DOI: 10.7641/CTA.2014.30974

非Lipschitz连续级联系统的稳定性分析及其应用

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摘要: 当考虑级联系统稳定性时, 一般都需要系统满足局部或者全局Lipschitz连续性条件. 与已有文献中的结果不同, 本文给出了一种处理满足非Lipschitz连续条件下级联系统的稳定性分析方法. 首先, 基于积分输入状态稳定的定义, 给出了级联系统全局稳定的Lyapunov形式条件. 基于此, 继续讨论了非Lipschitz连续情况下级联系统的有限时间稳定性. 然后, 利用上述稳定性分析结果, 讨论了一类驱动子系统具有上三角结构的级联系统的控制设计问题. 最后, 给出几个例子验证了上述结果的有效性.

关键词: 级联系统; 积分输入到状态稳定; 有限时间控制; 齐次性

中图分类号: TP273 **文献标识码:** A

Stability of non-Lipschitz continuous cascaded systems and its application

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Abstract: The local/global Lipschitz continuity is always required when considering the stability of the cascaded systems. Being different from the exiting methods proposed in the literature, we give a method to handle the non-Lipschitz continuous cascaded systems. By using the definition of iISS (integral input-to-state stability), the Lyapunov-like conditions for global stability of non-Lipschitz continuous cascaded systems are derived. Then, based on this, the finite-time stability for non-Lipschitz continuous cascaded systems is further studied. The stability analysis results are applied to the control design problem for a class of cascaded systems with upper-triangular deriving subsystem. Finally, some examples are proposed to validate the effectiveness of the proposed results.

Key words: cascaded system; iISS; finite-time control; homogeneity

1 Introduction

To verify the stability of a nonlinear control system, it is usually required to construct a Lyapunov function, while such a kind of Lyapunov functions are not easy to find, especially for some complex nonlinear systems. According to [1], if we can transform the nonlinear system into a system with cascaded structure, the controller design and stability analysis will become much easy. In this case, instead of looking for a Lyapunov function for the overall system, we only need to investigate the stability properties of two subsystems separately and exploit the structure of the interconnection.

Since the cascaded design can be used to reduce the complexity of controller design and stability analysis, the research on cascaded systems has been attracted much attention in recent years and various methods have been proposed in the literature (see, e.g., [2–11] and the reference therein). Among them, the representative methods can be summarized as Lyapunov method, ISS (input to state stability, introduced in [12]) method, passivity method, etc. The Lyapunov method has been widely used for control design problem of cascaded systems, and tremendous results have been obtained. This method is first applied to analyze autonomous cascaded systems, such as [3, 7, 13]. Later, it has also been applied to global stability of non-autonomous systems in

Received: 2013−09−16; Accepted: 2014−02−25.

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This work was supported by Natural Science Foundation of China (61203054, 61203014), the Priority Academic Program Development of Jiangsu Higher Education Institutions, Initial Research Fund of Highly Specialized Personnel from Jiangsu University (11JDG103) and Natural Science Foundation of Jiangsu Province (BK2012283).

[10, 14].

Another effective method is the ISS theory since it is an effective tool for verifying the boundedness of the trajectories. It is stated in [12] that when the driving subsystem is globally asymptotically stable, the cascaded system is globally asymptotically stable under the condition the driven subsystem is ISS with the states of the driving subsystem. In [9, 15], the partial-state feedback controllers are developed for global stabilization of cascaded systems by utilizing ISS properties. Later, it is pointed out in [16] that the ISS properties, to some extent, are restrictive. To this end, the integral ISS (iISS) property is studied for a class of nonlinear time-invariant cascaded systems in [2, 17], and some sufficient conditions for the preservation of the iISS property under a cascaded interconnection are presented. On the other hand, the passivity method is also an effective tool to analyze the stability of the cascaded systems. Two control schemes for a nonlinear system in cascade with a linear system is derived based on passivity property in [18] and [19], respectively. In [18], the analysis is carried out for the case of partially linear composite systems whose linear system is relative degree one. Then the results in [18] are generalized in [19] with the linear system being relative degree than one.

However, as can be seen from the above literature, all the mentioned results are based on one condition, i.e., the local or global Lipschitz continuity of the cascaded systems. It should be pointed out that there are many cascaded systems, which are not locally/globally Lipschitz continuous. This mainly attributes to two aspects. On one hand, there are many non-Lipschitz continuous dynamic systems, such as the frequencies of the oscillators considered in [20], which is apparently non-Lipschitz continuous when using cascaded method to test stability. On the other hand, to improve the disturbance rejection property, the non-smooth terms are always introduced in the controller, such as in [8]. In this circumstance, the closed-loop cascaded systems are also non-Lipschitz continuous. For the above mentioned non-Lipschitz continuous cascaded systems, the existing methods only for locally/globally Lipschitz continuous systems can not be applied directly to control design or stability analysis problems.

In this paper, we will propose a method to deal with the stability analysis problem for a class of non-Lipschitz continuous cascaded systems. By imposing the iISS assumption on the driven subsystem, sufficient conditions are derived to ensure the global asymptotic stability of the cascaded systems. Then, we also show that the cascaded system is globally finite-time stable if the zero dynamics of the driven subsystem and the driving subsystem are globally finite-time stable. Meanwhile, based upon the homogeneous theory[21], the proposed results are applied to stabilize a class of cascaded system with the driving subsystem having an upper-triangular structure, and show that the stabilization of the driving subsystem implies stabilization of the whole cascaded system.

2 Notations and definitions

Notation A continuous function $f: R_{\geqslant 0} \to R_{\geqslant 0}$ is of class $\mathcal{K}$ ($f \in \mathcal{K}$), if it is strictly increasing and $f(0) = 0$. A continuous function $g: R_{\geqslant 0} \to R_{\geqslant 0}$ is of class $\mathcal{L}$ ($g \in \mathcal{L}$) if it is decreasing and tends to zero as its argument tends to infinity. A function $h: R_{\geqslant 0} \times R_{\geqslant 0} \to R_{\geqslant 0}$ is said to be a class $\mathcal{KL}$ function ($h \in \mathcal{KL}$) if $h(\cdot, t) \in \mathcal{K}$ for any $t \in R_{\geqslant 0}$, and $h(s, \cdot) \in \mathcal{L}$ for any $s \in R_{\geqslant 0}$.

Then, we introduce the iISS definition for non-Lipschitz continuous systems.

Consider the following nonlinear system

$$\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{1}$$

where $f(x, u): \mathbb{R}^{n+m} \to \mathbb{R}^n$ is non-Lipschitz continuous but Hölder continuous. The Hölder continuity guarantees the existence of the solution, while it can not guarantee the uniqueness of the solution. Note that the conventional iISS is defined only for locally Lipschitz continuous systems. Therefore, we can not apply the conventional iISS to the non-Lipschitz continuous system directly.

For system (1), we let the set $U(x_0)$ including all the solutions denoted by $x(t, x_0)$ from the the initial state x_0 in forward time. Then by extending the iISS definition in [22], we have the iISS definition for system (1) as follows.

Definition 1 System (1) is said to be iISS if there exist functions $\beta(\cdot, t) \in \mathcal{KL}$ and $\gamma_1(\cdot), \gamma_2(\cdot) \in \mathcal{K}$ such that, for all $x_0 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, each solution $x(t, x_0) \in U(x_0)$ is defined for $t \geqslant 0$ and satisfies

$$\|x(t, x_0)\| \leqslant \beta(\|x_0\|, t) + \gamma_1\Big(\int_0^t \gamma_2(\|u(s)\|)\mathrm{d}s\Big).$$

Definition 2 It is called (α, μ)-iISS pair, if there are a C^1 positive-definite and proper Lyapunov function $V(x): \mathbb{R}^n \to \mathbb{R}$, and two continuous positive-definite functions $\alpha(\cdot)$ and $\mu(\cdot) \in \mathcal{K}$ such that

$$\dot{V}(x)|_{(1)} \leqslant -\alpha(\|x\|) + \mu(\|u\|).$$

According to Theorem 1 in [16], if there exists a

(α, μ)-iISS pair for system (1), then system (1) is iISS.

3 Main results

In this paper, we consider the following cascaded systems

$$\dot{x} = f(x, y), \tag{2a}$$

$$\dot{y} = g(y), \tag{2b}$$

with $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $f(x,y), g(y)$ are Hölder continuous in their arguments.

Regarding the state of driving subsystem (2b) as the input of driven subsystem (2a), by Sontag's ISS theory[12], the cascaded system (2) is globally asymptotically stable if subsystem (2a) is ISS and subsystem (2b) is globally asymptotically stable. It implies ISS+GAS=GAS. However, if ISS property of driven subsystem (2a) is relaxed to be iISS, the above property will not hold, i.e., iISS+GAS≠ GAS. To ensure the global stability of system (2), some additional conditions are also required, such as[2,16]. However, it can be observed from the literature that almost all the results for system (2) including[2,16] require that the vector fields at least should be locally Lipschitz continuity. If the cascaded systems are not local/global Lipschitz continuous, these methods can not be applied directly. In this paper, we will focus on deriving some sufficient conditions in terms of Lyapunov-like conditions to guarantee the global stability of the non-Lipschitz continuous cascaded system (2).

3.1 Global asymptotical stability based on iISS

We first show that under the iISS property for subsystem (2a), some Lyapunov-like sufficient conditions will be proposed to guarantee the global asymptotical stability of cascaded system (2).

Theorem 1 Assume that there is a (α, μ)-iISS pair for subsystem (2a) and subsystem (2b) is globally asymptotically stable. If there exist a constant k, a C^1 positive-definite Lyapunov function $V_2(y)$, a continuous positive-definite function $\omega(y)$ and a region Ω of the origin such that for $\forall y \in \Omega \setminus \{0\}$,

$$\dot{V}_2(y)|_{(2b)} \leqslant -\omega(y),\ \ \mu(\|y\|) \leqslant k\omega(y), \tag{3}$$

then the cascaded system (2) is globally asymptotically stable.

Proof Since there is a (α, μ)-iISS pair for subsystem (2a), it can be concluded that there exists a proper and positive-definite Lyapunov function $V_1(x)$ such that

$$\dot{V}_1(x) \leqslant -\alpha(\|x\|) + \mu(\|y\|). \tag{4}$$

Let $V(x,y) = V_1(x) + 2kV_2(y), k > 0$ and

$$Q = \{(x,y) : x \in \mathbb{R}^n,\ y \in \Omega \backslash \{0\}\}.$$

Then, by (3) and (4), we have

$$\dot{V}(x,y) \leqslant -\alpha(\|x\|) - k\omega(y),\ \forall (x,y) \in Q.$$

By strong stability theorem [23], system (2a)-(2b) is locally stable. Consequently, there exists an attractive region for system (2a)-(2b), denoted by

$$Q_1 = \{(x,y)^{\mathrm{T}} : \|x\| \leqslant r_1, \|y\| \leqslant r_1, r_1 > 0\}.$$

To prove the globally asymptotic stability, we only need to show the global attractivity.

First of all, we will prove that the states will converge to the region Q_1 from any initial state. Assume that the trajectory of cascaded system (2a)-(2b) starts from the initial state (x_0, y_0). Note that system (2b) is globally asymptotically stable. It is clear that there exist a time instant T_1 and a ball of the origin denoted by

$$\Omega_1 = \{y : \|y\| \leqslant r_2,\ r_1 > r_2 > 0\},$$

such that

$$y \in \Omega_1,\ \forall t \geqslant T_1. \tag{5}$$

Next, we show the state x will not escape to infinity in a finite time. Note that the state y is always bounded. It is clear that there exists a constant γ such that $\mu(\|y\|) \leqslant \gamma$. Then, by (4), we have

$$\dot{V}_1(x) \leqslant \mu(\|y\|) \leqslant \gamma.$$

Taking an integration from both sides of the above inequality, one has

$$V_1(x(t)) \leqslant V_1(x(0)) + \gamma t, \forall t > 0. \tag{6}$$

It implies that the state of system (2a) will not escape to infinity in a finite time.

In the following, we will prove the states will further converge to the region Q_1 in a finite time. Note from (5) that for $t > T_1$, the state y enters and stays in the region Ω_1. In addition, by (6), we know the state x is bounded during the time interval $[0, T_1]$. With this in mind, taking the derivative of $V(x,y)$ along system (2a)-(2b) again, we get for $t > T_1$,

$$\begin{aligned} &\dot{V}(x,y) \leqslant \\ &-\alpha(\|x\|) - 2k\omega(y) + \mu(\|y\|) \leqslant \\ &-\alpha(\|x\|) - k\omega(y), \forall (x,y) \in \mathbb{R}^n \times \Omega_1. \end{aligned}$$

This implies that there exists a time instant $T_2 > T_1$ such that for $t \geqslant T_2$

$$x(t) \in S = \{x : \|x\| \leqslant r_2\}. \tag{7}$$

In conclusion, by (5) and (7), when $t > T_2$, the states x and y will converge to the region

$$S \cup \Omega_1 = \{(x,y) : \|x\| \leqslant r_2, \|y\| \leqslant r_2\}.$$

Since $r_1 > r_2$, it is clear that the set $S \cup \Omega_1$ is included in the set Q_1, which also shows that the set $S \cup \Omega_1$ is an attractive region. Finally, the states will converge to the origin. This completes the proof.

Remark 1 When the driven subsystem (2a) is iISS, some results on global stability of cascaded system (2) have already been reported in [2, 16]. However, it can be observed that a precondition for the methods proposed in [2, 16] is the locally Lipschitz continuity. Consequently, these methods can not be applied to the non-Lipschitz continuous cascaded system (2) directly. As a matter of fact, when system (2) satisfies local Lipschitz continuity, a similar result to Theorem 1 can be found in [2]. Under the locally Lipschitz continuity assumption, if we replace the condition $\mu(\|y\|) \leqslant k\omega(y), \forall y \in \Omega$ with $\lim\limits_{y\to 0} \dfrac{\mu(\|y\|)}{\omega(y)} = k, k \geqslant 0, \forall y \in \Omega$, Theorem 1 will reduce to the result in [2]. In this paper, we relax the locally Lipschitz continuity to locally Hölder continuity. Noting that local Lipschitz continuity implies the local Hölder continuity, the result proposed in this paper can also be applied to the system considered in [2].

To apply Theorem 1, it is usually required to construct some Lyapunov-like conditions for subsystems (2a) and (2b). However, in many cases, it is not easy to construct such Lyapunov-like conditions as (3) and (4). For example, it is difficult to find a Lyapunov function for testing the following system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^{1/5} - x_2^{1/3}, \end{aligned} \tag{8}$$

although it is globally asymptotically stable. To this end, we relax the conditions given in Theorem 1 and propose the following theorem, whose proof is the same as that of Theorem 1.

Theorem 2 Assume that subsystem (2b) is globally asymptotically stable and there exist two C^1 positive-definite Lyapunov functions $V_1(x)$ and $V_2(y)$, a constant $k > 0$ and a region Ω of the origin such that for $\forall (x, y) \in \mathbb{R}^{n+m}$

$$\dot{V}_1(x)|_{(2a)} \leqslant -\bar{\alpha}(x) + \bar{\mu}(y), \tag{9}$$

and for $\forall y \in \Omega \backslash \{0\}$

$$\dot{V}_2(y)|_{(2b)} \leqslant -\bar{\omega}(y), \ \ \bar{\mu}(y) \leqslant k\bar{\omega}(y), \tag{10}$$

with positive-definite function $\bar{\alpha}(x)$ and semi-positive-definite functions $\bar{\omega}(y)$ and $\bar{\mu}(y)$. Then the cascaded system (2) is globally asymptotically stable.

The advantage of Theorem 2 lies in that the functions $\bar{\mu}(y)$ and $\bar{\omega}(y)$ are not required to be positive definite. This property leads to two advantages. Firstly, there is no need to find a positive-definite function to dominate the semi-positive-definite function $\bar{\mu}(y)$, which is required in Theorem 1. Secondly, it is not required to find the positive-definite functions $V_2(y)$ and $\omega(y)$ to test the stability of subsystem (2b). In many cases, we can only obtain a semi-positive-definite function $\bar{\omega}(y)$. To this end, compared with Theorem 1, this property allows Theorem 2 to be applied for a more general class of cascaded systems.

On the other hand, we can observe that the existing control methods proposed in the literature lead to at best the exponential convergence of the closed-loop systems. Compared with these existing control methods, it has been proved in [24–26] that the finite-time control method will yield better convergence performance and disturbance rejection property. Therefore, finite-time control of nonlinear systems has attracted much attention in recent years, e.g., [24, 26–28]. In this section, based on the iISS property of driven subsystem, we will show an interesting result on finite-time stability for cascaded system (2).

Theorem 3 Assume that there is a (α, μ)-iISS pair for subsystem (2a). If subsystems $\dot{x} = f(x, 0)$ and (2a) are globally finite-time stable, the cascaded system (2) is globally finite-time stable.

Proof Note that subsystem (2a) is iISS with respect to the state of subsystem (2b), and subsystem (2b) is globally finite-time stable. Similar to the local stability proof in Theorem 1, we know that the cascaded system (2a)-(2b) is locally stable. Since the local stability plus the global finite-time attractivity implies the global finite-time stability, to prove the global finite-time of system (2a)-(2b), we only required to prove the global finite-time attractivity.

From the global finite-time stability of subsystem (2b), it can be concluded that there exists a time instant $t_1^* < +\infty$ such that all the solutions from the initial state y_0, denoted by $y(t, y_0)$, satisfy

$$y(t, y_0) \equiv 0, \ \forall t \geqslant t_1^*.$$

In addition, consider that there is a (α, μ)-iISS pair for subsystem (2a), which indicates there exist positive-definite Lyapunov function $V_1(x)$, positive-definite functions $\alpha(\cdot)$ and $\mu(\cdot) \in \mathcal{K}$ such that

$$\dot{V}_1(x)|_{(2a)} \leqslant -\alpha(\|x\|) + \mu(\|y\|).$$

This together with the global finite-time stability of subsystem (2b) implies that the state x will not diverge to infinity in a finite time. Thus, the state of subsystem (2a) is bounded during the time interval $[0, t_1^*]$. As a consequence, for $t \geqslant t_1^*$, system (2a) reduces to subsystem $\dot{x} = f(x, 0)$, which is a globally finite-time stable system. Obviously, the state x will converge to the origin in finite time. This completes the proof of Theorem 3.

3.2 Stabilizing a class of cascaded systems

The control design problem for cascaded system has been paid considerable attention in the literature. Nevertheless, almost all the results are focused on the cascaded system with a lower-triangular driving subsystem. There are no results on the stabilization of cascaded systems with upper-triangular driving subsystem. The reason may be that it is not easy to verify the stability of the cascaded system with upper-triangular driving subsystem. It should be noted that there are many dynamical systems whose driving subsystems possess the upper-triangular structure, for example, the vertical take-off landing aircraft attitude control system [29]. Consequently, it is important to consider the control design problem for cascaded system with upper-triangular driving subsystem.

Consider the following cascaded systems described by

$$\dot{x} = f(x, y), \tag{11a}$$

$$\begin{cases} \dot{y}_1 = y_2 + g_1(y_2, \cdots, y_m), \\ \dot{y}_2 = y_3 + g_2(y_3, \cdots, y_m), \\ \quad\vdots \\ \dot{y}_{m-1} = y_m + g_{m-1}(y_m), \\ \dot{y}_m = u, \end{cases} \tag{11b}$$

where $f(x, y)$ and $g_i(y_{i+1}, \cdots, y_m), i = 1, \cdots, m-1$ are non-Lipschitz continuous functions. The driving subsystem (11b) satisfies the following assumption:

Assumption 1 In a neighborhood of the origin, the following holds

$$|g_i(y_{i+1}, \cdots, y_m)| \leqslant \rho(|y_{i+1}|^{q_{i,i+1}} + \cdots + |y_m|^{q_{im}}), \quad i = 1, \cdots, m-1, \tag{12}$$

for positive constants ρ and q_{ij} satisfying

$$q_{ij} > \frac{r_{i+1}}{r_j} > 0,\ i = 1, \cdots, m-1,\ j = i+1, \cdots, m. \tag{13}$$

where $r_i > 0,\ i = 1, \cdots, m$ are defined as

$$r_1 = 1,\ r_{i+1} = r_i + \tau > 0,\ \ i = 1, \cdots, m \tag{14}$$

with a positive constant τ being a ratio of even and odd numbers.

In addition, the driven subsystem (11a) satisfies the following condition:

Assumption 2 For system (11a), there exist positive-definite functions $\bar{\alpha}(x) \in C^0$ and $\bar{\mu}(y) = \mathcal{O}(\|y\|_\Delta^{2r_m})$ [1] with

$$\|y\|_\Delta = (|y_1|^{2/r_1} + \cdots + |y_m|^{2/r_m})^{1/2},$$

such that

$$\dot{V}_1(x) \leqslant -\bar{\alpha}(x) + \bar{\mu}(y).$$

By Assumption 2, it is obvious that subsystem (11a) is iISS with respect to the state of subsystem (11b).

Different from the considered cascaded systems in the literature, subsystem (11b) has an upper-triangular structure. It implies the conventional control design methods can not be applied to system (11). In this subsection, we will design a controller for system (11) under Assumptions 1-2. Before giving the main result, we first list the following two lemmas.

Lemma 1[30] There exist a small constant $\epsilon > 0$ and gains $\beta_m > \beta_{m-1} > \cdots > \beta_1 > 0$ such that the following controller

$$u = u_m(Y_m(t)) = -\beta_m \sigma^{\frac{r_{m+1}}{r_m}}(y_m - u_{m-1}(Y_{m-1})), \tag{15}$$

where

$$u_0 = 0, u_i(Y_i(t)) = -\beta_i \sigma^{\frac{r_{i+1}}{r_i}}(y_i - u_{i-1}(Y_{i-1})),$$
$$i = 1, \cdots, m-1,$$
$$\sigma(y) = \begin{cases} \epsilon^{r_i} \operatorname{sgn} y, & \text{for } |y| > \epsilon^{r_i}, \\ y, & \text{for } |y| \leqslant \epsilon^{r_i} \end{cases}$$

globally stabilizes system (11b).

Remark 2 Since τ is a ratio of an even integer and an odd integer, the parameters r_i in controller (15) have to be ratio of positive odd integers. As a matter of fact, similar to [4], we can extend τ to be any real number and relax this restriction by defining

$$< \cdot >^{\alpha} = \operatorname{sgn}(\cdot)|\cdot|^{\alpha},\ \alpha > 0. \tag{16}$$

Then controller (15) can be rewritten as

$$u = -\beta_m < \sigma(y_m - u_{m-1}(Y_{m-1})) >^{\frac{r_{m+1}}{r_m}}.$$

Lemma 2[31] Suppose that the positive-definite function $\nu(x): \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree r with respect to the dilation $(r_1, \cdots, r_n)$. If the positive-definite function $\psi(x): \mathbb{R}^n \to \mathbb{R}$ is also homogeneous of degree r with respect to the dilation $(r_1, \cdots, r_n)$, then there exist positive constants $\bar{c}$ and $\underline{c}$ such that $\underline{c}\psi(x) \leqslant \nu(x) \leqslant \bar{c}\psi(x)$.

Then, we have the following theorem.

Theorem 4 Under Assumptions 1–2, controller

[1] $b(x) = \mathcal{O}(a(x))$ means there exists positive constant c such that $\lim\limits_{x\to 0} \frac{b(x)}{a(x)} = c$

(15) globally stabilizes cascaded system (11).

Proof Note that Assumption 2 implies condition (9). Then, by Lemma 1, it is clear that the closed-loop system (11b)(15) is globally asymptotically stable. According to Theorem 2, to prove the global asymptotical stability of the closed-loop cascaded system (11)(15), we only need to show condition (10).

Let the candidate Lyapunov function as

$$V_2(y) = \frac{r_1}{2r_m-\tau}\xi_1^{\frac{2r_m-\tau}{r_1}} + \frac{r_2}{2r_m-\tau}\xi_2^{\frac{2r_m-\tau}{r_2}} + \cdots + \frac{r_m}{2r_m-\tau}\xi_m^{\frac{2r_m-\tau}{r_m}}. \tag{17}$$

with

$$y_1^* = 0,\ \xi_1 = y_1,\ y_i^* = -\beta_{i-1}\xi_{i-1}^{\frac{r_i}{r_{i-1}}},\quad \xi_i = y_i - y_i^*,\ i = 2,\cdots,m.$$

According to Lemma 2.1 in [30], with a proper selection of β_i's satisfying $\beta_m > \beta_{m-1} > \cdots > \beta_1$, the derivative of Lyapunov function $V_2(y)$ along the trajectory of the multiple integrators

$$\dot{y}_1 = y_2, \cdots, \dot{y}_m = u \tag{18}$$

yields

$$\dot{V}_2(y) \leqslant -(\xi_1^{2r_m/r_1} + \cdots + \xi_{n-1}^{2r_m/r_{m-1}} + \xi_m^{2r_m/r_m}). \tag{19}$$

By (17) and (19), the derivative of $V_2(y)$ along system (11b) under controller (15) is

$$\dot{V}_2(y) \leqslant -(\xi_1^{2r_m/r_1} + \cdots + \xi_{m-1}^{2r_m/r_{m-1}} + \xi_m^{2r_m/r_m}) + W_1(y)g_1(\cdot) + \cdots + W_{m-1}(y)g_{m-1}(\cdot), \tag{20}$$

where $W_i(y) = \dfrac{\partial V_2(y)}{\partial y_i}$, $i = 1,\cdots,m-1$. By homogeneous definition[21], it is easy to verify for $i = 2,\cdots,m$,

$$y_i^*(\varepsilon^{r_1}y_1,\cdots,\varepsilon^{r_{i-1}}y_{i-1}) = \varepsilon^{r_i}y_i^*(y_1,\cdots,y_{i-1})$$

and

$$V_2(\varepsilon^{r_1}y_1,\cdots,\varepsilon^{r_m}y_m) = \varepsilon^{2r_m-\tau}V_2(y).$$

In addition, we can also verify that

$$\begin{aligned} &\xi_i(\varepsilon^{r_1}y_1,\cdots,\varepsilon^{r_i}y_i) = \varepsilon^{r_i}\xi_i(y_1,\cdots,y_i),\\ &W_i(\varepsilon^{r_1}y_1,\cdots,\varepsilon^{r_m}y_m) = \varepsilon^{2r_m-\tau-r_i}W_i(y). \end{aligned} \tag{21}$$

Meanwhile, according to Assumption 1, one obtains

$$|g_i(y_{i+1},\cdots,y_m)| \leqslant \rho(|y_{i+1}|^{\frac{r_{i+1}}{r_{i+1}}} + \cdots + |y_m|^{\frac{r_{i+1}}{r_m}}) \times (|y_{i+1}|^{q_{i,i+1}-\frac{r_{i+1}}{r_{i+1}}} + \cdots + |y_m|^{q_{im}-\frac{r_{i+1}}{r_m}}).$$

Denote

$$\rho_1(y) = \xi_1^{\frac{2r_m}{r_1}} + \cdots + \xi_{m-1}^{\frac{2r_m}{r_{m-1}}} + \xi_m^{\frac{2r_m}{r_m}},$$

$$\rho_2(y) = W_1(y)(|y_2|^{\frac{r_2}{r_2}} + \cdots + |y_m|^{\frac{r_2}{r_m}}) + \cdots + W_{m-1}(y)|y_m|^{\frac{r_m}{r_m}}.$$

Taking (21) into account, we can easily verify that both $\rho_1(y)$ and $\rho_2(y)$ are homogeneous of degree $r = 2r_m$ with respect to the dilation $(r_1,\cdots,r_m)$. By Lemma 2, it can be concluded that there exists a positive constant $\bar{c}$ such that $\rho_2(y) \leqslant \bar{c}\rho_1(y)$. With the help of this relation, (20) becomes

$$\begin{aligned} &\dot{V}_2(y) \leqslant \\ &\rho\rho_2(y)\sum_{i=1}^{m-1}(|y_{i+1}|^{q_{i,i+1}-\frac{r_{i+1}}{r_{i+1}}} + \cdots + |y_m|^{q_{im}-\frac{r_{i+1}}{r_m}}) - \rho_1(y) \leqslant \\ &\bar{c}\rho\rho_1(y)\sum_{i=1}^{m-1}(|y_{i+1}|^{q_{i,i+1}-\frac{r_{i+1}}{r_{i+1}}} + \cdots + |y_m|^{q_{im}-\frac{r_{i+1}}{r_m}}) - \rho_1(y). \end{aligned} \tag{22}$$

Note from (13) that $q_{ij} > \dfrac{r_{i+1}}{r_j}$. So we can find a region

$$\Omega_0 = \{y : V_2(y) \leqslant \lambda_0\},$$

with $\lambda_0 > 0$ such that for all $y \in \Omega_0$,

$$\sum_{i=1}^{m-1}(|y_{i+1}|^{q_{i,i+1}-\frac{r_{i+1}}{r_{i+1}}} + \cdots + |y_m|^{q_{im}-\frac{r_{i+1}}{r_m}}) \leqslant 1/(2\bar{c}\rho).$$

By (22), it can be clearly seen that for all $y \in \Omega_0$,

$$\dot{V}_2(y) \leqslant -\rho_1(y)/2. \tag{23}$$

From Assumption 2, we know $\bar{\mu}(y) = \mathcal{O}(\|y\|_\Delta^{2r_m})$ as y converges to zero. It implies there exists a constant $\check{c} \geqslant 0$ such that

$$\lim_{y\to 0}\frac{\bar{\mu}(y)}{\|y\|_\Delta^{2r_m}} = \check{c}. \tag{24}$$

Note that $\|y\|_\Delta^{2r_m}$ is homogeneous of degree $r = 2r_m$. By Lemma 2 again, we get that there exist positive constants $\bar{c}$ and $\underline{c}$ such that $\|y\|_\Delta^{2r_m} \geqslant \underline{c}\rho_1(\|y\|)$ and $\|y\|_\Delta^{2r_m} \leqslant \bar{c}\rho_1(\|y\|)$. It implies there exists $\hat{c} > 0$ such that

$$\lim_{y\to 0}\frac{\|y\|_\Delta^{2r_m}}{\rho_1(y)} = \hat{c}. \tag{25}$$

It can be concluded that there exists a region $\Omega \subset \Omega_0$ of the origin such that for $\forall y \in \Omega$,

$$\begin{cases} \|y\|_\Delta^{2r_m} \leqslant 2\hat{c}\rho_1(y) = 4\hat{c}\bar{\rho}_1(y),\\ \dot{V}_2(y) \leqslant -\dfrac{1}{2}\rho_1(y) = -\bar{\rho}_1(y). \end{cases} \tag{26}$$

Combining Assumption 2 and (26), it can be concluded from Theorem 2 that the closed-loop system (11),(15) is globally asymptotically stable.

Remark 3 It should be pointed out that Theorem 4 is partially motivated by [4]. However, there are two main differences between [4] and this paper. The first one is that the driving subsystems are different. It can be clearly observed that the driving subsystem of the considered cascaded system in [4] has a lower-triangular structure, while the driving subsystem of the system considered in this paper is an upper-triangular system. The second one is that the assumptions for the driven subsystems are different. The former one is based upon the homogeneous assumption, and the latter one takes the iISS property as the precondition.

4 Illustrative examples

In this section, we will give four examples to show the effectiveness of the above mentioned results.

First of all, an example will be proposed to illustrate Theorem 1. Similar to [2], the Lyapunov function $V_2(y)$ in Theorem 1 is not required to be differential in the origin. This brings some flexibility in stability analysis, which can be reflected in the following example.

Example 1 Consider the following system

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 y_1^{1/3}, \\ \dot{y}_1 = -y_1^{3/5}. \end{cases} \tag{27}$$

Firstly, it can be verified that the subsystem $\dot{y}_1 = -y_1^{3/5}$ is globally asymptotically stable. Choose $V_1(x_1) = \frac{1}{2}\ln(1 + x_1^2)$ and $V_2(y_1) = \frac{3}{2}y_1^{2/3}$. Taking derivatives of $V_1(x_1)$ and $V_2(y_1)$ along system (27) yields

$$\dot{V}_1(x_1) = \frac{-x_1^2 + x_1^2 y_1^{1/3}}{1 + x_1^2} \leqslant \frac{-x_1^2}{1 + x_1^2} + |y_1|^{1/3}$$

and

$$\dot{V}_2(y_1) = -y_1^{4/15}.$$

Note that $4/15 < 1/3$. We conclude that there exists a small region $\{y_1 : \|y_1\| \leqslant 1\}$ such that $y_1^{4/15} > y_1^{1/3}$. According to Theorem 1, the cascaded system (27) is globally asymptotically stable.

Then, we will give an example to verify Theorem 2.

Example 2 Consider the following system

$$\dot{x}_1 = -\arctan(x_1) + x_1 y_2^{5/3}, \tag{28a}$$

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1^{1/5} - y_2^{1/3}. \end{cases} \tag{28b}$$

Usually, it is not easy to find the proper positive-definite fucntions $V_2(y)$ and $\omega(y)$ to test the global asymptotic stability of the driving subsystem (28b), although it is globally asymptotically stable. Therefore, Theorem 1 can not be used to handle cascaded system (28). However, we can show in the sequel that the stability analysis of cascaded system (28) is solvable by Theorem 2.

Let $V_1(x_1) = \frac{1}{2}\ln(1 + x_1^2)$ and $V_2(y_1, y_2) = \frac{5}{6}y_1^{6/5} + \frac{1}{2}y_2^2$. Taking the derivative of $V_1(x_1)$ along the driven subsystem (28a) yields

$$\dot{V}_1(x_1) = \frac{-x_1 \arctan(x_1) + x_1^2 y_2^{5/3}}{1 + x_1^2} \leqslant -\bar{\alpha}(x) + \bar{\mu}(y), \tag{29}$$

with $\bar{\alpha}(x) = \dfrac{x_1 \arctan(x_1)}{1 + x_1^2}$ and $\bar{\mu}(y) = |y_2|^{5/3}$. In addition, we also have for $\forall y \in \mathbb{R}^2$,

$$\dot{V}_2(y) = y_1^{1/5} y_2 + y_2(-y_1^{1/5} - y_2^{1/3}) = -y_2^{4/3} = -\bar{\omega}(y), \tag{30}$$

with $\bar{\omega}(y) = y_2^{4/3}$. Note that the driving subsystem (28b) is globally asymptotically stable. By letting $\Omega = \{y : \|y\| \leqslant 1\}$, it is clear that

$$\bar{\omega}(y) \geqslant \bar{\mu}(y),\ \forall y \in \Omega.$$

Till now, we have verified the sufficient conditions proposed in Theorem 2. According to Theorem 2, cascaded system (28) is globally asymptotically stable.

The following example shows how to verify the finite-time stability of a cascaded system by Theorem3.

Example 3 Consider the following academic example

$$\begin{cases} \dot{x}_1 = x_2 - x_1^{1/3} + x_2 y_1, \\ \dot{x}_2 = -x_1 - x_2^{1/3}, \end{cases} \tag{31a}$$

$$\dot{y}_1 = -y_1^{3/5}. \tag{31b}$$

By choosing $V_1(x_1, x_2) = \frac{1}{2}\ln(1 + x_1^2 + x_2^2)$, we have

$$\dot{V}_1(x_1, x_2)|_{(31a)} = -\frac{x_1^{4/3} + x_2^{4/3}}{1 + x_1^2 + x_2^2} + \frac{x_1 x_2 y_1}{1 + x_1^2 + x_2^2} \leqslant -\frac{x_1^{4/3} + x_2^{4/3}}{1 + x_1^2 + x_2^2} + y_1.$$

It follows that the driven subsystem (31a) is iISS. In addition, taking $V_0(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ and $V_2(y_1) = \frac{1}{2}y_1^2$, according to the finite-time Lyapunov theory in [24], it is easy to verify that subsystem

$$\dot{x}_1 = x_2 - x_1^{1/3},$$
$$\dot{x}_2 = -x_1 - x_2^{1/3},$$

and (31b) are globally finite-time stable. By Theorem 3, cascaded system (31) is globally finite-time stable.

Finally, we show how to use Theorem 4 to design a controller for a cascaded system with upper-triangular driving subsystem.

Example 4 Consider the following system

$$\begin{cases} \dot{x}_1 = -x_2 - x_1 + x_1^{1/3} y_3^2, \\ \dot{x}_2 = x_1 - x_2^{5/3} + x_2 y_2^4, \end{cases} \tag{32a}$$

$$\begin{cases} \dot{y}_1 = y_2 + y_2^2 + y_3^{2/3}, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = u. \end{cases} \tag{32b}$$

Let $\tau = 2$. It yields $r_1 = 1, r_2 = 3, r_3 = 5, r_4 = 7$. By taking $q_{12} = 3/2 > r_2/r_2 = 1, q_{13} = 2/3 > r_2/r_3 = 3/5$, one can easily verify that Assumption 1 holds. Then, the controller can be designed as

$$u = -\beta_3 \sigma^{\frac{7}{5}} (y_3 + \beta_2 \sigma^{5/3} (y_2 + \beta_1 \sigma^3 (y_1))). \tag{33}$$

In addition, let $V_1(x_1, x_2) = \dfrac{1}{2} \ln(1 + x_1^2 + x_2^2)$. Taking the derivative of $V_1(x_1, x_2)$ along subsystem (32a) yields

$$\dot{V}_1(x_1, x_2) = \frac{-x_1^2 - x_2^{8/3} + x_1^{4/3} y_3^2 + x_2^2 y_2^4}{1 + x_1^2 + x_2^2} \leqslant -\frac{x_1^2 + x_2^{8/3}}{1 + x_1^2 + x_2^2} + y_2^4 + y_3^2.$$

Let $\bar{\alpha}(x) = \dfrac{x_1^2 + x_2^{8/3}}{1 + x_1^2 + x_2^2}, \bar{\mu}(y) = y_2^4 + y_3^2$. Note that

$$\|y\|_\Delta = (|y_1|^{2/r_1} + |y_2|^{2/r_2} + |y_3|^{2/r_3})^{1/2} = (|y_1|^2 + |y_2|^{2/3} + |y_3|^{2/5})^{1/2}.$$

It can be easily verified that $\bar{\mu}(y) = \mathcal{O}(\|y\|_\Delta^6)$, which implies Assumption 2 holds. By Theorem 4, we conclude that system (32) can be globally stabilized by controller (33).

5 Conclusion

This paper has studied the stability analysis problem for a class of non-Lipschitz continuous cascaded systems. During the stability analysis, the iISS property of the driven subsystem plays an important role. It has been shown that the iISS of the driving subsystem, global stability of the driving subsystem plus a matching condition imply the global stability of the non-Lipschitz continuous cascaded systems. Furthermore, a special case of global stability, i.e., the finite-time stability for the non-Lipschitz continuous cascaded system are also studied. Finally, as an application, the control design problem for a class of cascaded system with driving subsystem being upper-triangular structure has been presented.

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