文章编号: 1000−8152(2011)12−1786−05

环境干扰下分形生长的定方位控制

张 丽, 刘树堂
(山东大学 控制科学与工程学院, 山东 济南 250061)

摘要: 针对复杂无序的实际分形生长系统, 研究了环境干扰下分形生长的定方位控制模型. 不同于以往对分形生长的定方位的研究仅限于对它的实验方面的讨论, 本文利用范数理论得到了干扰项为广义函数时控制生长粒子凝聚到不同方位的定量方法. 利用所提出的控制方法解决了干扰项为更一般函数形式的定方向生长问题, 并以非线性项为二次多项式、正弦函数为例, 运用所得的定量关系对该系统不同作用区域的生长粒子进行了控制, 仿真结果表明了该控制方法的有效性.

关键词: 环境干扰; 分形生长; 控制
中图分类号: TB112; O231 **文献标识码**: A

Directed control for fractal growth with environmental disturbance

ZHANG Li, LIU Shu-tang
(College of Control Science and Engineering, Shandong University, Jinan Shandong 250061, China)

Abstract: For a complex and disordered system of real fractal growth, the control model of the directed growth of the fractal growth with environmental disturbance is studied. Being different from the past studies on directed growth of the fractal growth based on their experiments, this work is concerned with the directed growth of the fractal growth theoretically. By the norm theory, we obtain a quantitative method for controlling the growing particles to aggregate in different regions when the environmental disturbance is a generalized function. The directed growth problem with the environmental disturbance in a more general functional form is solved by using the above obtained control method. Two examples of disturbances in quadratic polynomial and sinusoidal function are given, respectively. Simulation results illustrate that the growing particles will aggregate into the controlled regions when the corresponding controlled methods for associated growth regions are applied.

Key words: environmental disturbance; fractal growth; control

1 引言(Introduction)

在最近的几十年里, 人们对分形生长在非线性复杂体系中的作用和影响的研究越来越多[1]. 关于分形生长的研究, 不仅具有数学上的重要性, 而且能折射出深刻的物理规律. 它在理论物理和生物、医学等的研究中起着极其重要的作用. 分形生长的形态变化的模型主要有DLA(diffusion limited aggregation)模型[2]和DBM(dielectric breakdown model)模型[3]等. 由这两类模型导致的分形生长形态变化的研究已在工程技术、环境科学、气象学、材料科学等众多领域中取得了许多重要的成果[4~7]. 值得注意的是, 根据客观的要求, 往往需要制约粒子分形生长的方位, 但十分遗憾的是, 到目前为止, 对于分形生长的定方位的制约问题仅仅涉及到它的实验研究[8~11], 而对于分形生长的定方位控制在理论上的预测, 是目前还尚未涉及到的一个新的领域.

本文主要针对实际环境中的分形生长系统, 给出了分形生长的定方位控制理论模型, 利用范数理论获得了环境干扰项为广义函数时对分形生长的定方位控制的数量关系. 仿真结果表明了方法的有效性.

2 分形生长的定方位控制(Directed control for the fractal growth)

2.1 问题描述(Problem description)

注意到对于实际环境中复杂无序的分形凝聚生长问题, 通常其实际的生长概率满足如下形式的分布参数系统[12]:

$$\nabla^2\phi(x,y) = f(\phi(x,y), \frac{\partial\phi(x,y)}{\partial t}) + u(x,y), \quad (1)$$

这里: $\phi(x,y)$是生长概率(或凝聚浓度); $f(\phi(x,y), \dfrac{\partial\phi(x,y)}{\partial t}) + u(x,y)$是一个非线性函数, 代表环境干扰项, 一般称为强迫项, 其中$u(x,y)$是源项.

广义函数可以认为是由经典的函数逐步推广、扩大而来的, 而且广泛应用于物理和工程领域, 因此

收稿日期: 2010−10−16; 收修改稿日期: 2011−08−26.
基金项目: 国家自然科学基金资助项目(60874009, 10971120); 全国百篇优秀博士论文专项基金资助项目(200444).

本文考虑$f(\phi(x,y),\dfrac{\partial\phi(x,y)}{\partial t})+u(x,y)$为定义在基本函数空间$K$上的正则的广义函数.

通常对于非线性分布参数系统(1), 要得到其精确的解析解是非常困难的, 且大量用于描述现实世界中现象的微分方程并不具有足够光滑的解, 从而本文考虑系统(1)的弱解(也可称为广义解)性质来研究式(1)的生长概率$\phi(x,y)$与环境干扰项$f(\phi(x,y),\dfrac{\partial\phi(x,y)}{\partial t})+u(x,y)$之间的关系.

2.2 定方位控制的定量关系(Quantitative relationship of the directed control)

对空间
$$L_2=\{g(x,y)|\int_\Omega g^2(x,y)\mathrm{d}s<\infty\}$$
中的任意函数$g(x,y)$定义其范数为
$$\|g(x,y)\|_{L_2}=(\int_\Omega g^2(x,y)\mathrm{d}s)^{\frac{1}{2}}.$$
Sobolev空间$H^k(\Omega)$中的任意函数$\phi(x,y)$的范数为
$$\|\phi(x,y)\|_{H^k(\Omega)}=(\int_\Omega\sum_{k\geqslant1}|\nabla\phi(x,y)|^2\mathrm{d}s)^{\frac{1}{2}}.$$

定理 1　设$f(\phi(x,y),\dfrac{\partial\phi(x,y)}{\partial t})$为满足如下条件的广义函数:
$$\int_\Omega f(\phi(x,y),\frac{\partial\phi(x,y)}{\partial t})\phi(x,y)\mathrm{d}s\leqslant-\|\frac{\phi(x,y)}{\gamma}\|_{L_2},$$
则有
$$\|\phi(x,y)\|_{L^2}\leqslant\gamma\|u(x,y)\|_{L_2},$$
其中: $\gamma>0$, Ω为二维实空间中的任意区域.

证　首先证明存在一个函数$V(\phi)$, 使得这个函数满足如下的Hamilton-Jacobi方程[13]:
$$\begin{cases}\dfrac{\partial V(\phi)}{\partial\phi}\displaystyle\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s+\\ \dfrac{1}{2\gamma^2}(\dfrac{\partial V(\phi)}{\partial\phi})^2+\dfrac{1}{2}\phi^2(x,y)=0,\\ V(\phi)=0,\end{cases}\tag{2}$$
其中对于任何$\phi\in K$都有$V(\phi)>0$成立.

令$S=\dfrac{\partial V(\phi)}{\partial\phi}$, 则得到
$$S\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s+\frac{1}{2\gamma^2}S^2+\frac{1}{2}\phi^2=0,$$
以及解为
$$S_1=-\gamma^2\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s+|\gamma|\sqrt{\gamma^2(\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s)^2-\phi^2},$$
$$S_2=-\gamma^2\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s-|\gamma|\sqrt{\gamma^2(\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s)^2-\phi^2}.$$
即有
$$V_1(\phi)=\int_\Omega[-\gamma^2\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s+|\gamma|\sqrt{\gamma^2(\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s)^2-\phi^2}]\mathrm{d}s,$$
$$V_2(\phi)=\int_\Omega[-\gamma^2\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s-|\gamma|\sqrt{\gamma^2(\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s)^2-\phi^2}]\mathrm{d}s.$$
根据已知条件可知$\int_\Omega f(\phi,\dfrac{\partial\phi}{\partial t})\phi\mathrm{d}s<0$, 从而有以下不等式成立:
$$-\gamma^2\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s+|\gamma|\sqrt{\gamma^2(\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s)^2-\phi^2}>0.$$
又根据被积函数与其定积分的关系, 可知对任何$\phi\in K$都有$V_1(\phi)>0$.

下面证明$\|\phi\|_{L_2}\leqslant\gamma\|u\|_{L_2}$.

假设
$$W(\phi,\frac{\partial\phi}{\partial t})=\frac{\partial V_1(\phi)}{\partial\phi}(\int_\Omega f(\phi,\frac{\partial\phi}{\partial t})\phi\mathrm{d}s)+\frac{\partial V_1(\phi)}{\partial\phi}u,$$
由Hamilton-Jacobi方程(2), 可得
$$W(\phi,\frac{\partial\phi}{\partial t})=(-\frac{1}{2}\phi^2)-\frac{1}{2\gamma^2}(\frac{\partial V_1(\phi)}{\partial\phi})^2+\frac{\partial V_1(\phi)}{\partial\phi}u-\frac{1}{2}\gamma^2u^2+\frac{1}{2}\gamma^2u^2=(-\frac{1}{2}\phi^2)+\frac{1}{2}\gamma^2u^2-\frac{1}{2\gamma^2}(\frac{\partial V_1(\phi)}{\partial\phi})^2(-\frac{1}{2}\gamma^2u^2)+\frac{\partial V_1(\phi)}{\partial\phi}u.$$
积分上式, 得
$$\int_\Omega W(\phi,\frac{\partial\phi}{\partial t})\mathrm{d}s=\int_\Omega[-\frac{1}{2}\phi^2+\frac{1}{2}\gamma^2u^2-\frac{1}{2\gamma^2}(\frac{\partial V_1(\phi)}{\partial\phi})^2-\frac{1}{2}\gamma^2u^2+\frac{\partial V_1(\phi)}{\partial\phi}u]\mathrm{d}s=\frac{1}{2}(\gamma^2\|u\|^2_{L_2}-\|\phi(x,y)\|^2_{L_2})-\frac{\gamma^2}{2}\|u-\frac{1}{\gamma^2}\frac{\partial V_1(\phi)}{\partial\phi}\|^2_{L_2}.$$
由于$-\dfrac{\gamma^2}{2}\|u-\dfrac{1}{\gamma^2}\dfrac{\partial V_1(\phi)}{\partial\phi}\|^2_{L_2}\leqslant0$, 因此
$$\int_\Omega W(\phi,\frac{\partial\phi}{\partial t})\mathrm{d}s\leqslant\frac{1}{2}(\gamma^2\|u\|^2_{L_2}-\|\phi\|^2_{L_2}).$$
另外, 根据S_1与$\dfrac{\partial V_1(\phi)}{\phi}$的关系以及条件

$$\int_\Omega f(\phi, \frac{\partial\phi}{\partial t})\phi \mathrm{d}s < -\|\frac{\phi}{\gamma}\|_{L_2},$$

可得

$$\frac{\partial V_1(\phi)}{\partial\phi}(\int_\Omega f(\phi, \frac{\partial\phi}{\partial t})\phi \mathrm{d}s) + \frac{\partial V_1(\phi)}{\partial\phi}u > 0.$$

从而有

$$\int_\Omega [\frac{\partial V_1(\phi)}{\partial\phi}(\int_\Omega f(\phi, \frac{\partial\phi}{\partial t})\phi \mathrm{d}s) + \frac{\partial V_1(\phi)}{\partial\phi}u]\mathrm{d}s > 0$$

成立, 这意味着$\frac{1}{2}[\gamma^2\|u\|_{L_2}^2 - \|\phi\|_{L_2}^2] \geqslant 0$, 即

$$\|\phi\|_{L_2} \leqslant \gamma\|u\|_{L_2}. \tag{3}$$

证毕.

根据泛函偏微分方程弱解的性质[14], 可以得到如下的另外一个结论:

定理 2 设$\phi(x,y)$为式(1)的弱解, 则对于Ω的任意紧子集Ω', 有$\phi(x,y)\in H^2(\Omega')$, 满足

$$\|\phi\|_{H^2(\Omega')} \leqslant C\|\phi\|_{H^1(\Omega)} + \frac{C}{\gamma} + C\|u\|_{L_2(\Omega)},$$

其中C与$\mathrm{dist}(\Omega', \partial\Omega)$有关.

证 因为ϕ是式(1)的弱解, 则有

$$\int_\Omega(-\nabla\phi\nabla\varphi)\mathrm{d}s = \int_\Omega(f+u)\varphi\mathrm{d}s, \tag{4}$$

对$\forall\varphi\in H_0^1(\Omega)$. 令$\tau_h$为$x$方向的平移算子, $\tau_h\phi(x) = \phi(x+he_1)$, 差分算子$\Delta_h = \frac{\tau_h - I}{h}$. 设$v\in H_0^1(\Omega)$, v的支集$\mathrm{spt}\, v$是Ω的紧子集. 对$h < \frac{\mathrm{dist}(\mathrm{spt}\, v, \partial\Omega)}{2}$, 取检验函数$\varphi = \Delta_{-h}v$, 代入式(4)后, 得

$$\int_\Omega \Delta_h(-\nabla\phi)\nabla v\mathrm{d}s = -\int_\Omega(f+u)\Delta_{-h}v\mathrm{d}s.$$

因为$\Delta_h(-\nabla\phi) = \tau_h(-\Delta_h\nabla\phi) + \nabla\phi\Delta_h - 1$, 因此

$$\begin{aligned}&\int_\Omega \tau_h(-\nabla\Delta_h\phi\nabla v)\mathrm{d}s = \\ &-\int_\Omega(\Delta_h(-\nabla\phi)\nabla v) + (f+u)\Delta_{-h}v\mathrm{d}s \leqslant \\ &C(\|\phi\|_{H^1(\Omega)} + \|f\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)})\|\nabla v\|_{L_2(\Omega)}.\end{aligned}$$

取$\eta\in C_0^\infty(\Omega)$, 当$x\in\Omega'$时, $\eta(x) = 1$. 令$v = \eta^2\Delta_h\phi$, 则有

$$\begin{aligned}&\int_\Omega \eta^2\tau_h(-\nabla\Delta_h\phi\nabla\Delta_h\phi)\mathrm{d}s \leqslant \\ &-2\int_\Omega \eta\tau_h(-\nabla\Delta_h\phi(\nabla\eta)\Delta_h\phi\mathrm{d}s C(\|\phi\|_{H^1(\Omega)} + \\ &\|f\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)})(\|\eta\nabla\Delta_h\phi\|_{L_2(\Omega)} + \\ &2\|\Delta_h\phi\nabla\eta\|_{L_2(\Omega)}).\end{aligned}$$

利用Cauchy不等式, 则有

$$\begin{aligned}&\int_\Omega |\eta\Delta_h\nabla\phi|^2\mathrm{d}s \leqslant \\ &C\int_\Omega |\nabla\eta|^2|\Delta_h\phi|^2\mathrm{d}s + \\ &C(\|\phi\|_{H^1(\Omega)}^2 + \|f\|_{L_2(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2).\end{aligned}$$

所以

$$\begin{aligned}\|\eta\Delta_h\nabla\phi\|_{L_2(\Omega)}^2 \leqslant C(&\|\phi\|_{H^1(\Omega)}^2 + \\ &\|f\|_{L_2(\Omega)}^2 + \|u\|_{L_2(\Omega)}^2).\end{aligned}$$

适当选取η, 考虑任意方向的差分算子, 则有$\phi\in H^2(\Omega')$, 且满足如下估计式:

$$\begin{aligned}&\|\phi\|_{H^2(\Omega')} \leqslant \\ &C(\|\phi\|_{H^1(\Omega)} + \|f\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}).\end{aligned} \tag{5}$$

另一方面, $f(x,y)$为满足如下条件的广义函数:

$$\int_\Omega f(\phi, \frac{\partial\phi}{\partial t})\phi\mathrm{d}s < -\|\frac{\phi}{\gamma}\|_{L_2(\Omega)},$$

可知$|\int_\Omega f(\phi, \frac{\partial\phi}{\partial t})\phi(x,y)\mathrm{d}s| < \|\frac{\phi(x,y)}{\gamma}\|_{L_2(\Omega)}$. 又因为Hölder不等式, 有

$$\begin{aligned}&|\int_\Omega f(\phi, \frac{\partial\phi}{\partial t})\phi(x,y)\mathrm{d}s| \leqslant \\ &(\int_\Omega |f(\phi, \frac{\partial\phi}{\partial t})|^2\mathrm{d}s)^{\frac{1}{2}}(\int_\Omega |\phi|^2\mathrm{d}s)^{\frac{1}{2}} = \\ &\|f\|_{L_2(\Omega)}\|\phi\|_{L_2(\Omega)}.\end{aligned}$$

显然有$\|f\|_{L_2(\Omega)} \leqslant \frac{1}{\gamma}$成立, 代入式(5)立即可得结论.

证毕.

特别的, 当y为常值y_0时, 生长概率$\phi(x,y)$变为$\phi(x,y_0)$, 这时候生长概率为一维状态, 因此可以得到一维的状况:

推论 1 设$f(\phi(x,y_0), \frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}t})$为满足如下条件的广义函数:

$$\begin{aligned}&\int_\Omega f(\phi(x,y_0), \frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}t})\phi(x,y_0)\mathrm{d}x \leqslant \\ &-|\frac{\sqrt{\int_\Omega \phi^2(x,y_0)\mathrm{d}x}}{\gamma}|,\end{aligned}$$

则有$|\phi(x,y_0)| \leqslant \gamma|u(x,y_0)|$, 其中: $\gamma > 0$, Ω为一维实空间中的任意区域.

推论 2 设$\phi(x,y_0)$为如下系统:

$$\frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}x} = f(\phi(x,y_0), \frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}t}) + u(x,y_0)$$

的解, 则对于一维区间Ω的任意紧子集Ω', 有

$$\begin{aligned}&\sqrt{\int_{\Omega'}(\frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}x})^2\mathrm{d}x} \leqslant \\ &C\sqrt{\int_\Omega(\frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}x})^2\mathrm{d}x} + C\sqrt{\int_\Omega u^2(x,y_0)\mathrm{d}x} + \frac{C}{\gamma},\end{aligned}$$

其中C与$\mathrm{dist}(\Omega', \partial\Omega)$有关, $\frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}x}$满足如下关系:

$$\int_\Omega \phi(x,y_0)\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = -\int_\Omega v\frac{\mathrm{d}\phi(x,y_0)}{\mathrm{d}x}\mathrm{d}x,$$

对一切连续函数v都成立.

3　定方位控制的实现(Implementation of directed growth)

根据结论(3)可以看出, 当

$$\int_{\Omega} f(\phi(x,y), \frac{\partial \phi(x,y)}{\partial t})\phi(x,y)\mathrm{d}s \leqslant -\|\frac{\phi(x,y)}{\gamma}\|_{L_2(\Omega)}$$

时, 源项$u(x,y)$能抑制广义函数$f(\phi(x,y), \dfrac{\partial \phi(x,y)}{\partial t})$对实际分形生长形态的非线性作用, 从而可以控制生长粒子凝聚到不同的方位或区域, 实现定方位的控制.

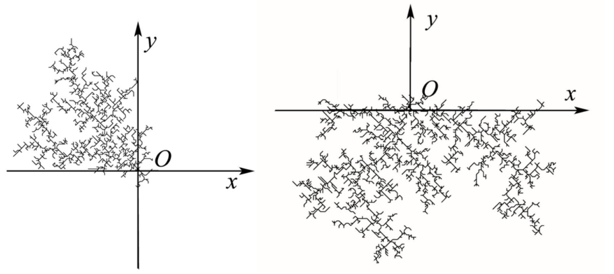
当分别取:

$$f(\phi(x,y), \frac{\partial \phi(x,y)}{\partial t}) = a\phi^2(x,y),$$

$$f(\phi(x,y), \frac{\partial \phi(x,y)}{\partial t}) = b\sin(\phi(x,y)),$$

源项$u(x,y)$的作用区域Γ为半平面、四分之一平面、直线、分段直线、圆域, $u(x,y)$的取值为常数k时, 利用文献[12]中的仿真方法所得的仿真图1～3与没有环境干扰的仿真图4的对比, 清楚的表达了源项对生长粒子凝聚到不同方位的控制作用, 体现了控制方法的有效性.

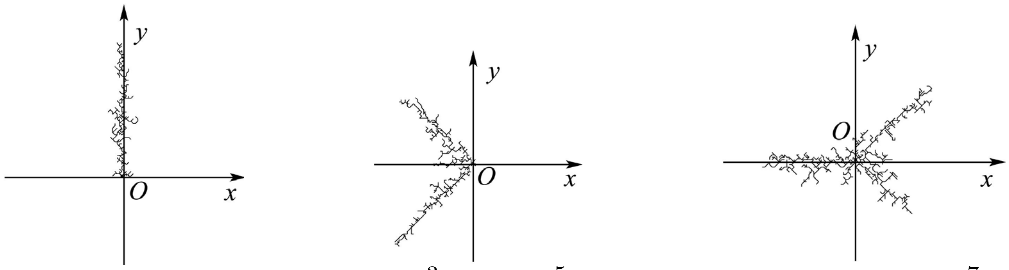
图5显示了用Sandbox方法[15]求得的分形生长簇的一致维数D之间的关系, 其中只有圆形控制域的生长簇的控制区域是$\Gamma = \{(r,\theta)|r变化, 0 \leqslant \theta < \dfrac{2}{\pi}\}$, 有分段控制的定方向生长簇的控制区域是$\Gamma = \Gamma_1 \cup \Gamma_2$, 其中: $\Gamma_1 = \{(r,\theta)|r \leqslant 15, 0 \leqslant \theta < \dfrac{\pi}{2}\}$, $\Gamma_2 = \{(r,\theta)|r \in \mathbb{R}, \theta = i\dfrac{\pi}{4}, i = 1, 2, \cdots\}$. 由图5可以看出, 只有圆形控制域的生长粒子凝聚的复杂度要大于除了有圆形控制域外还有在各方向控制的定方向生长凝聚的复杂度, 这说明, 虽然两种生长簇有共同的控制区域(圆域), 但是在该控制区域的范围之外的分段控制, 也即定方位生长的控制, 在控制了生长的方位的同时, 降低了生长的复杂度.



(a) $\dfrac{\pi}{2} < \theta \leqslant \pi$　　(b) $\pi < \theta \leqslant 2\pi$

图 1　当Γ中的r是实数, θ变化时对应的分形生长的变化

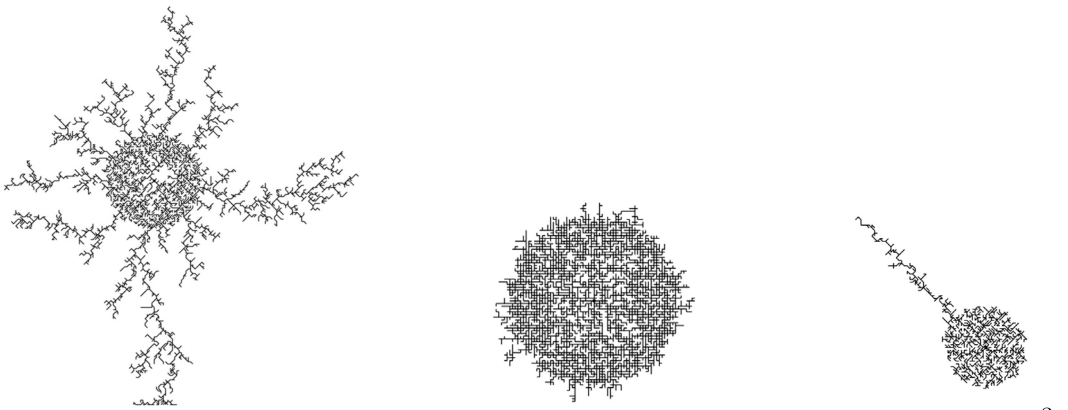
Fig. 1　The fractal growth variation when r is real and θ changes, where $(r,\theta) \in \Gamma$



$\theta = \dfrac{\pi}{2}$　　$\theta = \dfrac{3\pi}{4}$以及$\theta = \dfrac{5\pi}{4}$　　$\theta = \dfrac{\pi}{4}, \theta = \pi$以及$\theta = \dfrac{7\pi}{4}$

图 2　当Γ中的r是实数, θ变化时对应的分形生长的变化

Fig. 2　The fractal growth variation when r is real and θ changes, where $(r,\theta) \in \Gamma$



(a) $r \leqslant 20$　　(b) $r \leqslant 30$　　(c) $r \leqslant 15$, $\theta = \dfrac{3\pi}{4}$

图 3　当Γ中的r变化, $0 \leqslant \theta < 2\pi$时对应的分形生长的变化

Fig. 3　The fractal growth variation when r changes and $0 \leqslant \theta < 2\pi$, where $(r,\theta) \in \Gamma$

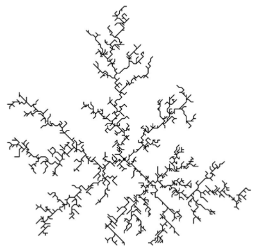


图4 没有环境干扰的分形生长变化

Fig. 4 The fractal growth variation without environmental disturbance

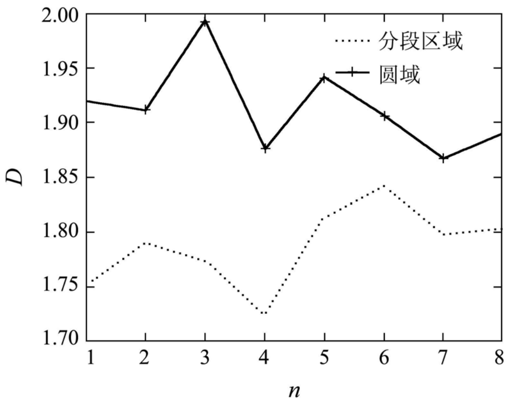


图5 有圆形控制域的分形生长的一致维数与有分段控制域的分形生长的一致维数之间的比较

Fig. 5 The comparison between the dimensions of the fractal growth with round region and that of the growth with piecewise region

4 结论(Conclusion)

本文中, 笔者将控制的思想和方法引入到分形生长的理论中, 并以广义函数这个更一般的函数形式为环境干扰项, 采用范数的理论进行了控制分析. 仿真结果表明了该控制方法是有效的、可行的, 这些理论分析必将有助于进一步加深分形生长在各领域中的理解与应用. 另外, 在笔者的理论当中, 只考虑了生长方向的控制问题并没有详细分析生长速度的控制问题. 原因是关于生长速度的控制研究是一个非常重要的问题, 需要很长的篇幅才能分析透彻, 因此基于文章篇幅的限制, 这方面的研究笔者将另文专门讨论.

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作者简介:

张 丽 (1980—), 女, 助教, 博士研究生, 目前研究方向为非线性系统的混沌控制、分形理论的应用与控制, E-mail: zhanglisdu2008@yahoo.com;

刘树堂 (1958—), 男, 教授, 博士生导师, 目前研究方向为复杂系统的定性理论与控制、非线性系统的混沌、分岔控制、分形理论的应用与控制、自适应控制, E-mail: stliu@sdu.edu.cn.