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# 切换奇异布尔网络的稳定性分析

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**摘要**: 基因调控网络的稳定性分析是系统生物学的研究热点问题之一. 本文利用矩阵半张量积方法研究了切换奇异布尔网络的稳定性问题. 首先给出了切换奇异布尔网络的代数表示, 基于该代数表示, 建立了系统解存在唯一的充要条件. 然后通过将切换奇异布尔网络转化为等价的切换布尔网络, 分别得到了系统在任意切换下稳定以及切换可稳的充要条件. 最后给出例子验证所得结果的有效性.

**关键词**: 系统生物学; 切换奇异布尔网络; 稳定; 矩阵半张量积

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# Stability analysis for switched singular Boolean networks

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**Abstract:** The stability analysis of gene regulatory networks is a hot topic in systems biology. We investigate the stability of switched singular Boolean networks (SSBNs) by using the semi-tensor product of matrices. First, the dynamics of SSBNs is converted to an algebraic form, based on which a necessary and sufficient condition is established for the uniqueness of solution of SSBNs. Second, several necessary and sufficient conditions are presented for the stability of SSBNs under arbitrary switching signal and the switching stabilizability of SSBNs, respectively, by converting an SSBN into an equivalent switched Boolean network. Two illustrative examples are presented to show that the main results obtained in this paper are effective in analyzing the stability of SSBNs.

**Key words:** systems biology; switched singular Boolean network; stability; semi-tensor product of matrices

## 1 Introduction

Boolean network is a proper model in the study of gene regulatory networks[1]. In this model, the gene expression is quantized as ‘1’ or ‘0’ to represent active or inactive. Unlike continuous model of gene regulatory networks that contains several parameters, the dynamics of Boolean networks is parameter free and much simpler, which has been extensively studied in many excellent works[2–4].

Recently, the semi-tensor product of matrices was proposed by Cheng[5–6] to analyze Boolean networks. Using this tool, Cheng and his colleagues convert the logic dynamics of a Boolean network into an equivalent algebraic form, which has the same form as a linear system. Based on the algebraic form, many fundamental results on the analysis and control of Boolean networks have been presented, which include the controllability and observability[7–13], the stability and stabilization[14–20], the disturbance decoupling[21–23], the optimal control[24–25] and the synchronization[26–27]. Besides, the semi-tensor product method has also been used in many other fields such as the general logical system[28–29], the fuzzy control[30–32], the calculation of Boolean derivative[33–34], the finite automata[35–36], the graph coloring[37] and the game theory[38–39].

It should be pointed out that, due to the external interventions and the asynchronous dynamics, the dynamics of gene regulatory networks in practice is often governed by different switching models. A typical example is the genetic switch in the bacteriophage $\lambda$, which contains two distinct models: lysis and lysogeny[40]. Boolean networks with switching models are called switched Boolean networks (SBNs), which have been studied in some recent works[17,26,41–44]. On the other hand, the singular Boolean network, which is a generalization of ordinary singular systems[45–48] to Boolean networks, was firstly proposed in[49] and then studied in[50]. Although there are many results on switched Boolean networks and singular Boolean networks, respectively, there are, to our best knowledge, fewer results on the study of switched singular Boolean networks (SSBNs). In fact, this is a very challenging topic and the existing methods on ordinary switched systems[51–55] can hardly be used.

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In this paper, using the semi-tensor product method, we investigate the stability of switched singular Boolean networks. Firstly, we convert the dynamics of SSBNs into an algebraic form, and establish a necessary and sufficient condition for the uniqueness of solution of SSBNs. Secondly, we propose the concept of switching point reachability and obtain a necessary and sufficient condition for the switching point reachability by converting the SSBN into an equivalent switched Boolean network. Thirdly, based on the switching point reachability, we present several necessary and sufficient conditions for the stability of SSBNs under arbitrary switching signal and the switching stabilizability of SSBNs, respectively.

The rest of the paper is organized as follows. Section 2 gives some necessary preliminaries on the semi-tensor product of matrices. Section 3 studies the stability of SSBNs and presents the main results of this paper. Two illustrative examples are given to show the effectiveness of the main results in Section 4, which is followed by a brief conclusion in Section 5.

## 2 Preliminaries

First, we introduce some notations, which will be used in the sequel.

· $\delta_k^i$ denotes the $i$-th column of the identity matrix $I_k$.

· $\Delta_k := \{\delta_k^i | i = 1, 2, \cdots, k\}$, and $\Delta := \Delta_2$.

· $\mathbf{1}_n := [\underbrace{1 \ \cdots \ 1}_{n}]$.

· $\mathcal{D} := \{1, 0\}$. To use the matrix expression, '1' and '0' are identified as $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, respectively, where '$\sim$' denotes two different forms of the same object.

· An $n \times t$ matrix $A$ is called a logical matrix, if $A = [\delta_n^{i_1} \ \delta_n^{i_2} \ \cdots \ \delta_n^{i_t}]$. We express $A$ briefly as $A = \delta_n[i_1 \ i_2 \ \cdots \ i_t]$. Denote the set of $n \times t$ logical matrices by $\mathcal{L}_{n\times t}$.

· $\mathrm{Col}_i(A)$ and $Row_j(A)$ denote the $i$-th column and the $j$-th row of the matrix $A$, respectively. The set of columns of $A$ is denoted by $\mathrm{Col}(A)$. $(A)_{ij}$ denotes the $(i, j)$-th entry of $A$.

Next, we recall some definitions and basic properties on the semi-tensor product of matrices. For details, please refer to [5].

**Definition 1** The semi-tensor product of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{1}$$

where $\alpha = lcm(n, p)$ is the least common multiple of $n$ and $p$, and $\otimes$ is the Kronecker product.

**Remark 1** One can see that when $n = p$, the semi-tensor product of matrices becomes the conventional matrix product. Thus, we can omit '$\ltimes$' if no confusion raises.

**Lemma 1** The semi-tensor product has the following properties:

i) Let $X \in \mathbb{R}^{t\times 1}$ be a column vector and $A \in \mathbb{R}^{m\times n}$. Then $X \ltimes A = (I_t \otimes A) \ltimes X$.

ii) Let $X \in \mathbb{R}^{m\times 1}$ and $Y \in \mathbb{R}^{n\times 1}$ be two column vectors. Then $Y \ltimes X = W_{[m,n]} \ltimes X \ltimes Y$, where $W_{[m,n]} \in \mathbb{R}^{mn\times mn}$ is the swap matrix.

In the following, we present a fundamental result on the matrix expression of logical functions, which is based on the semi-tensor product of matrices.

**Lemma 2** Let $f(x_1, x_2, \cdots, x_s) : \mathcal{D}^s \mapsto \mathcal{D}$ be a Boolean function. Then, there exists a unique matrix $M_f \in \mathcal{L}_{2\times 2^s}$, called the structural matrix of $f$, such that

$$f(x_1, x_2, \cdots, x_s) = M_f \ltimes_{i=1}^{s} x_i, \ x_i \in \Delta, \tag{2}$$

where $\ltimes_{i=1}^{s} x_i = x_1 \ltimes \cdots \ltimes x_s$.

Finally, we list the structural matrices of some basic logical operators which will be used later.

Negation ($\neg$): $M_n = \delta_2[2 \ 1]$; Conjunction ($\wedge$): $M_c = \delta_2[1 \ 2 \ 2 \ 2]$; Disjunction ($\vee$): $M_d = \delta_2[1 \ 1 \ 1 \ 2]$; Conditional ($\rightarrow$): $M_i = \delta_2[1 \ 2 \ 1 \ 1]$; Biconditional ($\leftrightarrow$): $M_e = \delta_2[1 \ 2 \ 2 \ 1]$; Exclusive Or ($\bar{\vee}$): $M_p = \delta_2[2 \ 1 \ 1 \ 2]$.

## 3 Main results

This section studies the stability of SSBNs, and presents the main results of this paper. First, the problem formulation is presented. Then the SSBN is converted into an equivalent SBN. Finally, based on the equivalent SBN, the stability of SSBNs under arbitrary switching signal and the switching stabilizability of SSBNs are investigated, respectively.

### 3.1 Problem formulation

Consider the following switched Boolean network with $n$ nodes and $m$ models:

$$\begin{cases} g_1^{\sigma(t)}(X(t+1)) = f_1^{\sigma(t)}(X(t)), \\ g_2^{\sigma(t)}(X(t+1)) = f_2^{\sigma(t)}(X(t)), \\ \quad \vdots \\ g_n^{\sigma(t)}(X(t+1)) = f_n^{\sigma(t)}(X(t)), \end{cases} \tag{3}$$

where $\sigma : \mathbb{N} \mapsto \mathcal{A} = \{1, 2, \cdots, m\}$ is the switching signal, $X(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathcal{D}^n$, and $f_i^j, g_i^j : \mathcal{D}^n \mapsto \mathcal{D}, \ i = 1, \cdots, n, \ j = 1, 2, \cdots, m$ are Boolean functions.

Using the vector form of logical variables and setting $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n}$, by Lemma 2, the SBN (3) can be expressed as

$$\begin{cases} Q_1^{\sigma(t)} x(t+1) = W_1^{\sigma(t)} x(t), \\ Q_2^{\sigma(t)} x(t+1) = W_2^{\sigma(t)} x(t), \\ \quad \vdots \\ Q_n^{\sigma(t)} x(t+1) = W_n^{\sigma(t)} x(t), \end{cases} \tag{4}$$

where $Q_i^{\sigma(t)} \in \mathcal{L}_{2\times 2^n}$ and $W_i^{\sigma(t)} \in \mathcal{L}_{2\times 2^n}$ are uniquely determined by $g_i^{\sigma(t)}$ and $f_i^{\sigma(t)}$, respectively. Multiplying the equations in (4) together yields the following algebraic form:

$$E_{\sigma(t)}x(t+1)=L_{\sigma(t)}x(t), \quad (5)$$

where $E_{\sigma(t)}, L_{\sigma(t)} \in \mathcal{L}_{2^n\times 2^n}$,

$$\mathrm{Col}_i(E_{\sigma(t)})=\ltimes_{j=1}^{n}\mathrm{Col}_i(Q_j^{\sigma(t)}),\ i=1,\cdots,2^n,$$

and

$$\mathrm{Col}_i(L_{\sigma(t)})=\ltimes_{j=1}^{n}\mathrm{Col}_i(W_j^{\sigma(t)}),\ i=1,\cdots,2^n.$$

When $\mathrm{rank}(E_i)<2^n,\ \forall\ i\in\mathcal{A}$, the system (3) is called switched singular Boolean network. In this case, (5) has the same form as the ordinary switched singular system[46,48]. Throughout this paper, we assume that $\mathrm{rank}(E_i)<2^n,\ \forall\ i\in\mathcal{A}$.

**Remark 2** It is noted that when $\mathrm{rank}(E_i)=2^n$, one can convert $E_ix(t+1)=L_ix(t)$ to $x(t+1)=(E_i^{-1}L_i)x(t)$, which is the algebraic form of a Boolean network.

**Remark 3** The system (3) and its algebraic form (5) are equivalent. One can obtain the logical form (3) from the algebraic form (5) by the following procedure:

1) Calculate $Q_i^{\sigma(t)}$ and $W_i^{\sigma(t)}$ from $E_{\sigma(t)}$ and $L_{\sigma(t)}$, respectively, as

$$Q_i^{\sigma(t)}=S_i^nE_{\sigma(t)},\ W_i^{\sigma(t)}=S_i^nL_{\sigma(t)}, \quad (6)$$

where $S_i^n=\mathbf{1}_{2^{i-1}}\otimes I_2\otimes\mathbf{1}_{2^{n-i}},\ i=1,2,\cdots,n$.

2) Partition $Q_i^{\sigma(t)}\in\mathcal{L}_{2\times 2^n}$ as

$$Q_i^{\sigma(t)}=[Q_{i,1}^{\sigma(t)}\ \ Q_{i,2}^{\sigma(t)}],$$

where $Q_{i,1}^{\sigma(t)},Q_{i,2}^{\sigma(t)}\in\mathcal{L}_{2\times 2^{n-1}}$. Then,

$$g_i^{\sigma(t)}(x_1,x_2,\cdots,x_n)=$$
$$(x_1\wedge g_{i,1}^{\sigma(t)}(x_2,\cdots,x_n))\vee(\neg x_1\wedge g_{i,2}^{\sigma(t)}(x_2,\cdots,x_n)),$$

where $Q_{i,1}^{\sigma(t)}$ and $Q_{i,2}^{\sigma(t)}$ are structural matrices for $g_{i,1}^{\sigma(t)}$ and $g_{i,2}^{\sigma(t)}$, respectively. Repeating this procedure, one can obtain $g_i^{\sigma(t)}$. $f_i^{\sigma(t)}$ can be obtained from $W_i^{\sigma(t)}$ by using the same procedure.

We give two examples to show the dynamics of switched singular Boolean networks.

**Example 1** Consider a game between a dealer and two players, and assume that the dealer and two players choose a bet from $\mathcal{D}$, respectively[41]. Denote by $x_i(t),\ i=1,2$ and $u(t)$ the action of the players and the dealer at the $t$-th step, respectively. Moreover, we assume that at each time, at least one of $x_i(t),\ i=1,2$ and $u(t)$ takes the bet '1'. Then, the dynamics of the game given in [41] becomes the following SSBN:

$$\begin{cases} g_1^{\sigma(t)}(X(t+1))=f_1^{\sigma(t)}(X(t)),\\ g_2^{\sigma(t)}(X(t+1))=f_2^{\sigma(t)}(X(t)),\\ g_3^{\sigma(t)}(X(t+1))=f_3^{\sigma(t)}(X(t)),\end{cases} \quad (7)$$

where $\sigma:\mathbb{N}\mapsto\{1,2\}$ is the switching signal, $X(t)=(x_1(t),x_2(t),u(t))$, $g_1^1=g_1^2=x_1$, $g_2^1=g_2^2=x_2$, $g_3^1=g_3^2=1$, $f_1^1=x_1\vee u$, $f_1^2=x_1\wedge x_2\wedge u$, $f_2^1=x_1\leftrightarrow x_2$, $f_2^2=\neg x_1\vee x_2\vee u$, and $f_3^1=f_3^2=x_1\vee x_2\vee u$.

**Example 2** Consider the following apoptosis network[4]:

$$\begin{cases} x_1(t+1)=\neg x_2(t)\wedge u(t),\\ x_2(t+1)=\neg x_1(t)\wedge x_3(t),\\ x_3(t+1)=x_2(t)\vee u(t),\end{cases} \quad (8)$$

where the concentration level (high or low) of the inhibitor of apoptosis proteins (IAP) is denoted by $x_1$, the concentration level of the active caspase 3 (C3a) by $x_2$, and the concentration level of the active caspase 8 (C8a) by $x_3$; the concentration level of the tumor necrosis factor (TNF, a stimulus) is regarded as the control input $u$.

When modeling the system (8) as the deterministic asynchronous Boolean network and keeping TNF in the high concentration level, one can convert it into the following SSBN:

$$\begin{cases} x_1(t+1)=f_1^{\sigma(t)}(x_1(t),x_2(t),x_3(t),u(t)),\\ x_2(t+1)=f_2^{\sigma(t)}(x_1(t),x_2(t),x_3(t),u(t)),\\ x_3(t+1)=f_3^{\sigma(t)}(x_1(t),x_2(t),x_3(t),u(t)),\\ 1=u(t),\end{cases} \quad (9)$$

where $\sigma:\mathbb{N}\mapsto\{1,2,\cdots,8\}$ is the switching signal, and

$$\begin{aligned} &f_1^1=x_1,\ f_2^1=x_2,\ f_3^1=x_3,\\ &f_1^2=\neg x_2\wedge u,\ f_2^2=x_2,\ f_3^2=x_3,\\ &f_1^3=x_1,\ f_2^3=\neg x_1\wedge x_3,\ f_3^3=x_3,\\ &f_1^4=x_1,\ f_2^4=x_2,\ f_3^4=x_2\vee u,\\ &f_1^5=\neg x_2\wedge u,\ f_2^5=\neg x_1\wedge x_3,\ f_3^5=x_3,\\ &f_1^6=\neg x_2\wedge u,\ f_2^6=x_2,\ f_3^6=x_2\vee u,\\ &f_1^7=x_1,\ f_2^7=\neg x_1\wedge x_3,\ f_3^7=x_2\vee u,\\ &f_1^8=\neg x_2\wedge u,\ f_2^8=\neg x_1\wedge x_3,\ f_3^8=x_2\vee u.\end{aligned}$$

Next, we give a necessary and sufficient condition for the uniqueness of solution of the SSBN (3) under arbitrary switching signal.

**Lemma 3** The solution of the system (3) is unique for any initial point and arbitrary switching signal, if and only if the following two conditions hold:

A1) $\mathrm{rank}([E_i\ L_i])=\mathrm{rank}(E_i),\ \forall\ i\in\mathcal{A}$;

A2) $\sum_{k=1}^{2^n}(L_i)_{jk}\neq 0\Rightarrow\sum_{k=1}^{2^n}(E_i)_{jk}=1,\ \forall\ i\in\mathcal{A},\ \forall\ j=1,2,\cdots,2^n$.

**Proof** It is easy to see that the system (3) has a unique solution for any initial point and arbitrary switching signal, if and only if for any $i\in\mathcal{A}$, the singular Boolean network $E_ix(t+1)=L_ix(t)$ has a unique solution for any initial point. Based on Theorem 6 in [50], the conclusion follows.

In the following, we always assume that A1) and A2) hold. The objective of this paper is to study the following two issues:

1) (stability under arbitrary switching signal) establishing a necessary and sufficient condition for the stability of the SSBN (3) under arbitrary switching signal;

2) (state feedback consistent stabilizability) designing a state feedback switching signal under which the SSBN (3) is consistently stabilizable to an equilibrium $X_{\rm e} = (x_1^{\rm e}, \cdots, x_n^{\rm e})$ (or in the vector form $x_{\rm e} = \ltimes_{i=1}^n x_i^e = \delta_{2^n}^{\mu}$).

### 3.2 Switching point reachability

This part introduces a concept of switching point reachability for SSBNs, which is an important tool for the stability analysis.

**Definition 2** (Switching point reachability) Consider the system (3). Let $X_0 = (x_1(0), \cdots, x_n(0)) \in \mathcal{D}^n$. Then, a point $X = (x_1, \cdots, x_n) \in \mathcal{D}^n$ is said to be switching reachable from $X_0$, if one can find an integer $k > 0$ and a switching signal $\sigma$, such that under the switching signal, the trajectory of the system (3) starting from $X_0$ reaches $X$ at time $k$.

To facilitate the analysis, we convert the system (5) into an equivalent switched Boolean network.

For each $i \in \mathcal{A}$, define $\hat{L}_i \in \mathcal{L}_{2^n \times 2^n}$ as

$$\mathrm{Col}_j(\hat{L}_i) = \delta_{2^n}^{k_j^i}, \text{ if } \mathrm{Col}_j(L_i) = \mathrm{Col}_{k_j^i}(E_i), \quad (10)$$
$$\forall\, j = 1, 2, \cdots, 2^n.$$

Then, we have the following result.

**Lemma 4** Assume that A1) and A2) hold. The system (5) is equivalent to the following switched Boolean network:

$$x(t+1) = \hat{L}_{\sigma(t)} x(t). \quad (11)$$

**Proof** For any initial point $x(0) = \delta_{2^n}^j$ and any switching signal $\sigma(t)$, denote the solution to the system (5) by $x(t; x(0), \sigma)$, and the solution to the system (11) by $\hat{x}(t; x(0), \sigma)$. We need to show that

$$x(t; x(0), \sigma) = \hat{x}(t; x(0), \sigma), \ \forall\, t \in \mathbb{Z}_+.$$

Next, we prove it by induction.

When $t = 1$, a simple calculation shows that

$$\hat{x}(1; x(0), \sigma) = \hat{L}_{\sigma(0)} x(0) = \mathrm{Col}_j(\hat{L}_{\sigma(0)}) = \delta_{2^n}^{k_j^{\sigma(0)}}.$$

On the other hand, since $E_{\sigma(0)} x(1; x(0), \sigma) = L_{\sigma(0)} x(0) = \mathrm{Col}_j(L_{\sigma(0)}) = \mathrm{Col}_{k_j^{\sigma(0)}}(E_{\sigma(0)})$, we have $x(1; x(0), \sigma) = \delta_{2^n}^{k_j^{\sigma(0)}} = \hat{x}(1; x(0), \sigma)$. Thus, $x(t; x(0), \sigma) = \hat{x}(t; x(0), \sigma)$ holds for $t = 1$.

Assume that the conclusion holds for $t = k$. Moreover, we set $x(k; x(0), \sigma) = \hat{x}(k; x(0), \sigma) = \delta_{2^n}^{j_1}$. We now consider the case of $t = k + 1$. In this case, for the system (5), since

$$E_{\sigma(k)} x(k+1; x(0), \sigma) = L_{\sigma(k)} x(k; x(0), \sigma) =$$
$$\mathrm{Col}_{j_1}(L_{\sigma(k)}) = \mathrm{Col}_{k_{j_1}^{\sigma(k)}}(E_{\sigma(k)}),$$

one can see that $x(k+1; x(0), \sigma) = \delta_{2^n}^{k_{j_1}^{\sigma(k)}}$. For the system (11), it is easy to obtain that

$$\hat{x}(k+1; x(0), \sigma) = \hat{L}_{\sigma(k)} x(k; x(0), \sigma) =$$
$$\mathrm{Col}_{j_1}(\hat{L}_{\sigma(k)}) = \delta_{2^n}^{k_{j_1}^{\sigma(k)}},$$

which implies that $x(k+1; x(0), \sigma) = \hat{x}(k+1; x(0), \sigma)$.

By induction, $x(t; x(0), \sigma) = \hat{x}(t; x(0), \sigma)$ holds for any $t \in \mathbb{Z}_+$.

Based on Lemma 4, and similar to the proof of Theorem 1 in [42], we have the following result on the switching point reachability of the system (3).

**Theorem 1** Assume that A1) and A2) hold. Then,

1) $x = \delta_{2^n}^p$ is switching reachable from $x(0) = \delta_{2^n}^q$ at time $k$, if and only if

$$(\hat{M}^k)_{pq} > 0, \quad (12)$$

where $\hat{M} = \sum\limits_{i=1}^m \hat{L}_i$, and $\hat{L}_i$ is defined in (10);

2) $x = \delta_{2^n}^p$ is switching reachable from $x(0) = \delta_{2^n}^q$, if and only if

$$\mathcal{R}_{pq} > 0, \quad (13)$$

where

$$\mathcal{R} = \sum_{k=1}^{2^n} \hat{M}^k. \quad (14)$$

### 3.3 Stability under arbitrary switching signal

Based on the switching point reachability, this subsection studies the stability of the system (3) under arbitrary switching signal. To this end, we need the following result.

**Lemma 5** Let $\hat{M} = \sum\limits_{i=1}^m \hat{L}_i$. Then,

$$\sum_{i=1}^{2^n} \left(\hat{M}^k\right)_{ij} = m^k, \ \forall\, j = 1, 2, \cdots, 2^n \quad (15)$$

holds for any $k \in \mathbb{Z}_+$, where $m$ is the number of subnetworks of the system (3).

**Proof** The proof of this lemma is similar to that of Proposition 4 in [42], and thus we omit it.

Lemma 5 tells us that starting from any initial point and under arbitrary switching signal, there are $m^k$ paths at time $k$. On the other hand, since the system (3) has $2^n$ different points in the state space, one can see that if the system (3) is globally stable at $x_{\rm e} = \delta_{2^n}^{\mu}$ under arbitrary switching signal, then, the trajectory starting from any initial point reaches $x_{\rm e}$ within time $2^n$ under any switching signal. Based on the above analysis, we have the following result.

**Theorem 2** Assume that A1) and A2) hold. Then, the system (3) is globally stable at $x_e = \delta_{2^n}^{\mu}$ under arbitrary switching signal, if and only if there exists a positive integer $k^* \leqslant 2^n$ such that

$$\mathrm{Row}_{\mu}(\hat{M}^{k^*}) = m^{k^*}\mathbf{1}_{2^n}, \tag{16}$$

where $\hat{M} = \sum_{i=1}^{m}\hat{L}_i$, and $m$ is the number of subnetworks for the system (3).

### 3.4 State feedback consistent stabilizability

In this part, based on the switching point reachability, we investigate the state feedback switching signal design for the consistent stabilizability of the system (3). Noting that the system (3) is equivalent to the system (11), we study this problem for the system (11).

Identifying $\sigma(t) \in \mathcal{A} \sim \Delta_m$, we have $\sigma(t) = i \sim \delta_m^i$. Let $\hat{L} = [\hat{L}_1 \ \cdots \ \hat{L}_m] = \delta_{2^n}[i_1 \ i_2 \ \cdots \ i_{m2^n}] \in \mathcal{L}_{2^n \times m2^n}$. For $x_e = \delta_{2^n}^{\mu}$ and $k \in \mathbb{Z}_+$, denote by $R_k(x_e)$ the set of all the initial states of the system (11) which reach $x_e$ at the $k$-th step, that is,

$$R_k(x_e) = \{x_0 \in \Delta_{2^n} : \text{there exists a switching signal } \sigma(t) \text{ such that } x(k; x_0, \sigma) = x_e\}. \tag{17}$$

Then, we have the following result.

**Theorem 3** Assume that A1) and A2) hold. The system (3) is consistently stabilizable to $x_e = \delta_{2^n}^{\mu}$ by a state feedback switching signal, if and only if there exists an integer $1 \leqslant \tau \leqslant 2^n$ such that

$$\begin{cases} x_e \in R_1(x_e), \\ R_{\tau}(x_e) = \Delta_{2^n}. \end{cases} \tag{18}$$

**Proof** Sufficiency. Assuming that (18) holds, we prove that the system (3) is consistently stabilizable to $x_e$ by a constructed state feedback switching signal.

Set

$$R_k^{\circ}(x_e) = R_k(x_e) \setminus R_{k-1}(x_e), \ k = 1, \cdots, \tau, \tag{19}$$

where $R_0(x_e) := \emptyset$.

One can see that $R_{k_1}^{\circ}(x_e) \cap R_{k_2}^{\circ}(x_e) = \varnothing, \forall\, k_1, k_2 \in \{1, \cdots, \tau\}$, $k_1 \neq k_2$, and $\bigcup_{k=1}^{\tau} R_k^{\circ}(x_e) = \Delta_{2^n}$. Thus, for any integer $1 \leqslant j \leqslant 2^n$, there exists a unique integer $1 \leqslant k_j \leqslant \tau$ such that $\delta_{2^n}^j \in R_{k_j}^{\circ}(x_e)$.

For $k_j = 1$, there exists an integer $1 \leqslant p_j \leqslant m$ such that $\delta_{2^n}^{i_{(p_j-1)2^n+j}} = x_e$; for $2 \leqslant k_j \leqslant \tau$, there exists an integer $1 \leqslant p_j \leqslant m$ such that $\delta_{2^n}^{i_{(p_j-1)2^n+j}} \in R_{k_j-1}(x_e)$.

Now, we set $G = \delta_m[p_1 \ p_2 \ \cdots \ p_{2^n}] \in \mathcal{L}_{m \times 2^n}$. Then, under the state feedback switching signal $\sigma(t) = Gx(t)$, along the trajectory of the system (11) starting from any initial state $x(0) = \delta_{2^n}^j \in \Delta_{2^n}$, it is easy to see that if $k_j = 1$, $x(1) = \hat{L}Gx(0)x(0) = \delta_{2^n}^{i_{(p_j-1)2^n+j}} = x_e$; otherwise, if $2 \leqslant k_j \leqslant \tau$, $x(1) = \hat{L}Gx(0)x(0) = \delta_{2^n}^{i_{(p_j-1)2^n+j}} \in R_{k_j-1}(x_e)$. Thus, $x(k_j) = x_e$, $\forall\, 1 \leqslant j \leqslant 2^n$. Since $x_e \in R_1(x_e)$, one can see that

$$x(t) = x_e, \ \forall\, t \geqslant \tau,$$

which together with Lemma 4 imply that the system (3) is consistently stabilizable to $x_e$ by the state feedback switching signal $\sigma(t) = Gx(t)$.

Necessity. Suppose that the system (3) is consistently stabilizable to $x_e$ by a state feedback switching signal, say, $\sigma(t) = Gx(t)$, $G \in \mathcal{L}_{m \times 2^n}$. Then, the closed-loop system consisting of the system (11) and $\sigma(t) = Gx(t)$ becomes

$$x(t+1) = \widetilde{L}x(t), \tag{20}$$

where $\widetilde{L} = \hat{L}G\varPhi_n$, and $\varPhi_n = \mathrm{diag}\{\delta_{2^n}^1, \delta_{2^n}^2, \cdots, \delta_{2^n}^{2^n}\} \in \mathcal{L}_{2^{2n} \times 2^n}$.

Obviously, the Boolean network (20) is globally stable at $x_e$. Thus, $x_e \in R_1(x_e)$. Let $T_t \leqslant 2^n$ be the transient time[5] of the system (20). Then, it is easy to see that (18) holds for $\tau = T_t \leqslant 2^n$. This completes the proof.

**Remark 4** One can check (18) via $\hat{M} = \sum_{i=1}^{m}\hat{L}_i$. Specifically, $x_e \in R_1(x_e)$ if and only if $\hat{M}_{\mu,\mu} > 0$, and $R_{\tau}(x_e) = \Delta_{2^n}$ if and only if $\mathrm{Row}_{\mu}(\hat{M}^{\tau})$ has no zero columns.

From the proof of Theorem 3, we can design state feedback switching signals as follows:

**Theorem 4** Let $\hat{L} = \delta_{2^n}[i_1 \ i_2 \ \cdots \ i_{m2^n}]$ be given. Suppose that there exists an integer $1 \leqslant \tau \leqslant 2^n$ such that (18) holds. For each integer $1 \leqslant j \leqslant 2^n$ which corresponds to a unique integer $1 \leqslant k_j \leqslant \tau$ such that $\delta_{2^n}^j \in R_{k_j}^{\circ}(x_e)$, let $1 \leqslant p_j \leqslant m$ be such that

$$\begin{cases} \delta_{2^n}^{i_{(p_j-1)2^n+j}} = x_e, \ \ k_j = 1, \\ \delta_{2^n}^{i_{(p_j-1)2^n+j}} \in R_{k_j-1}(x_e), \ 2 \leqslant k_j \leqslant \tau. \end{cases} \tag{21}$$

Then, the state feedback switching signal can be designed as $\sigma(t) = Gx(t)$ with

$$G = \delta_m[p_1 \ p_2 \ \cdots \ p_{2^n}]. \tag{22}$$

## 4 Illustrative examples

This section presents two illustrative examples to show how to use the results obtained in this paper to check the stability of SSBNs.

**Example 3** Consider the following SSBN:

$$\begin{cases} g_1^{\sigma(t)}(X(t+1)) = f_1^{\sigma(t)}(X(t)), \\ g_2^{\sigma(t)}(X(t+1)) = f_2^{\sigma(t)}(X(t)), \end{cases} \tag{23}$$

where $X(t) = (x_1(t), x_2(t))$, $g_1^1 = x_1 \vee x_2$, $g_2^1 = x_2$, $f_1^1 = \neg x_1 \wedge x_2$, $f_2^1 = 0$, $g_1^2 = \neg x_1 \wedge \neg x_2$, $g_2^2 = x_1 \wedge x_2$, $f_1^2 = x_1 \leftrightarrow x_2$ and $f_2^2 = x_1 \bar{\vee} x_2$. The objective is to check whether or not the system (23) is globally stable at $X_e = (0, 0)$ under arbitrary switching signal.

First, we can convert the system (23) into the following algebraic form:

$$E_{\sigma(t)}x(t+1) = L_{\sigma(t)}x(t), \tag{24}$$

where $E_1 = \delta_4[1\ 3\ 1\ 4]$, $L_1 = \delta_4[4\ 4\ 3\ 4]$, $E_2 = \delta_4[3\ 4\ 4\ 2]$ and $L_2 = \delta_4[2\ 3\ 3\ 2]$. Moreover, $X_{\rm e} \sim x_{\rm e} = \delta_4^4$. Obviously, (A1) and (A2) hold for the system (23).

Second, based on Lemma 4, we have the following equivalent system for (24):

$$x(t+1) = \hat{L}_{\sigma(t)}x(t), \tag{25}$$

where $\hat{L}_1 = \delta_4[4\ 4\ 2\ 4]$ and $\hat{L}_2 = \delta_4[4\ 1\ 1\ 4]$.

Set $\hat{M} = \hat{L}_1 + \hat{L}_2$. A simple calculation shows that

$$\hat{M}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 \end{bmatrix}.$$

By Theorem 2, the system (23) is globally stable at $x_{\rm e} = \delta_4^4 \sim X_{\rm e} = (0, 0)$ under arbitrary switching signal.

**Example 4** Consider the following SSBN:

$$\begin{cases} g_1^{\sigma(t)}(X(t+1)) = f_1^{\sigma(t)}(X(t)), \\ g_2^{\sigma(t)}(X(t+1)) = f_2^{\sigma(t)}(X(t)), \end{cases} \tag{26}$$

where $X(t) = (x_1(t), x_2(t))$, $g_1^1 = f_1^1 = \neg x_1 \vee \neg x_2$, $g_2^1 = \neg x_2$, $f_2^1 = 0$, $g_1^2 = x_1 \wedge x_2$, $g_2^2 = \neg x_1 \wedge \neg x_2$, $f_1^2 = \neg x_1 \wedge x_2$ and $f_2^2 = x_1 \vee \neg x_2$. Our objective is to design a state feedback switching signal which consistently stabilizes the system (26) to $X_{\rm e} = (1, 1)$.

The system (26) can be converted into the following algebraic form:

$$E_{\sigma(t)}x(t+1) = L_{\sigma(t)}x(t), \tag{27}$$

where $E_1 = \delta_4[4\ 1\ 2\ 1]$, $L_1 = \delta_4[4\ 2\ 2\ 2]$, $E_2 = \delta_4[2\ 4\ 4\ 3]$ and $L_2 = \delta_4[3\ 3\ 2\ 3]$. Moreover, $X_{\rm e} \sim x_{\rm e} = \delta_4^1$. It is easy to see that the system (26) satisfies A1) and A2).

Based on Lemma 4, we have the following equivalent system for (27):

$$x(t+1) = \hat{L}_{\sigma(t)}x(t), \tag{28}$$

where $\hat{L}_1 = \delta_4[1\ 3\ 3\ 3]$ and $\hat{L}_2 = \delta_4[4\ 4\ 1\ 4]$.

It is easy to see that $R_1(x_{\rm e}) = \{\delta_4^1, \delta_4^3\}$ and $R_2(x_{\rm e}) = \Delta_4$. Thus, (18) holds for $\tau = 2$.

A simple calculation shows that $p_1 = 1$, $p_2 = 1$, $p_3 = 2$ and $p_4 = 1$. Thus, by Theorem 4, we obtain a state feedback switching signal, that is, $\sigma(t) = \delta_2[1\ 1\ 2\ 1]x(t) = x_1(t) \vee \neg x_2(t)$.

## 5 Conclusion

In this paper, we have studied the stability of switched singular Boolean networks by using the semi-tensor product of matrices. Based on the algebraic form of SSBNs, we have obtained a necessary and sufficient condition for the uniqueness of solution of the system. In addition, we have presented several necessary and sufficient conditions for the stability of SSBNs under arbitrary switching signal and the switching stabilizability of SSBNs, respectively, by converting an SSBN into an equivalent switched Boolean network. The study of two illustrative examples showed that the main results obtained in this paper are effective in analyzing the stability of SSBNs.

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