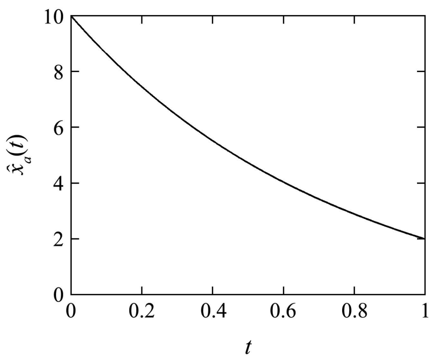


图 2 近似最优状态轨线

Fig. 2 The approximate optimal state trajectory

## 6 最后的话(Final remark)

对于仅假定$g$满足假设3中有关假设和假设10, 假设5以及

$$\exists \delta > 0, \forall u(\cdot) \in \mathcal{U}, (\nabla^2 I)(u(\cdot)) \geqslant \delta I_{\mathcal{U}} \tag{72}$$

之一般情形拟Riccati方程的解的存在唯一性和求LNQP的最优控制的有效算法等等是有待于进一步研究的较困难课题, 本文作者以后将另文予以深入详尽的讨论.

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