不确定性机械系统的最优鲁棒控制设计: 模糊法

甄圣超[1,2†], 赵　韩[1], 黄　康[1], 陈义华[2]

(1. 合肥工业大学 机汽学院, 安徽 合肥 230009; 2. 佐治亚理工学院 机械工程系, 亚特兰大 30332, 美国)

摘要: 针对不确定性机械系统, 提出了一种新的最优鲁棒控制方法. 本文用模糊法去描述机械系统中的不确定性. 机械系统的性能要求是确定的(保证最低要求), 同时也是模糊的(成本控制里用到). 所提出的控制方法是确定的, 而不是基于假设的规则. 经过严格的理论证明, 控制系统最终可达到理想的性能指标. 基于模糊信息, 本文设计了一个性能指标(综合成本, 包括系统的平均模糊性能和控制成本). 通过最小化此性能指标, 可解决控制的最优设计问题. 这种最优设计方法可得到唯一的解析形式的最优解. 总的来说, 这种最优鲁棒控制方法较为系统, 能够保证确定的系统性能得以实现, 同时控制成本最小. 最后, 本文选了一个机械系统作为例子.

关键词: 不确定性; 模糊理论; 鲁棒控制; 最优控制

中图分类号: TP273　　　**文献标识码:** A

Optimal robust control design of uncertain mechanical systems: a fuzzy approach

ZHEN Sheng-chao[1,2†], ZHAO Han[1], HUANG Kang[1], CHEN Yi-hua[2]

(1. School of Mechanical and Automotive Engineering, Hefei University of Technology, Hefei Anhui 230009, China;
2. Mechanical Engineering, Georgia Institute of Technology, Atlanta Georgia 30332, USA)

Abstract: A new optimal robust control is proposed for mechanical systems with fuzzy uncertainty. Fuzzy set theory is used to describe the bound of uncertainty. The desirable system performance is deterministic (assuring the bottom line) and also fuzzy (enhancing the cost consideration). The proposed control is deterministic but is not the usual if-then rules-based. The resulting controlled system is proved to be uniformly bounded and uniformly ultimately bounded via the Lyapunov minimax approach. A performance index (the composite cost which includes the average fuzzy system performance and the control effort) is proposed based on the fuzzy information. The optimal design problem associated with the control can then be solved by minimizing the performance index. The unique closed-form optimal gain and the cost are explicitly shown. The resulting control design is systematic and is able to guarantee the deterministic performance as well as the minimal cost. Finally, a mechanical system is chosen for demonstration.

Key words: uncertainty; fuzzy set theory; robust control; optimal control

1 Introduction

There always exists unnoticeable and unknown aspects of the real system in the dynamic model which captures prominent features of the mechanical system. Researches on mechanical system control have always been very active, especially on handling uncertainties in the system. Exploring uncertainty and determining what is known and what is unknown about the uncertainty is very important. Once the bound information of the uncertainty is clearly identified, we can use this known bound information to develop deterministic control approaches. The well-known H_2/H_∞ control[1–2], the Lyapunov-based control[3–4], the sliding mode control[5] and so on contribute to this deterministic approach. When the known portion cannot be completely isolated from the unknown, one may take the stochastic control approach. The classic linear-quadratic-Gaussian control[6] is in such domain.

The stochastic dynamical systems merge the probability theory with system theory and has been the most outstanding since the 50s. Kalman initiated the effort of looking into the estimation problem and control problem[7–8] in the state space framework when a system is under stochastic noise. Although the stochastic approach is quite self-contained and a impressive arena of practitioners. Concerns on the probability theory's validity in describing the real world does exist. That is to say, the link between the stochastic mathematical tool and the physical world might be loose. Kalman, among others, despite his early devotion to stochastic system

theory now contends that the probability theory might not be all that suitable to describe the majority of randomness[9].

The uncertainty in engineering is often acquired via observed data and then analyzed by the practitioner. However, the observed data are, by nature, always limited and the uncertainty is unlikely to be exactly repeated many times. The earthquake data can be an example for interpretation[10]. Hence, any interpretation via the frequency of occurrence, which requires a large number of repetitions, might sometimes be limited due to a lack of basis. As a result, the fuzzy view via the degree of occurrence may be considered as an alternative in certain applications. We can find more discussions on the relative advantages of fuzziness versus probability in[11].

Fuzzy theory was initially introduced to describe information (for example, linguistic information) that is in lack of a sharp boundary with its environment[12]. Most interest in fuzzy logic theory is attracted to fuzzy reasoning for control, estimation, decision-making, etc. Therefore, the merge between the fuzzy theory and system theory (fuzzy dynamical system) has been less focused on. Past efforts on fuzzy dynamical systems can be found in [13] and [14]. We stress that these are different from the very popular Takagi-Sugeno model or other fuzzy if-then rules-based models. In this paper, from a different angle, we employ the fuzzy theory to describe the uncertainty in the mechanical system and then propose optimal robust control design of fuzzy mechanical systems.

The main contributions are fourfold. First, we not only guarantee the deterministic performance (including uniform boundedness and uniform ultimate boundedness), but also explore fuzzy description of system performance should the fuzzy information of the uncertainty be provided. Second, we propose a robust control which is deterministic and is not the usual if-then rules-based. The resulting controlled system is uniformly bounded and uniformly ultimately bounded proved via the Lyapunov minimax approach. Third, a performance index (the combined cost, which includes average fuzzy system performance and control effort) is proposed based on the fuzzy information. The optimal design problem associated with the control can then be solved by minimizing the performance index. Fourth, the unique closed-form solution of optimal gain and the cost are explicitly presented. The resulting control design is systematic and is able to guarantee the deterministic performance as well as minimizing the cost.

2   Fuzzy mechanical systems

Consider the following uncertain mechanical system:

$$M(q(t), \sigma(t), t)\ddot{q}(t) + V(q(t), \dot{q}(t), \sigma(t), t) +$$

$$G(q(t), \sigma(t), t) + T(q(t), \dot{q}(t), \sigma(t), t) = \tau(t). \quad (1)$$

Here $t \in \mathbb{R}$ is the time (i.e., the independent variable), $q \in \mathbb{R}^n$ is the coordinate, $\dot{q} \in \mathbb{R}^n$ is the velocity, $\ddot{q} \in \mathbb{R}^n$ is the acceleration, $\sigma \in \mathbb{R}^p$ is the uncertain parameter, and $\tau \in \mathbb{R}^n$ is the control input, $M(q, \sigma, t) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $V(q, \dot{q}, \sigma, t) \in \mathbb{R}^n$ is the Coriolis/centrifugal force vector, $G(q, \sigma, t) \in \mathbb{R}^n$ is the gravitational force vector, and $T(q, \dot{q}, \sigma, t) \in \mathbb{R}^n$ is the friction force and external disturbance (we omit arguments of functions where no confusions may arise).

Assumption 1   The functions $M(\cdot)$, $V(\cdot)$, $G(\cdot)$, and $T(\cdot)$ are continuous (Lebesgue measurable in t). Furthermore, the bounding set Σ is known and compact.

Assumption 2   i) For each entry of q_0 (i.e., $q(t_0)$), namely q_{0i}, $i = 1, 2, \cdots, n$, there exists a fuzzy set Q_{0i} in a universe of discourse $\Xi_i \subset \mathbb{R}$ characterized by a membership function $\mu_{\Xi_i} : \Xi_i \to [0, 1]$. That is

$$Q_{0i} = \{(q_{0i}, \mu_{\Xi_i}(q_{0i})) | q_{0i} \in \Xi_i\}. \quad (2)$$

Here Ξ_i is known and compact. ii) For each entry of the vector $\sigma(t)$, namely $\sigma_i(t)$, $i = 1, 2, \cdots, p$, the function $\sigma_i(\cdot)$ is Lebesgue measurable. iii) For each $\sigma_i(t)$, there exists a fuzzy set S_i in a universe of discourse $\Sigma_i \subset \mathbb{R}$ characterized by a membership function $\mu_i : \Sigma_i \to [0, 1]$,

$$S_i = \{(\sigma_i, \mu_i(\sigma_i)) | \sigma_i \in \Sigma_i\}. \quad (3)$$

Here Σ_i is known and compact.

Remark 1   Assumption 2 imposes fuzzy restriction on the uncertainty q_0 and $\sigma(t)$. We employ the fuzzy description on the uncertainties in the mechanical system. This fuzzy description earns much more advantage than the probability avenue which often requires a large number of repetitions to acquire the observed data (always limited by nature).

Assumption 3   The inertia matrix $M(q, \sigma, t)$ in mechanical systems is uniform positive definite, that is, there exists a scalar constant $\underline{\gamma} > 0$ such that

$$M(q, \sigma, t) \geqslant \underline{\gamma}I, \quad (4)$$

for all $q \in \mathbb{R}^n$.

Remark 2   We emphasize that this is an assumption, not a fact. There are cases that the inertia matrix may be positive semi-definite (hence, $\underline{\gamma} = 0$). One example is documented in [15] where the generalized inertia matrix

$$M = \begin{bmatrix} ml_2^2 \cos^2 \theta_2 & 0 \\ 0 & ml_2^2 \end{bmatrix}. \quad (5)$$

Thus $\det[M] = 0$ if $\theta_2 = (2n + 1)\dfrac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \cdots$. That is to say, the generalized inertia matrix M is singular. When $\theta_2 = (2n+1)\dfrac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \cdots$, the kinetic energy $\dfrac{1}{2}\dot{q}^{\mathrm{T}} M \dot{q} = 0$, $\forall \dot{\theta}_1$ which means the rotation does not bring up kinetic energy.

Assumption 4   There is a constant $\bar{\gamma}$, such that for all $(q, t) \in \mathbb{R}^n \times \mathbb{R}$, $\sigma \in \Sigma$, the inertial matrix

$M(q, \sigma, t)$ is bounded as

$$\|M(q, \sigma, t)\| \leqslant \bar{\gamma}. \tag{6}$$

Unless otherwise stated, $\| \cdot \|$ always denotes the Euclidean norm (i.e., $\| \cdot \|_2$). The l_1-norm are sometimes used and indicated by subscript 1.

Theorem 1 There always exists the factorization

$$V(q, \dot{q}, \sigma, t) = C(q, \dot{q}, \sigma, t)\dot{q}, \tag{7}$$

such that $\dot{M}(q, \dot{q}, \sigma, t) - 2C(q, \dot{q}, \sigma, t)$ is skew symmetric[16]. Here $\dot{M}(q, \dot{q}, \sigma, t)$ is the time derivative of $M(q, \sigma, t)$.

Remark 3 To satisfy $V = C\dot{q}$, the matrix C may not be unique. But if you also want $\dot{M} - 2C$ to be skew symmetric, then the particular choice of C should be defined as

$$c_{ij} = \frac{1}{2} \sum_{k=1}^{n} \frac{\partial m_{ij}}{\partial q_k} \dot{q}_k + \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i} \right) \dot{q}_k, \tag{8}$$

where c_{ij} is the ij-element of matrix C.

3 Robust control design of fuzzy mechanical systems

We wish the mechanical system to follow a desired trajectory $q^d(t), t \in [t_0, t_1]$, with the desired velocity $\dot{q}^d(t)$. Assume $q^d(\cdot) : [t_0, \infty] \to \mathbb{R}^n$ is of class C^2 and $q^d(t), \dot{q}^d(t)$ and $\ddot{q}^d(t)$ are uniformly bounded. Let

$$e(t) := q(t) - q^d(t), \tag{9}$$

and hence $\dot{e}(t) = \dot{q}(t) - \dot{q}^d(t)$, $\ddot{e}(t) = \ddot{q}(t) - \ddot{q}^d(t)$. The system (1) can be rewritten as

$$M(e + q^d, \sigma, t)(\ddot{e} + \ddot{q}^d) + C(e + q^d, \dot{e} + \dot{q}^d, \sigma, t) \cdot$$
$$(\dot{e} + \dot{q}^d) + G(e + q^d, \sigma, t) +$$
$$T(e + q^d, \dot{e} + \dot{q}^d, \sigma, t) = \tau. \tag{10}$$

The functions $M(\cdot)$, $C(\cdot)$, $G(\cdot)$ and $T(\cdot)$ can be decomposed as

$$\begin{cases} M(e + q^d, \sigma, t) = \\ \bar{M}(e + q^d, t) + \Delta M(e + q^d, \sigma, t), \\ C(e + q^d, \dot{e} + \dot{q}^d, \sigma, t) = \\ \bar{C}(e + q^d, \dot{e} + \dot{q}^d, t) + \Delta C(e + q^d, \dot{e} + \dot{q}^d, \sigma, t), \\ G(e + q^d, \sigma, t) = \\ \bar{G}(e + q^d, t) + \Delta G(e + q^d, \sigma, t), \\ T(e + q^d, \dot{e} + \dot{q}^d, \sigma, t) = \\ \bar{T}(e + q^d, \dot{e} + \dot{q}^d, t) + \Delta T(e + q^d, \dot{e} + \dot{q}^d, \sigma, t), \end{cases} \tag{11}$$

where $\bar{M}, \bar{C}, \bar{G}$ and $\bar{T}$ are the nominal terms of corresponding matrix/vector and $\Delta M, \Delta C, \Delta G$, and ΔT are the uncertain terms which depend on σ. We now define a vector

$$\Phi(e, \dot{e}, \sigma, t) := -\Delta M(e + q^d, \sigma, t)(\ddot{q}^d - S\dot{e}) -$$
$$\Delta C(e + q^d, \dot{e} + \dot{q}^d, \sigma, t)(\dot{q}^d - Se) -$$
$$\Delta G(e + q^d, \sigma, t) -$$
$$\Delta T(e + q^d, \dot{e} + \dot{q}^d, \sigma, t), \tag{12}$$

where $S = \mathrm{diag}[s_i]_{n \times n}$, $s_i > 0$ is a constant, $i = 1, 2, \cdots, n$. Obviously $\Phi \equiv 0$ if all uncertain terms vanish.

Assumption 5 There are fuzzy numbers $\zeta_k(\dot{q}^d, \ddot{q}^d, e, \dot{e}, \sigma, t)$'s and scalars $\rho_k(\dot{q}^d, \ddot{q}^d, e, \dot{e}, \sigma, t)$'s, $k = 1, 2, \cdots, r$, such that

$$\|\Phi\| \leqslant \begin{bmatrix} \hat{\zeta}_1 & \hat{\zeta}_2 \cdots \hat{\zeta}_r \end{bmatrix} \begin{bmatrix} \hat{\rho}_1 & \hat{\rho}_2 \cdots \hat{\rho}_r \end{bmatrix}^{\mathrm{T}} =:$$
$$\hat{\zeta}^{\mathrm{T}}(e, \dot{e}, \sigma, t)\hat{\rho}(e, \dot{e}, t). \tag{13}$$

From Eq.(13), we have

$$\|\Phi\| \leqslant \|\hat{\zeta}\| \|\hat{\rho}\| =: \zeta\rho. \tag{14}$$

Remark 4 One can employ fuzzy arithmetic and decomposition theorem (see Appendix) to calculate the fuzzy number ζ based on the fuzzy description of σ_i's (Assumption 2).

We introduce the following desirable deterministic dynamical system performance.

Definition 1 Consider a dynamical system

$$\dot{\xi}(t) = f(\xi(t), t), \ \xi(t_0) = \xi_0. \tag{15}$$

The solution of the system (suppose it exists and can be continued over $[t_0, \infty)$) is uniformly bounded if for any $r > 0$ with $\|\xi_0\| \leqslant r$, there is $d(r) > 0$ with $\|\xi(t)\| \leqslant d(r)$ for all $t \geqslant t_0$. It is uniformly ultimately bounded if for $\|\xi_0\| \leqslant r$, there are $\bar{d}(r) > 0$ and $T(\bar{d}(r), r) \geqslant 0$ such that $\|\xi(t)\| \leqslant \bar{d}(r)$ for all $t \geqslant t_0 + T(\bar{d}(r), r)$.

Let

$$\underline{e}(t) := (e(t), \dot{e}(t))^{\mathrm{T}}. \tag{16}$$

The control design is to render the tracking error vector $\underline{e}(t)$ to be sufficiently small. We propose the control as

$$\tau(t) = \bar{M}(\ddot{q}^d - S\dot{e}) + \bar{C}(\dot{q}^d - Se) + \bar{G} + \bar{T} -$$
$$Pe - D\dot{e} - \gamma(\dot{e} + Se)\rho^2, \tag{17}$$

where P, D are positive definite diagonal matrices and the scalar $\bar{\gamma} := \gamma^r > 0$. The scalar γ is a constant design parameters.

Theorem 2 Subject to Assumptions 1–5, the control (17) renders $\underline{e}(t)$ of the system (10) to be uniformly bounded and uniformly ultimately bounded. In addition, the size of the ultimate boundedness ball can be made arbitrarily small by suitable choices of the design parameters.

Remark 5 The control $\tau(t)$ is based on the nominal system, the traking error $\underline{e}(t)$, the bound of uncertainty and the design parameters. Therefore, this proposed control is deterministic and is not if-then rules-based.

4 Proof of Theorem 2

The mechanical system with the proposed control is proved to be stable in this section. The chosen Lyapunov function candidate is shown to be legitimate and then the proof of stability follows via Lyapunov minimax approach[17–18].

The Lyapunov function candidate is chosen as

$$V(\underline{e}) = \frac{1}{2}(\dot{e} + Se)^{\mathrm{T}}M(\dot{e} + Se) +$$
$$\frac{1}{2}e^{\mathrm{T}}(P + SD)e. \quad (18)$$

To prove V is a legitimate Lyapunov function candidate, we shall prove that V is (globally) positive definite and decrescent.

By Eq.(4), we have

$$V(\underline{e}) \geqslant \frac{1}{2}\underline{\gamma}\|\dot{e} + Se\|^2 + \frac{1}{2}e^{\mathrm{T}}(P + SD)e =$$
$$\frac{1}{2}\underline{\gamma}\sum_{i=1}^{n}(\dot{e}_i^2 + 2s_i\dot{e}_ie_i + s_i^2e_i^2) + \frac{1}{2}\sum_{i=1}^{n}(p_i + s_id_i)e_i^2 =$$
$$\frac{1}{2}\sum_{i=1}^{n}[e_i \ \dot{e}_i]\Psi_i\begin{bmatrix} e_i \\ \dot{e}_i \end{bmatrix}, \quad (19)$$

where s_i, p_i's and d_i's are from Eqs.(12) and (17), e_i and $\dot{e}_i$ are the i-th components of e and $\dot{e}$, respectively, and

$$\Psi_i = \begin{bmatrix} \underline{\gamma}s_i^2 + p_{0_i} + s_id_{0_i} & \underline{\gamma}s_i \\ \underline{\gamma}s_i & \underline{\gamma} \end{bmatrix}. \quad (20)$$

It can be easily verified that $\Psi_i > 0$, $\forall i$. Thus by letting $\underline{\lambda} = \min(\frac{1}{2}\lambda_m(\Psi_1), \cdots, \frac{1}{2}\lambda_m(\Psi_n))$ (hence $\underline{\lambda} > 0$), V is shown to be positive definite

$$V \geqslant \frac{1}{2}\sum_{i=1}^{n}\lambda_m(\Psi_i)(e_i^2 + \dot{e}_i^2) \geqslant \underline{\lambda}\|\underline{e}\|^2. \quad (21)$$

By Assumption 4, we have

$$V \leqslant \|\dot{e} + Se\|^2\bar{\gamma} + e^{\mathrm{T}}(P + SD)e. \quad (22)$$

For the first term on the right-hand side,

$$\bar{\gamma}\|\dot{e} + Se\|^2 = \bar{\gamma}(\dot{e} + Se)^{\mathrm{T}}(\dot{e} + Se) =$$
$$\bar{\gamma}\begin{bmatrix} e & \dot{e} \end{bmatrix}\begin{bmatrix} S^2 & S \\ S & I \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} \leqslant$$
$$\bar{\gamma}\lambda_M\begin{bmatrix} S^2 & S \\ S & I \end{bmatrix}\|\underline{e}\|^2 =: \bar{\gamma}\bar{s}\|\underline{e}\|^2. \quad (23)$$

For the second term on the right-hand side, by Rayleigh's principle,

$$e^{\mathrm{T}}(P + SD)e \leqslant \lambda_M(P + SD)\|e\|^2. \quad (24)$$

With Inequalities (23) and (24) into Inequality (22), we have

$$V \leqslant \bar{\gamma}\bar{s}\|\underline{e}\|^2 + \lambda_M(P + SD)\|e\|^2 =: \bar{\lambda}\|\underline{e}\|^2, \quad (25)$$

where $\bar{\lambda} = \bar{\gamma}\bar{s} + \lambda_M(P + SD)$. Note that $\bar{\lambda}$ in Inequality (25) is a strictly positive constant, which implies that V is decrescent. From Inequalities (21) and (25), V is a legitimate Lyapunov function candidate.

Now, we prove the stability of the mechanical system with the proposed control. For any admissible $\xi(\cdot)$, the time derivative of V along the trajectory of the controlled mechanical system of Eq.(10) is given by

$$\dot{V} = (\dot{e} + Se)^{\mathrm{T}}M(\ddot{e} + S\dot{e}) + \frac{1}{2}(\dot{e} + Se)^{\mathrm{T}}\dot{M}(\dot{e} +$$

$$Se) + e^{\mathrm{T}}(P + SD)\dot{e}, \quad (26)$$

by applying $\ddot{e} = \ddot{q} - \ddot{q}^{\mathrm{d}}$ and Eq.(1), the first two terms become

$$(\dot{e}+Se)^{\mathrm{T}}M(\ddot{e}+S\dot{e}) + \frac{1}{2}(\dot{e}+Se)^{\mathrm{T}}\dot{M}(\dot{e}+Se) =$$
$$(\dot{e}+Se)^{\mathrm{T}}(M\ddot{q} - M\ddot{q}^{\mathrm{d}} + MS\dot{e} + \frac{1}{2}\dot{M}(\dot{e}+Se)) =$$
$$(\dot{e}+Se)^{\mathrm{T}}(\tau - C(\dot{e}+\dot{q}^{\mathrm{d}}) - G - T -$$
$$M\ddot{q}^{\mathrm{d}} + MS\dot{e} + 1/2\dot{M}(\dot{e}+Se)) =$$
$$(\dot{e}+Se)^{\mathrm{T}}(\tau - C(\dot{q}^{\mathrm{d}} - Se) - G - T -$$
$$M\ddot{q}^{\mathrm{d}} + MS\dot{e} - C(\dot{e}+Se) + \frac{1}{2}\dot{M}(\dot{e}+Se)) =$$
$$(\dot{e}+Se)^{\mathrm{T}}(\tau - C(\dot{q}^{\mathrm{d}} - Se) - G - T - M\ddot{q}^{\mathrm{d}} +$$
$$MS\dot{e}) + (\dot{e}+Se)^{\mathrm{T}}(\frac{1}{2}\dot{M} - C)(\dot{e}+Se). \quad (27)$$

With Theorem 1, Eqs.(11) and (17), we can get

$$(\dot{e}+Se)^{\mathrm{T}}M(\ddot{e}+S\dot{e}) + \frac{1}{2}(\dot{e}+Se)^{\mathrm{T}}\dot{M}(\dot{e}+Se) =$$
$$(\dot{e}+Se)^{\mathrm{T}}(\tau - C(\dot{q}^{\mathrm{d}} - Se) - G - T - M\ddot{q}^{\mathrm{d}} + MS\dot{e}) =$$
$$(\dot{e}+Se)^{\mathrm{T}}\{\bar{M}(\ddot{q}^{\mathrm{d}} - S\dot{e}) - M(\ddot{q}^{\mathrm{d}} - S\dot{e}) + \bar{C}(\dot{q}^{\mathrm{d}} -$$
$$Se) - C(\dot{q}^{\mathrm{d}} - Se) + \bar{G} - G + \bar{T} - T -$$
$$\gamma\rho^2(\dot{e}+Se) - Pe - D\dot{e}\} =$$
$$(\dot{e}+Se)^{\mathrm{T}}\{-\Delta M(\ddot{q}^{\mathrm{d}} - S\dot{e}) - \Delta C(\dot{q}^{\mathrm{d}} - Se) -$$
$$\Delta G - \Delta T - \gamma\rho^2(\dot{e}+Se) - Pe - D\dot{e}\}. \quad (28)$$

By Eqs.(12)–(14),

$$(\dot{e}+Se)^{\mathrm{T}}M(\ddot{e}+S\dot{e}) + \frac{1}{2}(\dot{e}+Se)^{\mathrm{T}}\dot{M}(\dot{e}+Se) =$$
$$(\dot{e}+Se)^{\mathrm{T}}[\Phi - \gamma\rho^2(\dot{e}+Se) - Pe - D\dot{e}] \leqslant$$
$$\|\dot{e}+Se\|\|\Phi\| - \gamma\rho^2\|\dot{e}+Se\|^2 -$$
$$(\dot{e}+Se)^{\mathrm{T}}(Pe + D\dot{e}) \leqslant$$
$$\zeta\rho\|\dot{e}+Se\| - \gamma\rho^2\|\dot{e}+Se\|^2 -$$
$$(\dot{e}+Se)^{\mathrm{T}}(Pe + D\dot{e}) \leqslant$$
$$\frac{\zeta^2}{4\gamma} - (\dot{e}+Se)^{\mathrm{T}}(Pe + D\dot{e}). \quad (29)$$

Since

$$-(\dot{e}+Se)^{\mathrm{T}}(Pe + D\dot{e}) =$$
$$-e^{\mathrm{T}}PSe - \dot{e}^{\mathrm{T}}D\dot{e} - e^{\mathrm{T}}(P + SD)\dot{e}, \quad (30)$$

we have

$$(\dot{e}+Se)^{\mathrm{T}}M(\ddot{e}+S\dot{e}) + \frac{1}{2}(\dot{e}+Se)^{\mathrm{T}}\dot{M}(\dot{e}+Se) \leqslant$$
$$\frac{\zeta^2}{4\gamma} - e^{\mathrm{T}}PSe - \dot{e}^{\mathrm{T}}D\dot{e} - e^{\mathrm{T}}(P + SD)\dot{e}. \quad (31)$$

Substituting Inequality (31) into Eq.(26), we get

$$\dot{V} \leqslant \frac{\delta}{\gamma} - e^{\mathrm{T}}PSe - \dot{e}^{\mathrm{T}}D\dot{e} -$$
$$e^{\mathrm{T}}(P + SD)\dot{e} + e^{\mathrm{T}}(P + SD)\dot{e} =$$
$$\frac{\zeta^2}{4\gamma} - e^{\mathrm{T}}PSe - \dot{e}^{\mathrm{T}}D\dot{e} \leqslant$$

$$\frac{\zeta^2}{4\gamma} - \lambda \|\underline{e}\|^2, \tag{32}$$

where $\lambda = \min\{\lambda_{\min}(PS),\ \lambda_{\min}(D)\}$. This in turn means that $\dot{V}$ is negative definite for all $\|\underline{e}\|$ such that

$$\frac{\delta}{\gamma} - \lambda \|\underline{e}\|^2 < 0, \tag{33}$$

where $\delta = \dfrac{\zeta^2}{4}$. Since all universes of discourse Σ_i's are compact (hence closed and bounded), δ is bounded. In addition, both γ and λ are crisp. Thus, $\dot{V}$ is negative definite for sufficiently large $\|\underline{e}\|$. The uniform boundedness performance follows [3]. That is, given any $r > 0$ with $\|\underline{e}(t_0)\| \leqslant r$, where t_0 is the initial time, there is a $d(r)$ given by

$$d(r) = \begin{cases} r[\bar{\lambda}/\underline{\lambda}]^{\frac{1}{2}}, & r > R, \\ R[\bar{\lambda}/\underline{\lambda}]^{\frac{1}{2}}, & r \leqslant R, \end{cases} \tag{34}$$

$$R = [\frac{\delta}{\lambda r}]^{\frac{1}{2}}, \tag{35}$$

such that $\|\underline{e}(t)\| \leqslant d(r)$ for all $t \geqslant t_0$. Uniform ultimate boundedness also follows. That is, given any $\bar{d}$ with

$$\bar{d} > R\left[\bar{\lambda}/\underline{\lambda}\right]^{\frac{1}{2}}, \tag{36}$$

we have $\|\underline{e}(t)\| \leqslant \bar{d},\ \forall t \geqslant t_0 + T(\bar{d}, r)$, with

$$T(\bar{d}, r) = \begin{cases} 0, & r \leqslant \bar{R}, \\ \dfrac{\bar{\lambda} r^2 - \underline{\lambda} \bar{R}^2}{\lambda \bar{R}^2 - \delta/\gamma}, & \text{otherwise}, \end{cases} \tag{37}$$

$$\bar{R} = \bar{d}\left[\underline{\lambda}/\bar{\lambda}\right]^{\frac{1}{2}}. \tag{38}$$

The stability of the mechanical system is guaranteed and tracking error $\|\underline{e}\|$ can be made arbitrarily small by choosing large λ and/or γ.

Remark 6   We have shown that fundamental properties explored in Section 3 are quite useful in constructing the legitimate Lyapunov function. The first five terms of the control scheme (17) are only for the nominal system (i.e. the system without uncertainty) while the last term is to compensate the uncertainty. For the last term, the magnitude γ is still free for we still have freedom on designing γ to determine the size of the ultimate boundedness. The larger the value of γ, the smaller the size. This stands for a trade-off between the system performance and the cost which suggests an interesting optimal quest for the control design. We will pursue the optimal design in the following section.

5   Optimal gain design

Sections 3 and 4 show that a system performance can be guaranteed by a deterministic control scheme. By the analysis, the size of the uniform ultimate boundedness region decreases as γ increases. As γ approaches to infinity, the size approaches to 0. This rather strong performance is accompanied by a (possibly) large control effort, which is reflected by γ (as-

suming r has been chosen). From the practical design point of view, the designer may be interested in seeking an optimal choice of γ for a compromise among various conflicting criteria. This is associated with the minimization of a performance index.

We first explore more on the deterministic performance of the uncertain mechanical system. Define

$$\kappa = \frac{\bar{\lambda}}{\lambda}, \tag{39}$$

where $\bar{\lambda}$ is from Inequality (25), λ is from Inequality (32) and $\kappa > 0$. Then by Inequalities (25) and (32), we get

$$\dot{V} \leqslant \frac{\delta}{\gamma} - \lambda \|\underline{e}\|^2 = \frac{\delta}{\gamma} - \frac{1}{\kappa} \bar{\lambda} \|\underline{e}\|^2 \leqslant \frac{\delta}{\gamma} - \frac{1}{\kappa} V, \tag{40}$$

with $V_0 = V(t_0) = V(\underline{e}(t_0))$. This is a differential inequality[19] whose analysis can be made according to [20] (see Appendix: analysis of differential inequality).

Therefore

$$V(t) \leqslant r(t), \tag{41}$$

or

$$V(t) \leqslant (V_0 - \frac{\kappa\delta}{\gamma}) \exp[-\frac{1}{\kappa}(t - t_0)] + \frac{\kappa\delta}{\gamma}, \tag{42}$$

for all $t \geqslant t_0$. By the same argument, we also have, for any t_s and any $\tau \geqslant t_s$,

$$V(\tau) \leqslant (V_s - \frac{\kappa\delta}{\gamma}) \exp[-\frac{1}{\kappa}(\tau - t_s)] + \frac{\kappa\delta}{\gamma}, \tag{43}$$

where $V_s = V(t_s) = V(\underline{e}(t_s))$. The time t_s is when the control scheme (17) starts to be executed. It does not need to be t_0.

By Inequality (21), $V(\underline{e}) \geqslant \underline{\lambda} \|\underline{e}\|^2$, the right-hand side of (43) provides an upper bound of $\underline{\lambda} \|\underline{e}\|^2$. This in turn leads to an upper bound of $\|\underline{e}\|^2$. For each $\tau \geqslant t_s$, let

$$\eta(\delta, \gamma, t, t_s) := (V_s - \frac{\kappa\delta}{\gamma}) \exp[-\frac{1}{\kappa}(\tau - t_s)], \tag{44}$$

$$\eta_\infty(\delta, \gamma) := \frac{\kappa\delta}{\gamma}. \tag{45}$$

Notice that for each $\delta, \gamma, t_s,\ \eta(\delta, \gamma, \tau, t_s) \to 0$ as $\tau \to \infty$.

One may relate $\eta(\delta, \gamma, t, t_0)$ to the transient performance and $\eta_\infty(\delta, \gamma)$ the steady state performance. Since there is no knowledge of the exact value of uncertainty, it is only realistic to refer to $\eta(\delta, \gamma, t, t_0)$ and $\eta_\infty(\delta, \gamma)$ while analyzing the system performance. We also notice that both $\eta(\delta, \gamma, t, t_0)$ and $\eta_\infty(\delta, \gamma)$ are dependent on δ. The value of δ is not known except that it is characterized by a membership function.

We now propose the following performance index: for any t_s, let

$$J(\gamma, t_s) := D[\int_{t_s}^{\infty} \eta^2(\delta, \gamma, \tau, t_s) \mathrm{d}\tau] +$$
$$\alpha D[\eta_\infty^2(\delta, \gamma)] + \beta\gamma^2 =:$$

$$J_1(\gamma, t_s) + \alpha J_2(\gamma) + \beta J_3(\gamma), \quad (46)$$

where $\alpha, \beta > 0$ are scalars. The performance index consists of three parts. The first part $J_1(\gamma, t_s)$ may be interpreted as the average (via the D-operation) of the overall transient performance (via the integration) from time t_s. The second part $J_2(\gamma)$ may be interpreted as the average (via the D-operation) of the steady state performance. The third part $J_3(\gamma)$ is due to the control cost. Both α and β are weighting factors. The weighting of J_1 is normalized to be unity. Our optimal design problem is to choose $\gamma > 0$ such that the performance index $J(\gamma, t_s)$ is minimized.

Remark 7   A standard linear-quadratic-Gaussian (LQG) (i.e., linear-quadratic-Gaussian) problem in stochastic control is to minimize a performance index, which is the average (via the expectation value operation in probability) of the overall state and control accumulation. The proposed new control design approach may be viewed, loosely speaking, as a parallel problem, though not equivalent, in fuzzy mechanical systems. However, one cannot be too careful in distinguishing the differences. For example, the Gaussian probability distribution implies that the uncertainty is unbounded (although a higher bound is predicted by a lower probability). In the current consideration, the uncertainty bound is always finite. Also, LQG does not take parameter uncertainty into account.

One can show that

$$\int_{t_s}^{\infty} \eta^2(\delta, \gamma, \tau, t_s) \mathrm{d}\tau =$$

$$(V_s - \frac{\kappa\delta}{\gamma})^2 \int_{t_s}^{\infty} \exp[-\frac{2}{\kappa}(\tau - t_s)]\mathrm{d}\tau =$$

$$(V_s - \frac{\kappa\delta}{\gamma})^2 (-\frac{\kappa}{2}) \exp[-\frac{2}{\kappa}(\tau - t_s)]\mathrm{d}\tau|_{t_s}^{\infty} =$$

$$(V_s - \frac{\kappa\delta}{\gamma})^2 \frac{\kappa}{2}. \quad (47)$$

Taking the D-operation yields

$$D[\int_{t_s}^{\infty} \eta^2(\delta, \gamma, \tau, t_s)\mathrm{d}\tau] =$$

$$D[(V_s - \frac{\kappa\delta}{\gamma})^2 \frac{\kappa}{2}] =$$

$$D[(V_s^2 - 2V_s\frac{\kappa\delta}{\gamma} + (\frac{\kappa\delta}{\gamma})^2)\frac{\kappa}{2}] =$$

$$(D[V_s^2] - 2\frac{\kappa}{\gamma}D[V_s\delta] + \frac{\kappa^2}{\gamma^2}D[\delta^2])\frac{\kappa}{2}. \quad (48)$$

The last equality is due to Lemma 1 in Appendix D-operation. Next, we analyze the cost $J_2(\gamma)$. Again by Lemma 1, we have

$$D[\eta_\infty^2(\delta, \gamma)] = D[(\frac{\kappa\delta}{\gamma})^2] = \frac{\kappa^2}{\gamma^2}D[\delta^2].$$
$$(49)$$

With Eqs.(48) and (49) into Eq.(46), we also have

$$J(\gamma, t_s) := (D[V_s^2] - 2\frac{\kappa}{\gamma}D[V_s\delta] +$$

$$\frac{\kappa^2}{\gamma^2}D[\delta^2])\frac{\kappa}{2} + \alpha\frac{\kappa^2}{\gamma^2}D[\delta^2] + \beta\gamma^2 =:$$

$$\kappa_1 - \frac{\kappa_2}{\gamma} + \frac{\kappa_3}{\gamma^2} + \alpha\frac{\kappa_4}{\gamma^2} + \beta\gamma^2, \quad (50)$$

where $\kappa_1 = (\kappa/2)D[V_s^2]$, $\kappa_2 = \kappa^2 D[V_s\delta]$, $\kappa_3 = (\kappa^3/2)D[\delta^2]$, $\kappa_4 = \kappa^2 D[\delta^2]$.

The optimal design problem is then equivalent to the following constrained optimization problem: for any t_s,

$$\min_{\gamma} J(\gamma, t_s) \text{ s.t. } \gamma > 0. \quad (51)$$

By using the performance index in Eq. (50), we then pursue the optimal solution (see Appendix: the closed-form solution of optimal gain) .

By using Eq. (A18), the cost J in Eq. (50) can be rewritten as

$$J = \kappa_1 - \frac{\kappa_2}{\gamma} + \frac{\kappa_3}{\gamma^2} + \alpha\frac{\kappa_4}{\gamma^2} + \beta\gamma^2 =$$

$$\kappa_1 - \frac{1}{\gamma^2}(\kappa_2\gamma + 2\beta\gamma^4) + \frac{\kappa_3}{\gamma^2} + \alpha\frac{\kappa_4}{\gamma^2} + 3\beta\gamma^2 =$$

$$\kappa_1 - \frac{1}{\gamma^2}[2(\kappa_3 + \alpha\kappa_4)] + \frac{\kappa_3}{\gamma^2} + \alpha\frac{\kappa_4}{\gamma^2} + 3\beta\gamma^2 =$$

$$\kappa_1 - \frac{\kappa_3}{\gamma^2} - \alpha\frac{\kappa_4}{\gamma^2} + 3\beta\gamma^2 =$$

$$\kappa_1 - \frac{1}{\gamma^2}(\kappa_3 + \alpha\kappa_4 - 3\beta\gamma^4). \quad (52)$$

With Eq.(A29), the minimum cost is given by

$$J_{\min} = \kappa_1 - \frac{4}{(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3})^2} \times$$

$$(\kappa_3 + \alpha\kappa_4 - \frac{3}{16}\beta(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3})^4).$$
$$(53)$$

Remark 8   Combining the results of Sections 4–7, the robust control scheme (17) using the optimal design of $\gamma > 0$ renders the tracking error $\underline{e}$ of the closed-loop mechanical system uniformly ultimately bounded (with the initial state $\underline{e}(t_s)$). In addition, the performance index J in Eq.(50) is globally minimized.

The optimal design procedure is summarized as follows:

Step 1   For a given inertia matrix M, obtain $\underline{\gamma}$ and $\bar{\gamma}$.

Step 2   According to $\|\varPhi\|$ in Eq.(12), obtain ζ and ρ in Eq.(14).

Step 3   Based on the $V(\underline{e})$ in Eq.(18), solve for the $\bar{\lambda}$ in Eq.(25). For given S, P's, and D's, solve for the λ in Inequality (32). Thus, κ is given in Eq.(39).

Step 4   Using the ζ obtained in Step 2 and the V_s in Inequality (43), calculate $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ in Eq.(50) based on the D-operation.

Step 5   For given α and β, solve for the γ_{opt} in Eq.(A29) and the minimum cost given in Eq.(53).

Step 6   The optimal robust control scheme is given

in Eq.(17).

6 Illustrative example

Consider a vehicle with an inverted pendulum hinged to the center as shown in Figure 1. Assume that there is no friction between the vehicle and the ground. The vehicle's mass is M (uncertain) and an external force F (the control) is imposed. The mass of the inverted pendulum is m (uncertain) and the length is l. An external torque τ (the control) is imposed on the pendulum.
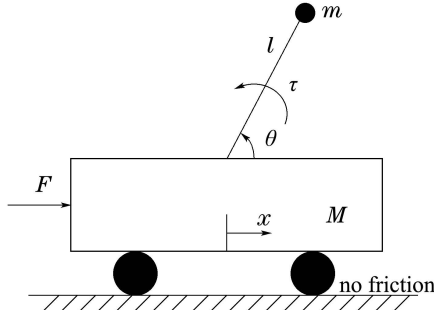

Fig. 1 Vehicle with an inverted pendulum

We choose two generalized coordinates $q := [q_1 \ q_2]^{\mathrm{T}} = [x \ \theta]^{\mathrm{T}}$ to describe the mechanical system, where x denotes the displacement of the vehicle and θ denotes the rotatory angle of the pendulum. The two coordinates are independent of each other. The kinetic energy of the mechanical system is

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{x}\dot{\theta}\sin\theta. \quad (54)$$

The potential energy is

$$V = mgl\sin\theta, \quad (55)$$

where g is gravitational acceleration. Then the Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = u \quad (56)$$

can be written out where the Lagrangian $L = T - V$, u is the external control force. The equation of motion can be written in matrix form from using Lagrange's equation as

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u, \quad (57)$$

where

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}, \ \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}, \ \ddot{q} = \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}, \ u = \begin{bmatrix} F \\ \tau \end{bmatrix},$$

$$H(q) = \begin{bmatrix} M+m & -ml\sin\theta \\ -ml\sin\theta & ml^2 \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -ml\dot{\theta}\cos\theta \\ 0 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} 0 \\ mgl\cos\theta \end{bmatrix}. \quad (58)$$

The desired trajectory $q^{\mathrm{d}}(t)$, the desired velocity and acceleration $\dot{q}^{\mathrm{d}}(t)$, $\ddot{q}^{\mathrm{d}}(t)$ are given by

$$\begin{cases} q^{\mathrm{d}}(t) = \begin{bmatrix} x^{\mathrm{d}} \\ \theta^{\mathrm{d}} \end{bmatrix} = \begin{bmatrix} \sin t \\ 1.5 - \cos t \end{bmatrix}, \\ \dot{q}^{\mathrm{d}}(t) = \begin{bmatrix} \dot{x}^{\mathrm{d}} \\ \dot{\theta}^{\mathrm{d}} \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \\ \ddot{q}^{\mathrm{d}}(t) = \begin{bmatrix} \ddot{x}^{\mathrm{d}} \\ \ddot{\theta}^{\mathrm{d}} \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}. \end{cases} \quad (59)$$

By using $q = e + q^{\mathrm{d}}$, $\dot{q} = \dot{e} + \dot{q}^{\mathrm{d}}$, $\ddot{q} = \ddot{e} + \ddot{q}^{\mathrm{d}}$, Eq.(57) can be rewritten as

$$H(e+q^{\mathrm{d}})\ddot{e} + H(e+q^{\mathrm{d}})\ddot{q}^{\mathrm{d}} + C(e+q^{\mathrm{d}}, \dot{e}+ \dot{q}^{\mathrm{d}})(\dot{e}+\dot{q}^{\mathrm{d}}) + G(e+q^{\mathrm{d}}) = u, \quad (60)$$

where

$$H(e+q^{\mathrm{d}}) = \begin{bmatrix} M+m \\ -ml\sin(e_2+1.5-\cos t) \end{bmatrix}$$
$$\begin{bmatrix} -ml\sin(e_2+1.5-\cos t) \\ ml^2 \end{bmatrix},$$

$$C(e+q^{\mathrm{d}}, \dot{e}+\dot{q}^{\mathrm{d}}) =$$
$$\begin{bmatrix} 0 & -ml(\dot{e}_2+\sin t)\cos(e_2+1.5-\cos t) \\ 0 & 0 \end{bmatrix},$$

$$G(e+q^{\mathrm{d}}) = \begin{bmatrix} 0 \\ mgl\cos(e_2+1.5-\cos t) \end{bmatrix}. \quad (61)$$

The masses M, m are uncertain with $M = \bar{M} + \Delta M(t)$, $m = \bar{m} + \Delta m(t)$, where $\bar{M}$, $\bar{m}$ are the constant nominal values and ΔM, Δm are the uncertainty. So, nominal matrices are given by

$$\hat{H} = \begin{bmatrix} \hat{M}+\hat{m} \\ -\hat{m}l\sin(e_2+1.5-\cos t) \end{bmatrix}$$
$$\begin{bmatrix} -\hat{m}l\sin(e_2+1.5-\cos t) \\ \hat{m}l^2 \end{bmatrix},$$

$$\hat{C} = \begin{bmatrix} 0 & -\hat{m}l(\dot{e}_2+\sin t)\cos(e_2+1.5-\cos t) \\ 0 & 0 \end{bmatrix},$$

$$\hat{G} = \begin{bmatrix} 0 \\ \hat{m}gl\cos(e_2+1.5-\cos t) \end{bmatrix}. \quad (62)$$

The uncertainty matrices are given by

$$\Delta H = \begin{bmatrix} \Delta M+\Delta m \\ -\Delta ml\sin(e_2+1.5-\cos t) \end{bmatrix}$$
$$\begin{bmatrix} -\Delta ml\sin(e_2+1.5-\cos t) \\ \Delta ml^2 \end{bmatrix},$$

$$\Delta C = \begin{bmatrix} 0 & -\Delta ml(\dot{e}_2+\sin t)\cos(e_2+1.5-\cos t) \\ 0 & 0 \end{bmatrix},$$

$$\Delta G = \begin{bmatrix} 0 \\ \Delta mgl\cos(e_2+1.5-\cos t) \end{bmatrix}. \quad (63)$$

We choose S to be a 2×2 identity matrix. Therefore, we can get

$$\Phi := -\Delta M(\ddot{q}^{\mathrm{d}} - S\dot{e}) - \Delta C(\dot{q}^{\mathrm{d}} - Se) - \Delta G, \quad (64)$$

$$\|\Phi\| \leqslant \hat{\zeta}^{\mathrm{T}}(e,\dot{e},\sigma,t)\hat{\rho}(e,\dot{e},t) \leqslant \|\hat{\zeta}\|\|\hat{\rho}\| = \zeta\rho, \quad (65)$$

where

$$\hat{\zeta} = \begin{bmatrix} \|\Delta M\| \\ \|\Delta m\| \end{bmatrix}, \ \hat{\rho} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}, \quad (66)$$

and

$$\rho_1 = \|\dot{e}_1 + \sin t\|,$$

$$\begin{aligned}
\rho_2 = {} & \|\dot{e}_1 + \sin t\| + \|l\sin(e_2 + 1.5 - \cos t)(\dot{e}_2 - \\
& \cos t)\| + \|l(\dot{e}_2 + \sin t)\cos(e_2 + 1.5 - \\
& \cos t)(\dot{e}_2 - \sin t)\| + \|l\sin(e_2 + 1.5 - \\
& \cos t)(\dot{e}_1 + \sin t)\| + \|l^2(\dot{e}_2 - \cos t)\| + \\
& \|gl\cos(e_2 + 1.5 - \cos t)\|.
\end{aligned} \tag{67}$$

For the mechanical system, we choose $\bar{M} = 1$, $\bar{m} = 1$ and assume the uncertainties ΔM, Δm are all 'close to 0' and governed by the membership function

$$\mu_{\Delta M, \Delta m} = \begin{cases} 1 + 10\nu, & -0.1 \leqslant \nu \leqslant 0, \\ 1 - 10\nu, & 0 \leqslant \nu \leqslant 0.1. \end{cases} \tag{68}$$

Set the design parameters P, and D to be the identity matrix $I_{2\times 2}$. Follow the design procedure, we have $\bar{\lambda} = 24.33$, $\lambda = 1$ and $\kappa = 24.33$. By using the fuzzy arithmetic and decomposition theorem, we obtain $\kappa_1 = 5761.30$, $\kappa_2 = 48.56$, $\kappa_3 = 0.24$, $\kappa_4 = 0.02$. By selecting five sets of weighting α and β, the optimal gain γ_{opt} and the corresponding minimum cost $J_{\min}$ are summarized in Table 1.

Table 1 Weighting/optimal gain/minimum cost

(α, β)	α/β	γ_{opt}	$J_{\min}$
(1, 1)	1	3.3097	5793
(1, 10)	0.1	1.5386	5832
(1, 100)	0.01	0.7165	5915
(10, 1)	10	3.3128	5794
(100, 1)	100	3.3428	5795

We choose $t_s = 0$ and the initial condition $\underline{e}(0) = [1\ 0\ 0\ 0.1]^{\text{T}}$. For numerical simulation, we choose the uncertainties as $\Delta M = 0.1$, $\Delta m = 0.1\sin(10t)$. Two different classes of uncertainties (i.e., constant, high frequency) are used to test the proposed control scheme. Simulation results are as follows.

Figure 2 shows comparison of the tracking error norm $\|\underline{e}\|$ trajectory with the proposed control (under $\gamma_{\text{opt}} = 3.3097$ when $\alpha = \beta = 1$), without any control and with the nominal PD control (without the part of control that governs the uncertainty i.e., $\gamma = 0$). The trajectory $\|\underline{e}\|$ with the proposed control enters a much smaller region around 0 after some time (hence ultimately bounded) than the trajectory only with the nominal PD control. The uncontrolled trajectory $\|\underline{e}\|$ moves far away from 0.

Figure 3 shows the corresponding trajectories of the proposed control $\tau = [u_1\ u_2]^{\text{T}}$.

Figure 4 shows the tracking error norm trajectories for all five γ_{opt}'s (by using different (α, β) combinations). The use of $\gamma = 0$ (not optimal) is also shown for a comparison.

Figure 5 shows the corresponding histories of control efforts $\|\tau(t)\|$.
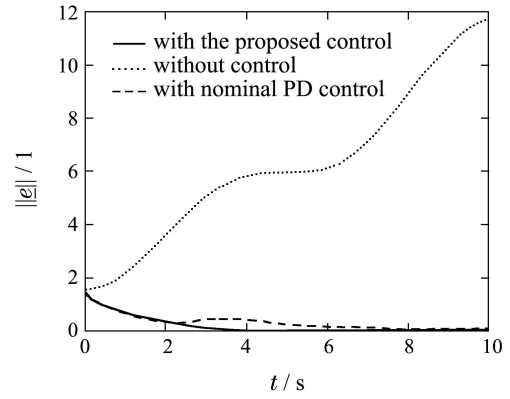

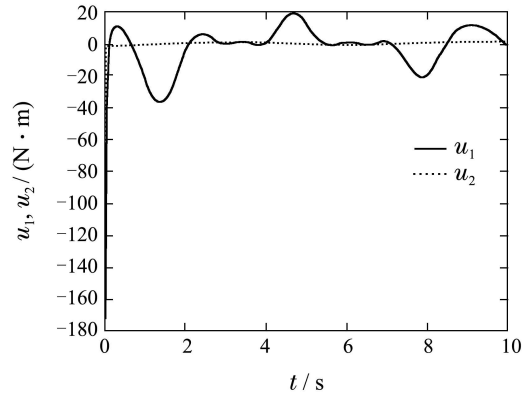Fig. 2 Comparison of fuzzy mechanical system performances
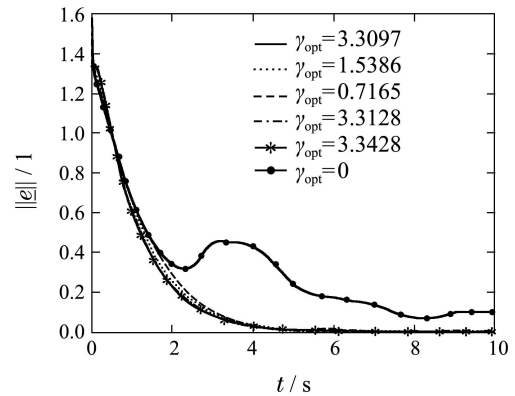

Fig. 3 Control histories

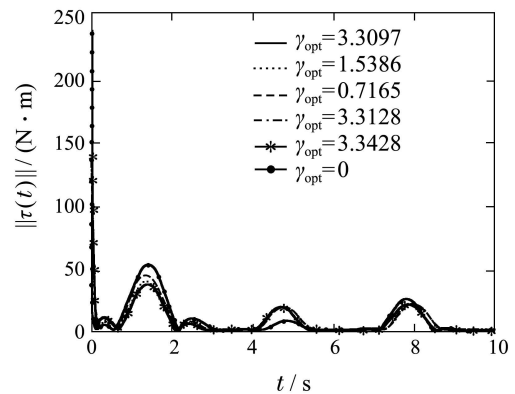Fig. 4 Comparison of system performances under different γ_{opt}

Fig. 5 Comparison of control efforts under different γ_{opt}

In order to show the results explicitly, we list the area S between the tracking error norm trajectory (under different γ_{opt}) and the coordinate axes, and also the area R between the control effort trajectory (under different γ_{opt}) and the coordinate axes in Table 2.

Table 2 S and R under different γ_{opt}

γ_{opt}	3.3097	1.5386	0.7165	3.3128	3.3428	0
S	1.6502	1.7140	1.7852	1.6488	1.6471	3.2715
R	80.8343	81.5461	84.2688	80.8033	80.7629	90.3669

7   Conclusions

Fuzzy description of uncertainties in mechanical systems is employed and we incorporate fuzzy uncertainty and fuzzy performance into the control design. A new robust control scheme is proposed to guarantee the deterministic performance (including uniform boundedness and uniform ultimately boundedness). The control is deterministic and is not if-then rules-based. The resulting controlled system is stable proved via the Lyapunov minimax approach[21]. A performance index is proposed and by minimizing the performance index, the optimal design problem associated with the control can be solved. The solution of the optimal gain is unique and closed-form. The resulting control design is systematic and is able to guarantee the deterministic performance as well as minimizing the cost.

References:

[1] LI X P, CHANG B C, BANDA S S. Robust control system design using H_∞ optimization theory [J]. *Journal of Guidance, Control and Dynamics*, 1992, 15(4): 944 – 952.

[2] SHEN T, TAMURA K. Robust H_∞ control of uncertain nonlinear system via state feedback [J]. *IEEE Transactions on Automatic Control*, 1995, 40(4): 766 – 768.

[3] CHEN Y H, LEITMANN G. Robustness of uncertain systems in the absence of matching assumptions [J]. *International Journal of Control*, 1987, 45(5): 1527 – 1542.

[4] SCHOENWALD D A, OZGUNNER I. Robust stabilization of nonlinear systems with Parametric uncertainty [J]. *IEEE Transactions on Automatic Control*, 1994, 39(8): 1751 – 1755.

[5] UTKIN V. Sliding mode control in mechanical systems [C] //*20th International Conference on Industrial Electronics, Control and Instrumentation*, 1994, 3(2): 1429 – 1431.

[6] STENGEL R F. *Optimal Control and Estimation* [M]. New York: Dover Publications, 1994.

[7] KALMAN R E. A new approach to linear filtering and prediction problems [J]. *ASME Transactions on Journal of Basic Engineering*, 1960, 82(1): 35 – 45.

[8] KALMAN R E. Contributions to the theory of optimal control [J]. *Boletin de la Sociedad Mastematica Mexicana*, 1960, 5(1): 102 – 119.

[9] KALMAN R E. Randomness reexamined [J]. *Modeling, Identification and Control*, 1994, 15(3): 141 – 151.

[10] SOONG T T. *Active Structural Control: Theory and Practice* [M]. England: Longman Scientific and Technical, Essex, 1990.

[11] BEZDEK J C. Special issue on fuzziness vs. probability–the N-th round [J]. *IEEE Transactions on Fuzzy Systems*, 1994, 2(1): 1 – 42.

[12] ZADEH L A. Fuzzy sets [J]. *Information and Control*, 1965, 8(3): 338 – 353.

[13] HANSS M. *Applied Fuzzy Arithmetic: an Introduction with Engineering Applications* [M]. Berlin: Springer-Verlag, 2005.

[14] BEDE B, RUDAS I J, FODOR J. Friction model by using fuzzy differential equations [M] //*Foundations of Fuzzy Logic and Soft Computing*. Berlin: Springer-Verlag, 2007: 338 – 353.

[15] MCKERROW P J. *Introduction to Robotics* [M]. Sydney: Addison-Wesley, 1991.

[16] SLOTINE J E, LI W. *Applied Nonlinear Control* [M]. New Jersey: Prentice Hall, Englewood Cliffs, 1991.

[17] CORLESS M. Control of uncertain nonlinear systems [J]. *Journal of Dynamic Systems, Measurement and Control*, 1993, 115(2B): 362 – 372.

[18] LEITMANN G. On one approach to the control of uncertain systems [J]. *Journal of Dynamic Systems, Measurement and Control*, 1993, 115(2B): 373 – 380.

[19] HALE J K. *Ordinary Differential Equations* [M]. New York: Wiley-Interscience, 1969.

[20] CHEN Y H. Performance analysis of controlled uncertain systems [J]. *Dynamics and Control*, 1996, 6(2): 131 – 142.

[21] ZHEN S C, ZHAO H, HUANG K, et al. A new Lyapunov based robust control for uncertain mechanical systems [J]. *Acta Automatica Sinica*, 2014, 40(5): 875 – 882.

[22] KLIR G J, YUAN B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications* [M]. New Jersey: Pretice-Hall, Englewood Cliffs, 1995.

[23] CHEN Y H. Fuzzy dynamical system approach to the observer design of uncertain systems [J]. *Journal of System and Control Engineering*, 2010, 224(5): 509 – 520.

[24] HUANG J. Robust control for fuzzy dynamical systems: uniform ultimate boundedness and optimality [J]. *IEEE Transactions on Fuzzy Systems*, 2012, 20(6): 1022 – 1031.

[25] CHEN Y H. Optimal design of robust control for uncertain systems: a fuzzy approach [J]. *Nonlinear Dynamics and Systems Theory*, 2001, 1(2): 133 – 143.

[26] CHEN Y H. A new approach to the control design of fuzzy dynamical systems [J]. *Journal of Dynamic Systems, Measurement and Control*, 2011, 133(6): 1 – 9.

[27] BRONSHTEIN I N, SEMENDYAYEV K A. *Handbook of Mathematics* [M]. New York: Van Nostrand Reinhold, 1985.

Appendix   Fuzzy mathematics

We briefly review some preliminaries regarding fuzzy numbers and their operations[22]:

1) Fuzzy number. Let G be a fuzzy set in $\mathbb{R}$, the real number. G is called a fuzzy number if: i) G is normal, ii) G is convex, iii) the support of G is bounded, and iv) all α-cuts are closed intervals in $\mathbb{R}$.

Throughout, we shall always assume the universe of discourse of a fuzzy set number to be its 0-cut.

2) Fuzzy arithmetic. Let G and H be two fuzzy numbers and $G_\alpha = [g_\alpha^-, g_\alpha^+], H_\alpha = [h_\alpha^-, h_\alpha^+]$ be their α-cuts, $\alpha \in [0, 1]$. The addition, substraction, multiplication, and division of G and H are given by, respectively,

$$(G + H)_\alpha = [g_\alpha^- + h_\alpha^-, g_\alpha^+ + h_\alpha^+], \tag{A1}$$

$$(G - H)_\alpha = [\min(g_\alpha^- - h_\alpha^-, g_\alpha^+ - h_\alpha^+),$$
$$\max(g_\alpha^- - h_\alpha^-, g_\alpha^+ - h_\alpha^+)], \tag{A2}$$

$$(G \cdot H)_\alpha = [\min(g_\alpha^- h_\alpha^-, g_\alpha^- h_\alpha^+, g_\alpha^+ h_\alpha^-, g_\alpha^+ h_\alpha^+),$$
$$\max(g_\alpha^- h_\alpha^-, g_\alpha^- h_\alpha^+, g_\alpha^+ h_\alpha^-, g_\alpha^+ h_\alpha^+)], \tag{A3}$$

$$(G/H)_\alpha = [\min(g_\alpha^-/h_\alpha^-, g_\alpha^-/h_\alpha^+, g_\alpha^+/h_\alpha^-, g_\alpha^+/h_\alpha^+),$$

$$\max(g_\alpha^-/h_\alpha^-, g_\alpha^-/h_\alpha^+, g_\alpha^+/h_\alpha^-, g_\alpha^+/h_\alpha^+)]. \quad \text{(A4)}$$

3) Decomposition theorem. Define a fuzzy set $\tilde{V}_\alpha$ in U with the membership function $\mu_{\tilde{V}_\alpha} = \alpha I_{\tilde{V}_\alpha}(x)$ where $I_{\tilde{V}_\alpha}(x) = 1$ if $x \in \tilde{V}_\alpha$ and $I_{\tilde{V}_\alpha}(x) = 0$ if $x \in U - \tilde{V}_\alpha$. Then the fuzzy set V is obtained as

$$V = \bigcup_{\alpha \in [0,1]} \tilde{V}_\alpha, \quad \text{(A5)}$$

where $\bigcup$ is the union of the fuzzy sets (that is, sup over $\alpha \in [0,1]$).

Based on these, after the operation of two fuzzy numbers via their α-cuts, one may apply the decomposition theorem to build the membership function of the resulting fuzzy number.

Appendix　Analysis of differential inequality

The analysis of differential inequality[19] can be made according to [20]:

Definition 2　If $w(\psi, t)$ is a scalar function of the scalars ψ, t in some open connected set D, we say a function $\psi(t)$, $t_0 \leqslant t \leqslant \bar{t}$, $\bar{t} > t_0$ is a solution of the differential inequality[19]

$$\dot{\psi}(t) \leqslant w(\psi(t), t) \quad \text{(A6)}$$

on $[t_0, \bar{t}]$ if $\psi(t)$ is continuous on $[t_0, \bar{t}]$ and its derivative on $[t_0, \bar{t}]$ satisfies Inequality (A6).

Theorem 3　Let $w(\phi(t), t)$ be continuous on an open connected set $D \in \mathbb{R}^2$ and such that the initial value problem for the scalar equation[19]

$$\dot{\phi}(t) = w(\phi(t), t), \; \phi(t_0) = \phi_0 \quad \text{(A7)}$$

has a unique solution. If $\phi(t)$ is a solution of Eq.(A7) on $t_0 \leqslant t \leqslant \bar{t}$ and $\psi(t)$ is a solution of Inequality (A6) on $t_0 \leqslant t \leqslant \bar{t}$ with $\psi(t_0) \leqslant \phi(t_0)$, then $\psi(t) \leqslant \phi(t)$ for $t_0 \leqslant t \leqslant \bar{t}$.

Instead of exploring the solution of the differential inequality, which is often non-unique and not available, the above theorems suggests that it may be feasible to study the upper bound of the solution. The reason is, however, based on that the solution of Eq.(A7) is unique.

Theorem 4　Consider the differential inequality Inequality (A6) and the differential equation Eq.(A7). Suppose that for some constant $L > 0$, the function $w(\cdot)$ satisfies the Lipschitz condition[20]

$$|w(v_1, t) - w(v_2, t)| \leqslant L|v_1 - v_2|, \quad \text{(A8)}$$

for all points $(v_1, t), (v_2, t) \in D$. Then any function $\psi(t)$ that satisfies the differential Inequality (A6) for $t_0 \leqslant t \leqslant \bar{t}$ satisfies also the inequality

$$\psi(t) \leqslant \phi(t), \quad \text{(A9)}$$

for $t_0 \leqslant t \leqslant \bar{t}$.

We consider the differential equation

$$\dot{r}(t) = -\frac{1}{\kappa} r(t) + \frac{\delta}{\gamma}, \quad r(t_0) = V_0. \quad \text{(A10)}$$

The right-hand side satisfies the global Lipschitz condition with $L = 1/\kappa$. We proceed with solving the differential equation (A10). This results in

$$r(t) = (V_0 - \frac{\kappa\delta}{\gamma}) \exp[-\frac{1}{\kappa}(t - t_0)] + \frac{\kappa\delta}{\gamma}. \quad \text{(A11)}$$

Appendix　D-operation

We can see D-operation in [23]:

Definition 3　Consider a fuzzy set

$$N = \{(v, \mu_N(v))|v \in N\}. \quad \text{(A12)}$$

For any function $f : N \to \mathbb{R}$, the D-operation $D[f(v)]$ is given by

$$D[f(v)] = \frac{\int_N f(v)\mu_N(v)\mathrm{d}v}{\int_N \mu_N(v)\mathrm{d}v}. \quad \text{(A13)}$$

Remark 9　In a sense, the D-operation $D[f(v)]$ takes an average value of $f(v)$ over $\mu_N(v)$. In the special case that $f(v) = v$, this is reduced to the well-known center-of-gravity defuzzification method [22]. Particularly, if N is crisp (i.e., $\mu_N(v) = 1$ for all $v \in N$), $D[f(v)] = f(v)$.

Lemma 1　For any crisp constant $a \in \mathbb{R}$,

$$D[af(v)] = \frac{\int_N af(v)\mu_N(v)\mathrm{d}v}{\int_N \mu_N(v)\mathrm{d}v} =$$
$$a\frac{\int_N f(v)\mu_N(v)\mathrm{d}v}{\int_N \mu_N(v)\mathrm{d}v} = aD[f(v)]. \quad \text{(A14)}$$

Appendix　The closed-form solution of optimal gain

The closed-form solution of optimal gain can be seen in [23–26]:

For any t_s, taking the first order derivative of J with respect to γ

$$\frac{\partial J}{\partial \gamma} = \frac{\kappa_2}{\gamma^2} - 2\frac{\kappa_3}{\gamma^3} - 2\alpha\frac{\kappa_4}{\gamma^3} + 2\beta\gamma =$$
$$\frac{1}{\gamma^3}(\kappa_2\gamma - 2\kappa_3 - 2\alpha\kappa_4 + 2\beta\gamma^4). \quad \text{(A15)}$$

That

$$\frac{\partial J}{\partial \gamma} = 0, \quad \text{(A16)}$$

leads to

$$\kappa_2\gamma - 2\kappa_3 - 2\alpha\kappa_4 + 2\beta\gamma^4 = 0, \quad \text{(A17)}$$

or

$$\kappa_2\gamma + 2\beta\gamma^4 = 2(\kappa_3 + \alpha\kappa_4). \quad \text{(A18)}$$

Eq.(A18) is a quartic equation.

Theorem 5　Suppose $D[\delta] \neq 0$. For given $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, the solution $\gamma > 0$ to Eq.(A18) always exists and is unique, which globally minimizes the performance index (50).

Proof　Let $\theta(\gamma) := \kappa_2\gamma + 2\beta\gamma^4$. Then $\theta(0) = 0$ and $\theta(\cdot)$ is continuous in γ. In addition, since $\kappa_2 \geqslant 0$ and $\beta > 0$, $\theta(\cdot)$ is strictly increasing in γ. Since $D[\delta] \neq 0$, we have $D[\delta] > 0$, $D[\delta^2] > 0$, $\kappa_3, \kappa_4 > 0$, and therefore $2(\kappa_3 + \alpha\kappa_4) > 0$ (notice that $\alpha, \kappa > 0$). As a result, the solution $\gamma > 0$ to Eq.(A18) always exists and is unique. For the unique solution $\gamma > 0$ that solves Eq.(A18),

$$\frac{\partial^2 J}{\partial \gamma^2} = -\frac{3}{\gamma^4}(\kappa_2\gamma - 2\kappa_3 - 2\alpha\kappa_4 + 2\beta\gamma^4) +$$
$$\frac{1}{\gamma^3}(\kappa_2 + 8\beta\gamma^3) = \frac{1}{\gamma^3}(\kappa_2 + 8\beta\gamma^3) > 0. \quad \text{(A19)}$$

Therefore the positive solution $\gamma > 0$ of the quartic Eq.(A18) solves the constrained minimization problem (51).

Remark 10　In the special case that the fuzzy sets are crisp, $D[\delta] = \delta$, $D[\delta^2] = \delta^2$. The current setting still applies. The optimal design can be found by solving Eq.(A18).

The solutions of the quartic Eq.(A18) depend on the cubic resolvent[27]

$$z^3 + (-4r_1)z - r_2^2 = 0, \quad \text{(A20)}$$

where

$$r_1 = -\frac{1}{\beta}(\kappa_3 + \alpha\kappa_4), \tag{A21}$$

$$r_2 = \frac{\kappa_2}{2\beta}. \tag{A22}$$

Let $p_1 := -4r_1, p_2 := -r_2^2$. The discriminant H of the cubic resolvent is given by

$$H = (\frac{p_1}{3})^3 + (\frac{p_2}{2})^2. \tag{A23}$$

Since $r < 0$, $H > 0$. The solutions of the cubic resolvent are given by

$$z_1 = u + w, \tag{A24}$$

$$z_2 = -\frac{(u+w)}{2} + (u-w)i\sqrt{\frac{3}{2}}, \tag{A25}$$

$$z_3 = -\frac{(u+w)}{2} - (u-w)i\sqrt{\frac{3}{2}}, \tag{A26}$$

where

$$u = (-\frac{p_2}{2} + \sqrt{H})^{\frac{1}{3}}, \tag{A27}$$

$$w = (-\frac{p_2}{2} - \sqrt{H})^{\frac{1}{3}}. \tag{A28}$$

The cubic resolvent possesses one real solution and two complex conjugate solutions. This in turn implies that the quartic equation has two real solutions and one pair of complex conjugate solutions. The maximum real solution, which is positive and is therefore the optimal solution to the constrained optimization problem, of the quartic equation is given by

$$\gamma_{\mathrm{opt}} = \frac{1}{2}(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}). \tag{A29}$$

Appendix Some limiting performance

This can be seen in [23–26]:

As was shown earlier, the tracking error $\underline{e}$ of the controlled system enters the uniform ultimate boundedness region after a finite time and stays within the region thereafter. Thus it is interesting to consider, in the limiting case, only the cost associated with this portion of performance.

In the limiting case, the transient performance cost $J_1(\gamma, t_s)$ is not considered. The cost is then dictated by that of the steady state performance and the control gain:

$$J(\gamma) = \alpha J_2(\gamma) + \beta J_3(\gamma) = \alpha\frac{\kappa_4}{\gamma^2} + \beta\gamma^2. \tag{A30}$$

The quartic Eq.(A18) is reduced to

$$\beta\gamma^4 = \alpha\kappa_4. \tag{A31}$$

Its positive solution is given by

$$\gamma_{\mathrm{opt}} = (\frac{\alpha}{\beta}\kappa_4)^{\frac{1}{4}}. \tag{A32}$$

With Eq.(A31) into Eq.(A30),

$$J = \frac{1}{\gamma^2}(\alpha\kappa_4 + \beta\gamma^4) = \frac{2\alpha\kappa_4}{\gamma^2}. \tag{A33}$$

Using Eq.(A32), the minimum cost is then

$$J_{\min} = 2\sqrt{\alpha\beta\kappa_4}. \tag{A34}$$

If the weighting $\alpha \to \infty$, then in the quartic equation, the positive solution $\gamma \to \infty$. This simply means that the relative cost of the ultimate boundedness region (as is given by $\alpha J_2(\gamma)$) is high and the control gain, which is relatively cheap, is turned high.

If $\beta \to \infty$, then in the quartic equation, the positive solution $\gamma \to 0$. This shows the other extreme case when the control is very expensive.

作者简介:

甄圣超　(1988–), 男, 博士研究生, 研究领域包括分析力学、多体系统动力学、最优控制、鲁棒控制、自适应控制、模糊工程、不确定性处理等, E-mail: zhenshengchao@gmail.com;

赵　韩　(1957–), 博士生导师, 国际IFToMM教育委员会委员、机械工程学报会员, 研究领域包括机械传动、磁力机械、汽车、数字化设计与制造、信息系统、动力学与控制等, E-mail: hanzhaoff@qq.com;

黄　康　(1968–), 博士生导师, 研究领域包括机械传动、磁力机械、汽车电子、数字化设计、机器人、动力学、控制等, E-mail: hfhuang98@hotmail.com;

陈义华　(1958–), 博士, 教授, IEEE、ASME、Sigma Xi成员, Nonlinear Dynamics and System Theory北美地区编辑, International Journal of Intelligent Automation and Soft Computing副编辑, 研究领域包括机械臂的先进控制方法、神经网络、模糊工程、不确定系统的自适应鲁棒控制、不确定性处理等, E-mail: yehwa.chen@me.gatech.edu.