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一类非线性系统有限时间流形吸引的浸入与不变控制

黄显林, 张 旭†, 卢鸿谦
(哈尔滨工业大学 控制理论与制导技术研究中心, 黑龙江 哈尔滨 150001)

摘要: 本文对一类非线性系统, 提出了一种设计渐近稳定控制律的有效方法. 其中, 通过更新系统浸入与不变流形理论的应用方法, 流形的吸引坐标可以在有限时间内收敛到平衡点. 为了得到闭环系统的稳定性, 增广系统的各个信号被证明是有界的. 本文得出的一个重要成果是流形吸引有限时间的计算方法. 此外, 在施加了有限时间流形吸引控制器之后, 流形对外部有界未知扰动具有不敏感性. 最后利用车摆系统来论述所提出的控制方法的设计步骤, 以及通过仿真验证控制器的性能.

关键词: 非线性系统; 系统浸入; 不变流形; 有限时间; 外部扰动
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Finite-time attractivity-based immersion and invariance control for a class of nonlinear systems

HUANG Xian-lin, ZHANG Xu†, LU Hong-qian
(Harbin Institute of Technology, Center for Control Theory and Guidance Technology, Harbin Heilongjiang 150001, China)

Abstract: We propose an effective approach for designing asymptotically stabilizing control laws for a class of nonlinear systems. In this approach, by modifying the application method of the immersion and invariance (I & I) theorem, the off-the-manifold coordinates are ensured to converge to the equilibrium point in finite time. In order to obtain the stability of closed-loop system, all trajectories in the augmented system are proved bounded. An important result we obtained is the computation method for the finite time of the manifold attractivity. Moreover, the application of the finite-time manifold-attractivity controller makes the manifold insensitive to all external bounded unknown disturbances. The design procedures are detailed by designing a controller for a cart-pendulum system, and the controller performances are validated by simulations.

Key words: nonlinear systems; system immersion; manifold invariance; finite time; external disturbance

1 Introduction

Over the past few decades, stabilization control in finite time for nonlinear systems, for affine nonlinear systems in particular, has played an important role in the control field. Natural nonlinearities and diversity of nonlinear systems are the most significant constraints on applicability. Many research groups have devoted themselves to identifying reliable stabilizing control laws and applying them successfully to various systems. Acosta et al., for example, proposed an interconnection and damping assignment passivity-based control (IDA–PBC) for underactuated mechanical systems. This algorithm was derived from the passivity-based theory in order to provide a natural procedure that shapes kinetic and potential energy[1–3]. The most crucial issue which remains is effectively solving a partial differential equation (PDE). As degree of freedom (DOF) increases, IDA–PBC implementation becomes increasingly complex. Thus, considering calculation difficulties, this algorithm is only applicable for the lower-order nonlinear systems. A sliding mode control was proposed to stabilize a class of underactuated systems in cascaded form[4–5].

One of the promising and effective control methods is the immersion and invariance (I&I) theorem, proposed by Astolfi and Ortega in 2003[6–7], which has been further developed in additional studies[8–11]. System immersion is based on the nonlinear regular theory, where the required mappings can integrate the desired dynamical behavior with the reduced-order systems toward the high-order systems. Manifold invariance is derived from geometric nonlinearity to ensure the stability of closed-loop systems. Immersion and invariance control is not needed for the Lyapunov candidate function in the controller design phase. A notable advantage of this approach is the perfect decoupling calculation between manifold attractivity and invariance. Recently, researchers have evaluated the use of I&I for tracking control on a pneumatic actuator with a proven tracking theorem[12]. Manjarekar et al. applied I&I to stabilize a single machine infinite bus (SMIB) system using a controllable series capacitor (CSC)[13]. This technique has also been applied to tendon-controlled systems with variable stiffness[14]. An adaptive-state feedback controller was designed for n-dimensional nonlinear systems in feed-

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†Corresponding author. E-mail: alonsobobo@gmail.com; Tel.: +86 451-86402204.
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back form, as well[15]. An adaptive regulation via state feedback for discrete-time nonlinear systems in a parametric strict-feedback form has also been proposed[16].

In actuality, the off-the-manifold coordinate converges to equilibrium in the exponential form such that the distance from the state space of closed-loop systems to the manifold is rendered asymptotically attractive. However, low convergence rate may decrease the transient performance of closed-loop systems. Furthermore, robustness of stabilization cannot be theoretically ensured due to external disturbance. Compared to asymptotically stable manifold, finite-time attractivity controls possess the following properties: first, it shows better convergence performance around the equilibrium point, and second, it has better disturbance rejection erformance[17]. In one relative study[18], two globally stable control algorithms for robust stabilization of spacecraft in the presence of control input saturation, parametric uncertainty, and external disturbances were proposed, and fast and accurate response was designed. In another study[19], a finite-time control technique for a rigid spacecraft with external disturbances was proposed.

This work mainly focuses on the finite-time attractivity-based immersion and invariance stabilizing control for a class of nonlinear systems. A detailed procedure for designing the immersion and invariance controller is provided. Mapping can be obtained by selecting a target system and solving a partial differential equation. The stability proof of closed-loop systems accounting for the boundedness of actual states is described.The finite time is computed, in which the manifold is rendered attractive; The manifold does remain insensitive to bounded unknown disturbance under the proposed method. Simulation results validate the stabilizing control laws, based on this novel technique, through a cart-pendulum system. The proposed algorithms are computationally simple and involve straightforward tuning.

Preliminary results of the immersion and invariance theorem for an affine nonlinear system are described in Section 2. Section 3 describes the primary results of the finite-time attractivity-based immersion and invariance controller design for a nonlinear system class, and provides the stability proof and the computations of finite time. An analysis of the disturbance rejection is also described in Section 3. The controller performance is demonstrated by a cart-pendulum system in Section 4, and Section 5 provides concluding remarks.

2 System description and preliminary results

2.1 System description

In this section, a class of nonlinear systems are considered as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f_2(x_1) + g_2(x_1)u, \\ \dot{x}_3 = f_3(x_1) + u \end{cases} \tag{1}$$

with state $x_i \in \mathbb{R}$ for $i = 1, 2, 3$, input $u \in \mathbb{R}$, the vector fields $f_2(x_1)$, $f_3(x_1)$, $g_2(x_1) : \mathbb{R} \to \mathbb{R}$ and an equilibrium $x_*(t)$ to be asymptotically stabilized. The functions $f_2(x_1)$, $g_2(x_1)$, and $f_3(x_1)$ are assumed to be known. The finite-time attractivity of manifold based immersion and invariance control requires the following assumption on the continuity of $f_2(x_1)$, $g_2(x_1)$, and $f_3(x_1)$.

Assumption 1 For each $i = 2, 3$, the functions $f_i(x_1)$, $g_2(x_1)$ and their derivatives are continuous and bounded on any compact set $D \subset \mathbb{R}$.

The main objective for this work is to design a control law to asymptotically stabilize system (1) based on the I & I theorem, while making the manifold attractive in finite time.

2.2 Preliminary results

This section recalled the fundamental theorem, serving as principal tool of the immersion and invariance approach.

Theorem 1[6] Consider an affine nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{2}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and an equilibrium point $x_* \in \mathbb{R}^n$ to be stabilized. Let $p < n$, and assume that there exist smooth mappings $\alpha(\cdot)\colon \mathbb{R}^p \to \mathbb{R}^p$, $\pi(\cdot)\colon \mathbb{R}^p \to \mathbb{R}^n$, $c(\cdot)\colon \mathbb{R}^p \to \mathbb{R}^m$, $\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n-p}$, $\psi(\cdot) : \mathbb{R}^{n\times(n-p)} \to \mathbb{R}^m$, such that the following hold.

H1) Target system. The target system

$$\dot{\xi} = \alpha(\xi) \tag{3}$$

with $\xi \in \mathbb{R}^p$ has a globally asymptotically stable equilibrium $\xi_* \in \mathbb{R}^p$ and $x_* = \pi(\xi_*)$.

H2) Immersion condition. For all $\xi \in \mathbb{R}^p$,

$$\pi(\xi(0)) = x(0), \tag{4}$$

$$\pi(0) = 0, \tag{5}$$

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi). \tag{6}$$

H3) Implicit manifold. The following set identity

$$\begin{aligned} \mathcal{M} = \{x \in \mathbb{R}^n | \phi(x) = 0\} = \\ \{x \in \mathbb{R}^n | x = \pi(\xi), \xi \in \mathbb{R}^p\} \end{aligned} \tag{7}$$

holds.

H4) Manifold attractivity and trajectory boundedness. All trajectories of the system

$$\dot{z} = \frac{\partial \phi}{\partial x}(f(x) + g(x)\psi(x, z)), \tag{8}$$

$$\dot{x} = f(x) + g(x)\psi(x, z) \tag{9}$$

are bounded and satisfy

$$\lim_{t\to\infty} z(t) = 0. \tag{10}$$

Then x_* is a globally asymptotically stable equilibrium of the closed-loop system

$$\dot{x} = f(x) + g(x)\psi(x, \phi(x)). \tag{11}$$

Taken from [20], this theorem provides us an explicit procedure to design stabilizing control laws for a class of nonlinear systems. The objective is to find a manifold $\mathcal{M} = \{x \in \mathbb{R}^n | x = \pi(\xi), \xi \in \mathbb{R}^p\}$ based on the system (2) and the target dynamics (3). This manifold can be rendered invariant and attractive, and such that the well-defined restriction of the closed-loop system to $\mathcal{M}$ is described by the target system. Note that the control input

u that makes the manifold invariant is not unique, since it is uniquely defined only on $\mathcal{M}$. One possible control, that drives the off-the-manifold coordinates z to zero and keeps the system bounded, is selected. The I&I concept is illustrated for $p = 2$ and $n = 3$ in Fig. 1.

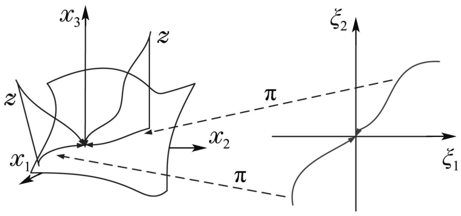

Fig. 1 Illustration of the immersion and invariance approach

3 Main results

This section details the procedures for modified immersion and invariance control for a class of nonlinear systems.

3.1 Controller design

First, the conditions of Theorem 1 must be verified.

Step 1 The objective here is to find a target dynamics and immerse it into the original system (1). The dimension of this target dynamics is strictly lower than the original system. From Eq. (1), the target dynamics are specified as follows:

$$\begin{aligned}\dot{\xi}_1 &= \xi_2,\\ \dot{\xi}_2 &= -\frac{\partial V(\xi_1)}{\partial \xi_1} - R(\xi_1,\xi_2)\xi_2,\end{aligned} \tag{12}$$

where ξ_1, $\xi_2 \in \mathbb{R}$ are two states of system (12), $V(\xi_1) : \mathbb{R} \to \mathbb{R}$ is potential energy, and $R(\xi_1,\xi_2) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a damping injection function. To ensure that the target dynamics have an asymptotically stable equilibrium at the origin, the following assumption is required.

Assumption 2 1) The damping injection function $R(\xi_1,\xi_2)$ is positive definite, i.e. $R(0,0) \geqslant 0$.

2) The potential energy function $V(\xi_1)$ satisfies $V(\xi_1)|_{\xi_1=0} = 0$, $\frac{\partial V}{\partial \xi_1}|_{\xi_1=0} = 0$ and $\frac{\partial^2 V}{\partial \xi_1^2}|_{\xi_1=0} > 0$.

A Lyapunov function is defined as follows:

$$H_1(\xi_1,\xi_2) = \frac{1}{2}\xi_2^2 + V(\xi_1). \tag{13}$$

The derivative of this positive function along the solution of (12) is

$$\dot{H}_1 = \xi_2\dot{\xi}_2 + \frac{\partial V}{\partial \xi_1}\dot{\xi}_1 = -R(\xi_1,\xi_2)\xi_2^2. \tag{14}$$

According to Assumption 2, $\dot{H}_1 \leqslant 0$ for $t > 0$. Therefore, the target dynamic system (12) has an asymptotically stable equilibrium at $\xi_1 = 0$, $\xi_2 = 0$.

Step 2 From the system (1), the immersion condition of Theorem 1 requires $\operatorname{rank}(\pi(\xi)) = 2$, so that a natural choice of the mapping $\pi(\xi)$ can be provided by

$$\pi(\xi) = \begin{bmatrix}\pi_1(\xi)\\ \pi_2(\xi)\\ \pi_3(\xi)\end{bmatrix} = \begin{bmatrix}\xi_1\\ \xi_2\\ \pi_3(\xi_1,\xi_2)\end{bmatrix}, \tag{15}$$

where $x_1 = \pi_1(\xi) = \xi_1$, $x_2 = \pi_2(\xi) = \xi_2$, and $\pi_3(\xi_1,\xi_2) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is an unknown mapping. From Eqs. (4)−(6), the relationships on the manifold $\mathcal{M}$ defined in (22) are as follows:

$$f_2(x_1) + g_2(x_1)c(\pi(\xi)) = -\frac{\partial V}{\partial \xi_1} - R(\xi_1,\xi_2)\xi_2, \tag{16}$$

$$f_3(x_1) + c(\pi(\xi)) = \frac{\partial \pi_3}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \pi_3}{\partial \xi_2}\dot{\xi}_2, \tag{17}$$

where $c(\pi(\xi))$ is a controller that ensures the manifold $\mathcal{M}$ is invariant. This controller is not used, however, because the manifold $\mathcal{M}$ cannot be rendered attractive. In other words, $c(\pi(\xi))$ cannot make an off-the-manifold coordinate, the distance from x to $\pi(\xi)$, converge to zero in finite time. After replacing the controller $c(\pi(\xi))$ from (17) into (16) and making a few rearrangements, a partial differential equation (PDE) to be solved is as follows:

$$\begin{aligned}&(g_2(x_1)\frac{\partial \pi_3}{\partial x_1} + \Delta(x)R(x_1,x_2))x_2 = \\ &g_2(x_1)f_3(x_1) - f_2(x_1) - \Delta(x)\frac{\partial V}{\partial x_1},\end{aligned} \tag{18}$$

where the funtion $\Delta(x)$ is defined as

$$\Delta(x_1) = 1 - g_2(x_1)\frac{\partial \pi_3}{\partial x_2}. \tag{19}$$

According to Theorem 1, the PDE (18) could be solved only if $\frac{\partial V}{\partial x_1}$ and $R(x_1,x_2)$ are known. Two degrees of freedom (DOF), $\frac{\partial V}{\partial x_1}$ and $R(x_1,x_2)$, are then added. The mapping $\pi_3(\xi)$ plays an important role in controller design, where the following assumption is necessary.

Assumption 3 There exist a mapping π_3 and positive constant $\sigma > 0$ such that $\frac{\partial \pi_3}{\partial x_2}$ is independent of x_2, and consequently, $|\Delta(0)| \geqslant \sigma > 0$.

If Assumption 3 holds, the PDE (18) can be solved by selecting

$$R(x_1,x_2) = -\Delta^{-1}(x_1)g_2(x_1)\frac{\partial \pi_3}{\partial x_1}, \tag{20}$$

$$\frac{\partial V}{\partial x_1} = \Delta^{-1}(x_1)(g_2(x_1)f_3(x_1) - f_2(x_1)). \tag{21}$$

Eqs.(21) and (22) provide a selection of 1-dimensional mapping functions π_3 so that Assumption 3 holds. The explicit form of π_3 can be obtained in terms of the specified nonlinear systems.

Step 3 The application of the set identity (7) permits the derivation of the following implicit manifold:

$$\begin{aligned}\mathcal{M} = \{x \in \mathbb{R}^3 | \phi(x) = 0\} = \\ \{x \in \mathbb{R}^3 | x = \pi(\xi), \xi \in \mathbb{R}^2\}.\end{aligned} \tag{22}$$

Step 4 The off-the-manifold coordinate $z = \phi(x)$ is defined as follows:

$$z = \phi(x) = x_3 - \pi_3(x_1,x_2). \tag{23}$$

Then, considering a Lyapunov candidate function:

$$H_2(z) = \frac{1}{2}z^2, \tag{24}$$

which can be differentiated along the solutions of (1) and (12), and produces the following after straightforward computations:

$$\begin{aligned}\dot{H}_2(z) = z(\dot{x_3} - \dot{\pi}_3) = \\ z\{\Delta(x_1)\psi(x,z) + f_3(x_1) - \\ \frac{\partial \pi_3}{\partial x_1}x_2 - \frac{\partial \pi_3}{\partial x_2}f_2(x_1)\}.\end{aligned} \tag{25}$$

The stabilizing controller $\psi(x, \phi(x))$ for the system (1) is obtained from Eq.(25) as such that the off-the-manifold coordinate converges to equilibrium at $z = 0$ in finite time. The explicit form of the controller follows:

$$\begin{aligned}\psi(x,z) = \Delta^{-1}(x_1)\{-\gamma \mathrm{sgn}(z) - f_3(x_1) + \\ \frac{\partial \pi_3}{\partial x_1}x_2 + \frac{\partial \pi_3}{\partial x_2}f_2(x_1)\},\end{aligned} \tag{26}$$

where the parameter γ is a positive constant, and $\mathrm{sgn}(\cdot)$ denotes a sign function, i.e.

$$\mathrm{sgn}\, z = \begin{cases} 1, & z > 0, \\ 0, & z = 0, \\ -1, & z < 0. \end{cases} \tag{27}$$

Substituting (26) into (25) yields

$$\dot{H}_2(z) = -\gamma|z| < 0. \tag{28}$$

Therefore, $\lim\limits_{t\to\infty} z(t) = 0$.

All trajectories of the closed-loop system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f_2(x_1) + g_2(x_1)\psi(x,z), \\ \dot{x}_3 = f_3(x_1) + \psi(x,z) \end{cases} \tag{29}$$

remain bounded, because the stabilizing controller is obtained. In the following section, a summary of of this method is proposed and the proof of stability is described.

3.2 Stability analysis

Stabilization can be summarized via the finite-time attractivity-based immersion and invariance control as follows.

Proposition 1 Consider a class of nonlinear systems (1) that satisfies Assumption 1, and equilibrium $x_*(t)$. Then, all trajectories of closed-loop system (1) and off-the-manifold dynamics are bounded, and $\lim\limits_{t\to\infty} x(t) = x_*(t)$ holds.

Prior to the stability proof, an assumption is necessary.

Assumption 4 There exists ϵ_1, ϵ_2, $\epsilon_3 > 0$, such that $|\Delta(x_1)| \geqslant \epsilon_1, R(x_1, x_2) \geqslant \epsilon_2$, and $|g_2(x_1)| \leqslant \epsilon_3$.

Proof Stability analysis is completed by proving that there exists a set of initial conditions $(x(0), z(0))$ such that the corresponding trajectories $x(t)$ of (29) are bounded. From (20) and (21) and after some simple calculations, all trajectories of the system (29) can be rewritten as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\dfrac{\partial V}{\partial x_1} - R(x_1,x_2)x_2 - \gamma g_2(x_2)\Delta^{-1}(x_1)\mathrm{sgn}\, z, \\ \dot{z} = -\gamma \mathrm{sgn}\, z. \end{cases} \tag{30}$$

Consider a positive definite function

$$H_3(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1), \tag{31}$$

the derivative of which, along the trajectories of (30), is

$$\dot{H}_3(z) = -R(x_1,x_2)x_2^2 - \gamma\Delta^{-1}(x_1)x_2 g_2(x_1)\mathrm{sgn}\, z. \tag{32}$$

All trajectories of the system (29) are bounded on $t \in (0, \infty)$. Two cases should be considered: 1) $t < \dfrac{z_0}{\gamma}$, and 2) $t \geqslant \dfrac{z_0}{\gamma}$.

Case 1 $t < \dfrac{z_0}{\gamma}$. In this case, $z(t) < z(0)$, and Eq.(32) satisfies the following:

$$\dot{H}_3(z) \leqslant -\epsilon_2 x_2^2 + \frac{\epsilon_3}{\epsilon_1}|x_2|\gamma \leqslant \frac{\epsilon_3^2\gamma^2}{4\epsilon_2\epsilon_1^2}, \tag{33}$$

where the second inequality follows from Young's inequality. In other words, the above inequality shows that the system energy is bounded.

Case 2 $t \geqslant \dfrac{z_0}{\gamma}$. In this case, $z(t) = 0$, and $\dot{H}_3(z) \leqslant -\epsilon_2 x_2^2$ is easily obtained.

Hence there exists a ball around zero and a finite time t_f, where all trajectories starting from all initial conditions converge in the ball at t_f, then converge to equilibrium asymptotically. The boundedness of x_3 is expressed as

$$x_3(t) = z(t) + \pi_3(x_1(t), x_2(t)) - z(0) + x_3(0). \tag{34}$$

According to the boundedness of x_1, x_2, then $x_3 \in \mathcal{L}_\infty$.

3.3 Speed of manifold response

In this section, the finite time t_f, from which the off-the-manifold coordinate z can be rendered attractive, is computed.

Proposition 2 Considering a manifold (7), all trajectories of the n-dimensional closed-loop systems converge to the manifold $\mathcal{M}$ in finite time t_f for any initial condition $z(0) = z_0$, and the attractivity of the manifold is described by $\lim\limits_{t\to t_\mathrm{f}} z(t) = 0$.

Proof Due to the fact that

$$\sum_{i=1}^{n} |z_i| \geqslant \|z\|, \tag{35}$$

where $\|\cdot\|$ represents the Euclidean norm, $\underline{\gamma}$ is defined by

$$\underline{\gamma} = \min\{\gamma_i\} \tag{36}$$

for $\gamma_i > 0$, $i = 1, \cdots, n$. Thus

$$-\sum_{i=1}^{n} |z_i|\gamma_i \leqslant -\|z\|\gamma_i \leqslant -\|z\|\underline{\gamma}. \tag{37}$$

Considering a positive definite function

$$H(z) = \frac{1}{2}z^\mathrm{T} z = \frac{1}{2}\|z\|^2, \tag{38}$$

the derivative along the trajectories of $\dot{z} = -\Sigma(z)\gamma$ is provided by

$$\dot{H}(z) = (-\Sigma(z)\gamma)^\mathrm{T} z \tag{39}$$

with

$$\Sigma(z) = \begin{pmatrix} \mathrm{sgn}\, z_1 & 0 & \cdots & 0 \\ 0 & \mathrm{sgn}\, z_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \mathrm{sgn}\, z_n \end{pmatrix}. \tag{40}$$

Then,

$$\dot{H}(z) = -\sum_{i=1}^{n} \gamma_i |z_i| \leqslant -\sqrt{2H}\underline{\gamma}. \tag{41}$$

Integrating the differential inequality,

$$H(z(t)) \leqslant (\sqrt{H(z(0))} - \frac{1}{\sqrt{2}}\underline{\gamma}t)^2, \tag{42}$$

$$\|z\| \leqslant \sqrt{2H(z(0))} - \underline{\gamma}t. \tag{43}$$

According to $H(z(0)) = \frac{1}{2}\|z_0\|^2$,

$$0 \leqslant \|z\| \leqslant \|z_0\| - \underline{\gamma}t. \tag{44}$$

Therefore, the finite time t_{f} is provided by

$$t_{\mathrm{f}} = \frac{\|z_0\|}{\underline{\gamma}} \tag{45}$$

and the off-the-manifold coordinate $z(t)$ converges to equilibrium at t_{f} from an initial condition z_0.

Remark 1 The dimension of the manifold dynamics (23) is one, so that the attractivity of finite time is $t_{\mathrm{f}} = \frac{z_0}{\gamma}$.

3.4 Disturbance rejection

This section analyzes the disturbance rejection using the proposed finite-time control law. A class of nonlinear systems in the presence of bounded unknown disturbance is described as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f_2(x_1) + g_2(x_1)u, \\ \dot{x}_3 = f_3(x_1) + u + d(t), \end{cases} \tag{46}$$

where all variables are defined above. With regards to this study's objective, an Assumption is necessary.

Assumption 5 In general, the external disturbance $d(t)$ is time-varying and bounded, i.e., $\|d(t)\| \leqslant \delta$, where δ is a positive constant.

By selecting the controller (26), the derivative of $H_2(z)$ is rewritten as

$$\dot{H}_2(z) = -\gamma\|z\| + d(t)z \leqslant -(\gamma - \delta)\|z\|. \tag{47}$$

The design parameter γ is set to $\gamma \geqslant \delta$ to ensure that the term $(\gamma - \delta)\|z\|$ is positive, so $\dot{H}_2(z) \leqslant 0$.

When computing the finite time, the disturbance must also be considered. From Eq.(45), the finite time t_{f} is expressed as follows:

$$t_{\mathrm{f}} = \frac{\|z_0\|}{\underline{\gamma} - \delta}, \tag{48}$$

where $\underline{\gamma}$ is defined as $\underline{\gamma} = \min\{\gamma_i\}$. Therefore, the attractivity of finite time for the manifold coordinate (23) is $t_{\mathrm{f}} = \frac{z_0}{\gamma - \delta}$.

Remark 2 The manifold is insensitive to unknown bounded disturbance. Accordingly, the robustness of stabilization of the nonlinear system (1) with the proposed finite-time controller can be ensured.

4 A cart-pendulum system

This section describes the construction of a cart-pendulum system as an example of mastering the modified I & I technique (see Fig. 2).

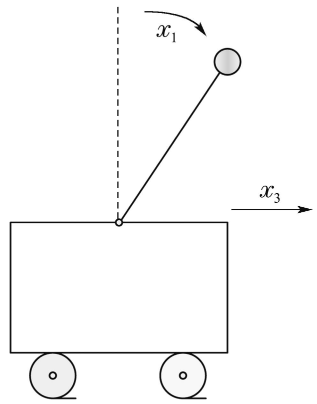

Fig. 2 The cart-pendulum system

A partial feedback linearization stage is assumed to have been applied[21]. After normalization, the state equation becomes:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = a\sin x_1 + b\cos x_1 u, \\ \dot{x}_3 = u, \end{cases} \tag{49}$$

where x_1 and $x_2 \in \mathbb{R}$ are the pendulum angle with respect to the upright vertical axis and its velocity, respectively, $x_3 \in \mathbb{R}$ is the velocity of the cart, and $u \in \mathbb{R}$ is the control input. The positive constants $a > 0$ and $b > 0$ are physical parameters. The equilibrium to be stabilized is the upward position of the pendulum after the cart stops, which corresponds to $x_* = 0$. The state equation (49) has the same form as the system (1), i.e.:

$$\begin{cases} f_2(x_1) = a\sin\, x_1, \\ g_2(x_1) = -b\cos\, x_1, \\ f_3(x_1) = 0, \end{cases} \tag{50}$$

and Assumption 1 is automatically satisfied. From Eq.(26), the I & I stabilizing control law is specified by the following:

$$\psi(x,z) = \frac{1}{\varDelta(x_1,x_2)}(-\varSigma(z)\gamma + \frac{\partial\pi_3}{\partial x_1}x_2 + \frac{\partial\pi_3}{\partial x_2}a\sin x_1), \tag{51}$$

where π_3 and $\varDelta$ are obtained based on Assumptions 2–3. $\varDelta$ plays a fundamental role in the stabilization of closed-loop systems. Select the following:

$$\pi_3 = -k_1x_1 - k_2\frac{x_2}{\cos\, x_1}, \tag{52}$$

$$\varDelta = 1 - k_2 b \tag{53}$$

with $k_1 > 0,\ k_2 > 0$, so the stabilizing control law for the cart-pendulum system becomes

$$\psi(x) = -\frac{1}{1-k_2b}\{-\mathrm{sgn}(x_3 - \pi_3)\gamma + (k_1 + \frac{k_2x_2\sin x_1}{\cos^2 x_1})x_2 + k_2 a\tan x_1\} \tag{54}$$

with $\gamma > 0$. Replacing the explicit form $f_2(x)$, $g_2(x)$ and $f_3(x)$, the stability proof can be completed. (This procedure is omitted for brevity.)

Note that V has an isolated global minimum at zero and Δ is a constant. The controller is not globally defined because x_1 has a singularity at $\frac{\pi}{2}$. The proposed stabilizing control law was implemented on a MATLAB Simulink simulation. It was assumed that $a = 24.5,\ b = 2.5$. The domain of attraction for the cart-pendulum system was $(-\frac{\pi}{2}, \frac{\pi}{2})$. The initial conditions were $x(0) = (\frac{\pi}{2} - 0.1, 0, 0)$, and parameters were $k_1 = 5,\ k_2 = 3$, and $\gamma = 5$. Simulation results are shown in Figs. 3−5. Fig. 3 shows that the state variables x_1, x_2, and x_3 converge to zero at time $t = 5\,\mathrm{s}$, with the proposed I & I method. Convergence to zero at a high speed was achieved. Fig. 4 displays the curves of the corresponding input of the system (1). Fig. 5 shows that the off-the-manifold coordinate z converged to equilibrium in finite time t_{f}. From $z(0) = x_3(0) - \pi_3(x_1(0), x_2(0)) = 7.354$, the finite time $t_{\mathrm{f}} = 1.47\,\mathrm{s}$ is obtained by replacing the initial condition in (45).

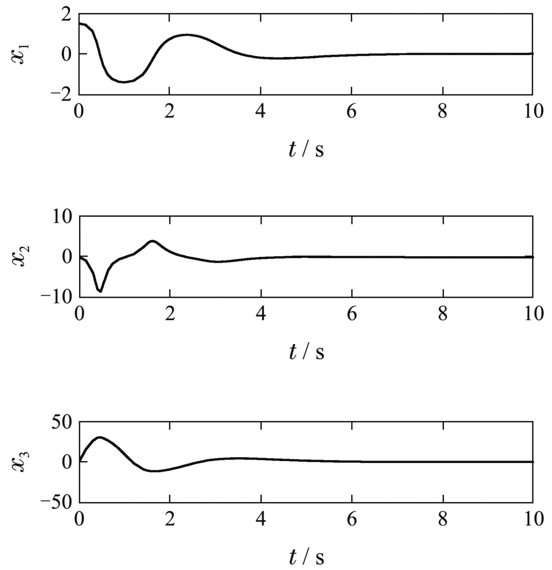

Fig. 3 Simulation of trajectories of states for the cart-pendulum system

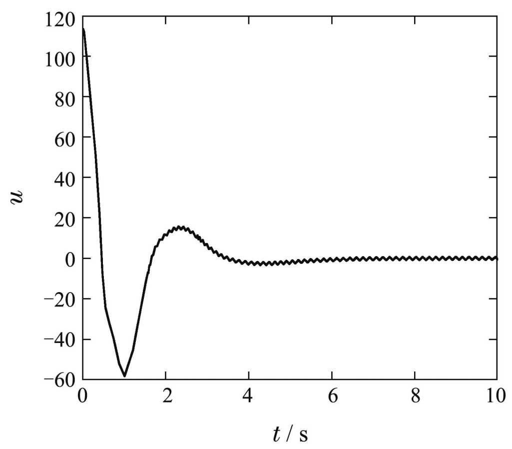

Fig. 4 Simulation of control input for the cart-pendulum system

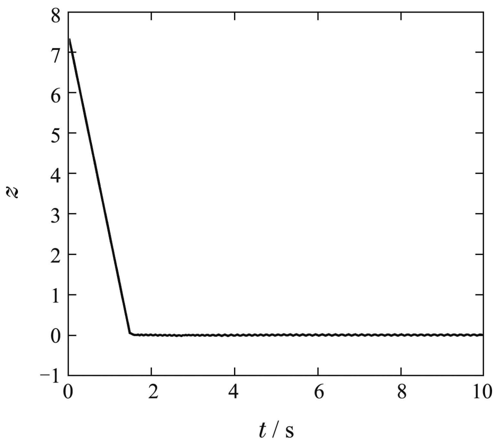

Fig. 5 Simulation of the off-the-manifold coordinate for the cart-pendulum system

Simulation results conducted for disturbance $d(t) = 5\cos(1.5t + \frac{\pi}{6})$ at $10 \sim 15\,\mathrm{s}$ are shown in Figs. 6−8. Parameters were $k_1 = 5,\ k_2 = 3$, and $\gamma = 9$. A slight fluctuation in x_1 can be observed at $10 \sim 15\,\mathrm{s}$ in Fig. 6, the amplitude of which is about 0.5 rad. Fig. 7 shows that the control input converged to zero after a sine disturbance at $10 \sim 15\,\mathrm{s}$, whereas the response curve oscillated at the zero point. Fig. 8 shows that manifold z was rendered insensitive to bounded disturbance.

The mode of the system response was inclined to chatter along $z = 0$, as shown in Figs. 4 and 7. The reason for manifold chatter is that the attractivity speed was limited, and inertia existed in the system. Rejecting manifold chatter is the primary objective of our laboratory's future research.

Remark 3 The off-the-manifold coordinate depicted in Figs. 5 and 8 converged to zero in finite time with the proposed I & I method. That said, manifold chatter directly affected the control input and reduced the performance of the closed-loop system (1).

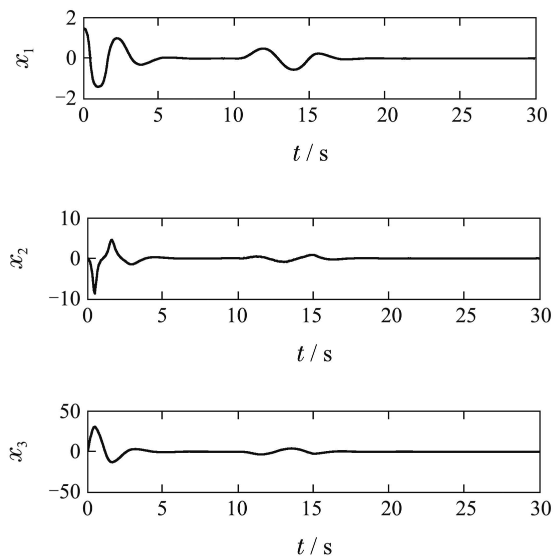

Fig. 6 Simulation of trajectories of states for the cart-pendulum system with a bounded disturbance

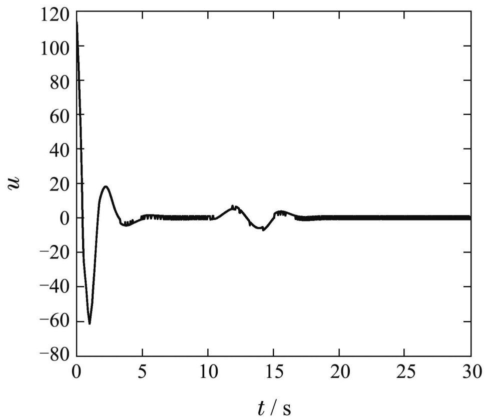

Fig. 7 Simulation of control input for the cart-pendulum system with a bounded disturbance

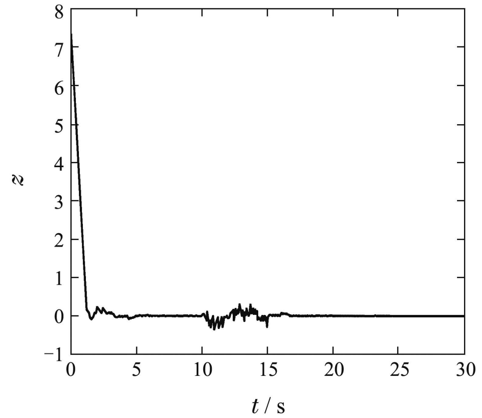

Fig. 8 Simulation of the off-the-manifold coordinate for the cart-pendulum system with a bounded disturbance

5 Conclusions

This study developed a finite-time attractivity-based immersion and invariance control for a class of nonlinear systems. This novel approach modified the standard immersion and invariance theorem and focused on computations of finite time. The controller design was detailed above, and stability proofs were provided. A manifold was successfully designed to be insensitive to bounded unknown disturbance by implementing the finite-time attractivity controller. Controller performance was demonstrated using a cart-pendulum system with various simulations. Clearly, the proposed control algorithm is effective for this class of nonlinear systems.

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作者简介:

黄显林 (1956–), 男, 教授, 博士生导师, 目前研究方向为飞行器导航、制导与控制技术、非线性系统鲁棒控制、复杂系统自适应控制方法、欠驱动控制系统的研究, E-mail: xlinhuang163@163.com;

张　旭 (1986–), 男, 博士研究生, 目前研究方向为高阶非线性系统控制方法、系统浸入与不变流形理论、拮抗肌腱式机械手的控制方法、超视距捷联/红外复合制导技术, E-mail: alonsobobo@gmail.com;

卢鸿谦 (1975–), 男, 副教授, 目前研究方向包括非线性控制与应用、组合导航与控制技术、恒拉力控制系统, E-mail: luhongqian@163.com.