第 32 卷第 9 期
2015 年 9 月

控 制 理 论 与 应 用
Control Theory & Applications

Vol. 32 No. 9
Sep. 2015

DOI: 10.7641/CTA.2015.50247

具有死区输入的分数阶混沌系统的有限时间同步

田小敏†, 费树岷, 柴 琳
(东南大学 自动化学院 复杂系统测量与控制教育部重点实验室, 江苏 南京 210096)

摘要: 本文提出一种新型分数阶滑模控制方案来实现两个不同分数阶混沌系统的有限时间同步. 首先, 根据实际情况, 在研究系统同步过程中, 充分考虑死区非线性输入、参数不确定性、模型不确定性以及外界扰动对系统的影响, 然后, 采用Lyapunov稳定理论证明滑模阶段和趋近阶段均是有限时间收敛的, 最后, 给出一个仿真实例充分验证本文所提出控制策略的有效性和可行性.

关键词: 有限时间同步; 不确定分数阶混沌系统; 滑模控制; Lyapunov稳定理论

中图分类号: TP273 **文献标识码**: A

Finite-time synchronization of fractional-order chaotic systems by considering dead-zone phenomenon

TIAN Xiao-min†, FEI Shu-min, CHAI Lin
(Key Laboratory of Measurement and Control of CSE, Ministry of Education, School of Automation Southeast University, Nanjing Jiangsu 210096, China)

Abstract: A novel fractional-order sliding-mode control (FSMC) scheme is proposed to realize the finite-time synchronization between two different fractional-order chaotic systems. We assume that both master system and slave system are perturbed by parameter uncertainties, model uncertainties and external disturbances. Moreover, the effects of dead-zone nonlinearities in control inputs are also taken into consideration. Lyapunov's stability theory is applied to prove that both reaching phase and sliding phase are of finite-time convergence. A simulation example is provided to validate the effectiveness and feasibility of the proposed scheme.

Key words: finite-time synchronization; uncertain fractional-order chaotic system; sliding mode control; Lyapunov's stability theory

1 Introduction

Although fractional calculus is a mathematical topic with more than 300 years history, its applications in the fields of physics and engineering have attracted lots of attentions only in the recent years. It was found that, with the help of fractional calculus, many systems in interdisciplinary fields can be described more accurately[1–4]. That is to say, fractional calculus provide a superb instrument for the description of memory and hereditary properties of various materials and processes.

The research for fractional-order chaotic systems has grown significantly over past decades and has become a popular topic. For example, Odibat[5] proposed nonlinear feedback control scheme to realize the synchronization of two non-identical fractional-order chaotic systems. Targhvafard et al.[6] developed an active control method to proceed phase and anti-phase synchronization of fractional-order chaotic systems. Zhang et al.[7] used the single driving variable method to realize the adaptive stabilization of chaotic systems. Lu[8] introduced a nonlinear observer to synchronize the chaotic systems. Chen et al.[9–10] researched the chaos synchronization of fractional-order chaotic neural network.

However, almost of the above mentioned works are merely focus on the asymptotic stability of fractional-order systems. In practice, it is more favorable to stabilize/synchronize the fractional-order systems in a given time rather than asymptotically. The finite-time control techniques have demonstrated better robustness and disturbance rejection properties. On the other hand, it is worth noting that the dead-zone properties is frequently encountered in various engineering systems and can be a reason of instability. Thus, in designing and implementing the controller, the effect of dead-zone nonlinearities in control inputs cannot be neglected. However, to the best of the authors' knowledge, up until now, there is no information available about the finite-time

Received 28 March 2015; accepted 21 July 2015.
†Corresponding author. E-mail: tianxiaomin100@163.com.
Supported by National Natural Science Foundation of China (61273119, 61374038).

synchronization of two uncertain fractional-order systems with dead-zone nonlinear inputs, so it is still an interesting and challenging problem.

Motivated by the above discussions, in this paper, our goal is to design a robust controller to achieve finite-time synchronization for two different uncertain fractional-order chaotic systems. It is well known that sliding mode control (SMC) approach is a very effective robust nonlinear control technique. When the controlled system operates on sliding surface, the closed-loop system dynamics reduced to the sliding surface dynamics which has desired characteristics such as good stability, disturbance rejection ability, and tracking capability. With this in mind, in this paper, we applied SMC technique to realize our goal.

2 Preliminaries

In this section, a commonly used definition and some lemmas are given to analyze the fractional-order system.

2.1 Definition and Lemmas

Definition 1 For $n-1<\alpha\leqslant n$, the Riemann-Liouville fractional derivative of order α is defined as

$$
\begin{aligned}
&{}_{t_0}D_t^{\alpha}f(t)=\frac{\mathrm{d}^{\alpha}f(t)}{\mathrm{d}t^{\alpha}}=\\
&\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau,
\end{aligned}
\tag{1}
$$

when $\alpha>0$, ${}_{t_0}D_t^{\alpha}$ represents fractional order derivative, when $\alpha<0$, ${}_{t_0}D_t^{\alpha}$ represents fractional order integral, in this paper, we use D^{α} to denote ${}_0D_t^{\alpha}$.

Lemma 1(see [11]) In Riemann-Liouville derivatives if $p>q\geqslant 0$, m and n are integers such that $0\leqslant m-1\leqslant p<m$, $0\leqslant n-1\leqslant q<n$, then we obtain

$$
{}_aD_t^{p}({}_aD_t^{-q}f(t))={}_aD_t^{p-q}f(t).
$$

Lemma 2(see [11]) In Riemann-Liouville derivatives if $p,\ q\geqslant 0$, m and n are integers such that $0\leqslant m-1\leqslant p<m$, $0\leqslant n-1\leqslant q<n$, then we obtain

$$
\begin{aligned}
&{}_aD_t^{p}({}_aD_t^{q}f(t))=\\
&{}_aD_t^{p+q}f(t)-\sum_{j=1}^{n}\left[{}_aD_t^{q-j}f(t)\right]_{t=a}\times\frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}.
\end{aligned}
$$

Lemma 3(see [12]) Consider the system

$$
\dot{x}(t)=f(x(t)),\ f(0)=0,\ x(t)\in\mathbb{R}^n,\tag{2}
$$

where $f:D\rightarrow\mathbb{R}^n$ is continuous on an open neighborhood $D\subset\mathbb{R}$. Suppose there exists a continuous differential positive definite function $V(x(t)):D\rightarrow\mathbb{R}$, real numbers $p>0$, $0<\eta<1$, such that

$$
\dot{V}(x(t))+pV^{\eta}(x(t))\leqslant 0,\ \forall X(t)\in D.
$$

Then the origin of system (2) is a locally finite time stable equilibrium, and the settling time, depending on the initial state $x(0)=x_0$, satisfies $T(x_0)=\dfrac{V^{1-\eta}(x_0)}{p(1-\eta)}$. In addition, if $D=\mathbb{R}^n$ and $V(x(t))$ is also radially unbounded, then the origin is a globally finite time stable equilibrium of system (2).

2.2 Problem statement

Consider two fractional-order chaotic systems with parameter uncertainties, model uncertainties and external disturbances, described by

Master system:

$$
D^{\alpha}x_i=(A_i+\Delta A_i)x+f_i(x)+\Delta f_i(x)+d_i^{\mathrm{m}}(t).\tag{3}
$$

Slave system

$$
\begin{aligned}
D^{\alpha}y_i=\ &(B_i+\Delta B_i)y+g_i(y)+\Delta g_i(y)+\\
&d_i^{\mathrm{s}}(t)+h_i(u_i(t)),
\end{aligned}
\tag{4}
$$

where $i=1,2,\cdots,n$, $\alpha\in(0,1)$, $x=(x_1\ \ x_2\ \ \cdots\ \ x_n)^{\mathrm{T}}$, $y=(y_1\ y_2\ \cdots\ y_n)^{\mathrm{T}}\in\mathbb{R}^n$ are the state vectors of the system (3) and (4), respectively. $A_i,B_i\in\mathbb{R}^{1\times n}$ are the ith row of two $n\times n$ constant matrices, respectively. $\Delta A_i,\Delta B_i\in\mathbb{R}^{1\times n}$ are the ith row of two $n\times n$ parameter uncertainties matrices, respectively. $f_i(x),g_i(y):\mathbb{R}^n\rightarrow\mathbb{R}$ are two continuous nonlinear functions, $\Delta f_i(x),\Delta g_i(y):\mathbb{R}^n\rightarrow\mathbb{R}$ and $d_i^{\mathrm{m}}(t),d_i^{\mathrm{s}}(t)\in\mathbb{R}$ are model uncertainties and external disturbances in system (3) and (4), respectively. $u(t)=(u_1(t)\ \ u_2(t)\ \ \cdots\ \ u_n(t))^{\mathrm{T}}\in\mathbb{R}^n$ is the vector of controller to be designed later, and $h_i(u_i(t))$, $i=1,2,\cdots$, n is a dead-zone nonlinear function, determined by

$$
\begin{aligned}
&h_i(u_i(t))=\\
&\begin{cases}
(u_i(t)-u_{+i})h_{+i}(u_i(t)), u_i(t)>u_{+i},\\
0, \qquad\qquad\qquad\qquad u_{-i}\leqslant u_i(t)\leqslant u_{+i},\\
(u_i(t)-u_{-i})h_{-i}(u_i(t)), u_i(t)<u_{-i},
\end{cases}
\end{aligned}
\tag{5}
$$

where $h_{+i}(\cdot)$ and $h_{-i}(\cdot)$ are nonlinear functions of $u_i(t)$, u_{+i} and u_{-i} are given constants. Besides, the nonlinear input $h_i(u_i(t))$ outside of the dead-band has gain reduction tolerances β_{+i} and β_{-i}, which satisfy the following inequalities:

$$
\begin{cases}
(u_i(t)-u_{+i})h_i(u_i(t))\geqslant\beta_{+i}(u_i(t)-u_{+i})^2,\\
\qquad\qquad\qquad\qquad\qquad u_i(t)>u_{+i};\\
0, \qquad\qquad\qquad\qquad\quad u_{-i}\leqslant u_i(t)\leqslant u_{+i};\\
(u_i(t)-u_{-i})h_i(u_i(t))\geqslant\beta_{-i}(u_i(t)-u_{-i})^2,\\
\qquad\qquad\qquad\qquad\qquad u_i(t)<u_{-i},
\end{cases}
\tag{6}
$$

where β_{+i} and β_{-i} are positive constants.

Subtracting (3) from (4), we can get the synchronization error system as

$$
\begin{aligned}
D^{\alpha}e_i=\ &(B_i+\Delta B_i)y+g_i(y)+\Delta g_i(y)+d_i^{\mathrm{s}}(t)-\\
&(A_i+\Delta A_i)x-f_i(x)-\Delta f_i(x)-d_i^{\mathrm{m}}(t)+\\
&h_i(u_i(t)).
\end{aligned}
\tag{7}
$$

In order to make the proposed scheme is more effective and reasonable, an assumption is provided.

Assumption 1 It is assumed that the uncertainties $\Delta A_i,\ \Delta B_i, \Delta f_i(x), \Delta g_i(y)$ and disturbances $d_i^{\rm m}(t), d_i^{\rm s}(t)$ are bounded by

$$||\Delta A_i|| \leqslant \theta_i,\quad ||\Delta B_i|| \leqslant \psi_i,$$
$$|\Delta f_i(x)| \leqslant \delta_i,\ |\Delta g_i(y)| \leqslant \rho_i,$$
$$|d_i^{\rm m}(t)| \leqslant L_i^{\rm m},\ |d_i^{\rm s}(t)| \leqslant L_i^{\rm s},$$

where $\theta_i, \psi_i, \rho_i, \delta_i$ are given positive constants, $L_i^{\rm m}, L_i^{\rm s}$ are two known positive constants.

3 Controller design

Generally, the design of sliding mode controller for stabilizing the uncertain fractional-order error system (7) has two steps. First, an appropriate sliding surface with the desired dynamics need to be constructed. Second, a robust control law is designed to ensure the existence of the sliding motion. In this paper, we select the following fractional-order integral type sliding surface

$$s_i = D^{\alpha-1}e_i + D^{\alpha-2}[\frac{\lambda}{\mu}e_i + (e^{\rm T}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)}], \tag{8}$$

where $e = (e_1\ \ e_2\ \ \cdots\ \ e_n)^{\rm T},\ \lambda, \mu > 0$.

When system (7) operates in sliding mode, the following equations are satisfied

$$s_i = 0,\ \dot s_i = 0.$$

Taking the time derivate of (8), it yields

$$\dot s_i = D^{\alpha}e_i + D^{\alpha-1}[\frac{\lambda}{\mu}e_i + (e^{\rm T}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)}]. \tag{9}$$

That is, the sliding mode dynamics is obtained as

$$D^{\alpha}e_i = -D^{\alpha-1}[\frac{\lambda}{\mu}e_i + (e^{\rm T}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)}]. \tag{10}$$

Theorem 1 Consider the sliding mode dynamics (10), the system is stable and its state trajectories converge to zero in a finite time T_1, given by

$$T_1 = \frac{1}{2\lambda}\ln(1+\frac{\lambda}{\mu}(e^{\rm T}(0)e(0))^{\mu}).$$

Proof Choosing the Lyapunov function in the form of

$$V_1(t) = \sum_{i=1}^{n} e_i^2.$$

Taking the time derivative of $V_1(t)$ along the trajectories of (10), we obtain

$$\begin{aligned}
&\dot V_1(t) = 2\sum_{i=1}^{n} e_i\dot e_i(t) = \\
&2\sum_{i=1}^{n} e_i[D^{1-\alpha}(D^{\alpha}e_i) + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)}] = \\
&2\sum_{i=1}^{n} e_i[D^{1-\alpha}(-D^{\alpha-1}(\frac{\lambda}{\mu}e_i + (e^{\rm T}e)^{-\mu}e_i + \\
&D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})) + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)}] = \\
&2\sum_{i=1}^{n} e_i[-\frac{\lambda}{\mu}e_i - (e^{\rm T}e)^{-\mu}e_i - D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)} + \\
&D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)}] = \\
&2\sum_{i=1}^{n} e_i(-\frac{\lambda}{\mu}e_i - (e^{\rm T}e)^{-\mu}e_i) = \\
&2\sum_{i=1}^{n}(-\frac{\lambda}{\mu}e_i^2 - (e^{\rm T}e)^{-\mu}e_i^2) = \\
&-2\frac{\lambda}{\mu}e^{\rm T}e - 2(e^{\rm T}e)^{1-\mu} = \\
&-2\frac{\lambda}{\mu}V_1(t) - 2V_1^{1-\mu}(t).
\end{aligned} \tag{11}$$

Multiplying both sides of (11) by $\mu V_1^{\mu-1}(t)$, we have

$$\mu V_1^{\mu-1}(t)\dot V_1(t) + 2\lambda V_1^{\mu}(t) = -2\mu. \tag{12}$$

Further, we can multiplying both sides of (12) by ${\rm e}^{2\lambda t}$, it yiels

$$\begin{aligned}&{\rm e}^{2\lambda t}(\mu V_1^{\mu-1}(t)\dot V_1(t) + 2\lambda V_1^{\mu}(t)) = \\ &{\rm d}({\rm e}^{2\lambda t}V_1^{\mu}(t)) = -2\mu{\rm e}^{2\lambda t}.\end{aligned} \tag{13}$$

Integrating both sides of (13) from zero to t, one has

$${\rm e}^{2\lambda t}V_1^{\mu}(t) - V_1^{\mu}(0) = -\frac{\mu}{\lambda}{\rm e}^{2\lambda t} + \frac{\mu}{\lambda}.$$

It is easy to deduce that

$$V_1^{\mu}(t) = (\frac{\mu}{\lambda} + V_1^{\mu}(0)){\rm e}^{-2\lambda t} - \frac{\mu}{\lambda}. \tag{14}$$

If $V_1^{\mu}(T_1) \equiv 0$, then according to (14), we have

$${\rm e}^{2\lambda T_1} = \frac{\lambda}{\mu}\big(\frac{\mu}{\lambda} + V_1^{\mu}(0)\big) = 1 + \frac{\lambda}{\mu}V_1^{\mu}(0).$$

Consequently, one obtains

$$T_1 = \frac{1}{2\lambda}\ln(1+\frac{\lambda}{\mu}V_1^{\mu}(0)).$$

Thus, the state trajectories of error system (10) converge to zero in a finite time $T_1 = \frac{1}{2\lambda}\ln\big(1+\frac{\lambda}{\mu}V_1^{\mu}(0)\big)$.

Once the appropriate sliding surface is designed, the next step is to design a controller to steer the state trajectories of error system go on to the sliding surface in a given time and stay on it forever. In this paper, we design the controller as

$$u_i(t) = \begin{cases} -\gamma_i\zeta_i{\rm sgn}\, s_i + u_{-i}, & s_i > 0, \\ 0, & s_i = 0, \\ -\gamma_i\zeta_i{\rm sgn}\, s_i + u_{+i}, & s_i < 0, \end{cases} \tag{15}$$

where "sgn" is the sign function,

$$\begin{aligned}
&\gamma_i = \frac{1}{\beta_i},\ \beta_i = \min\{\beta_{-i}, \beta_{+i}\}, \\
&\zeta_i = |B_iy + g_i(y) - A_ix - f_i(x)| + \sigma_i + \\
&\quad |D^{\alpha-1}(\frac{\lambda}{\mu}e_i + (e^{\rm T}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})| + \\
&\quad k_i > 0,
\end{aligned}$$

k_i is a positive constant, $\sigma_i = \theta_i||x|| + \psi_i||y|| + \delta_i + \rho_i + L_i^{\mathrm{m}} + L_i^{\mathrm{s}}$, λ and μ are defined in (8).

Theorem 2 Consider the synchronization error system (7) with dead-zone nonlinear inputs, if the system is controlled by the controller (15), then the system trajectories can converge to the sliding surface $s_i = 0$ in the finite time T_2 , determined by

$$T_2 \leqslant \frac{\sqrt{\sum\limits_{i=1}^{n} s_i^2(0)}}{k},$$

where $k = \min\{k_i, i = 1, 2, \cdots, n\}$.

Proof Selecting the following positive definite Lyapunov function

$$V_2(t) = \frac{1}{2}\sum_{i=1}^{n} s_i^2. \tag{16}$$

Taking the derivative of (16) with respect to time, it yields

$$\dot{V}_2(t) = \sum_{i=1}^{n} s_i\dot{s}_i. \tag{17}$$

Inserting $\dot{s}_i$ from (9) into (17), one has

$$\dot{V}_2(t) = \sum_{i=1}^{n} s_i[D^{\alpha}e_i + D^{\alpha-1}(\frac{\lambda}{\mu}e_i + (e^{\mathrm{T}}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})]. \tag{18}$$

Replacing $D^{\alpha}e_i$ from (7) into (18), and according to Assumption 1, one obtains

$$\begin{aligned}
&\dot{V}_2(t) = \\
&\sum_{i=1}^{n} s_i[(B_i + \Delta B_i)y + g_i(y) + \Delta g_i(y) + \\
&d_i^{\mathrm{s}}(t) - (A_i + \Delta A_i)x - f_i(x) - \Delta f_i(x) - \\
&d_i^{\mathrm{m}}(t) + h_i(u_i(t)) + D^{\alpha-1}(\frac{\lambda}{\mu}e_i + \\
&(e^{\mathrm{T}}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})] \leqslant \\
&\sum_{i=1}^{n} |s_i|[|B_iy + g_i(y) - A_ix - f_i(x)| + |\Delta B_iy - \\
&\Delta A_ix + \Delta g_i(y) - \Delta f_i(x) + d_i^{\mathrm{s}}(t) - d_i^{\mathrm{m}}(t)| + \\
&|D^{\alpha-1}(\frac{\lambda}{\mu}e_i + (e^{\mathrm{T}}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})|] + \\
&\sum_{i=1}^{n} s_ih_i(u_i(t)) \leqslant \\
&\sum_{i=1}^{n} |s_i|[|B_iy + g_i(y) - A_ix - f_i(x)| + \theta_i||x|| + \\
&\psi_i||y|| + \delta_i + \rho_i + L_i^{\mathrm{m}} + L_i^{\mathrm{s}} + \\
&|D^{\alpha-1}(\frac{\lambda}{\mu}e_i + (e^{\mathrm{T}}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})|] + \\
&\sum_{i=1}^{n} s_ih_i(u_i(t)).
\end{aligned} \tag{19}$$

When $s_i < 0$, from (15), it is obvious that $u_i(t) > u_{+i}$, and then according to (6),

$$\begin{aligned}
&(u_i(t) - u_{+i})h_i(u_i(t)) = \\
&-\gamma_i\zeta_i\mathrm{sgn}\, s_ih_i(u_i(t)) \geqslant \beta_{+i}(u_i(t) - u_{+i})^2 = \\
&\beta_{+i}\gamma_i^2\zeta_i^2\mathrm{sgn}^2 s_i \geqslant \beta_i\gamma_i^2\zeta_i^2\mathrm{sgn}^2 s_i.
\end{aligned} \tag{20}$$

Owing to $\gamma_i = \dfrac{1}{\beta_i} > 0,\ \zeta_i > 0$, the above inequality can be rewritten as

$$-\mathrm{sgn}\, s_ih_i(u_i(t)) \geqslant \zeta_i\mathrm{sgn}^2 s_i. \tag{21}$$

Multiplying both sides of (21) by $|s_i|$, and according to $|s_i|\mathrm{sgn}\, s_i = s_i,\ \mathrm{sgn}^2 s_i = 1$, it yields

$$s_ih_i(u_i(t)) \leqslant -\zeta_i|s_i|. \tag{22}$$

When $s_i > 0$, through the similar operation, the inequality (22) still holds. Substituting (22) into (19), we have

$$\begin{aligned}
&\dot{V}_2(t) \leqslant \\
&\sum_{i=1}^{n} |s_i|[|B_iy + g_i(y) - A_ix - f_i(x)| + \\
&\theta_i||x|| + \psi_i||y|| + \delta_i + \rho_i + L_i^{\mathrm{m}} + \\
&L_i^{\mathrm{s}} + |D^{\alpha-1}(\frac{\lambda}{\mu}e_i + (e^{\mathrm{T}}e)^{-\mu}e_i + \\
&D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})|] - \sum_{i=1}^{n}\zeta_i|s_i| = \\
&\sum_{i=1}^{n} |s_i|[|B_iy + g_i(y) - A_ix - f_i(x)| + \theta_i||x|| + \\
&\psi_i||y|| + \delta_i + \rho_i + L_i^{\mathrm{m}} + L_i^{\mathrm{s}} + |D^{\alpha-1}(\frac{\lambda}{\mu}e_i + \\
&(e^{\mathrm{T}}e)^{-\mu}e_i + D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})|] - \\
&\sum_{i=1}^{n} |s_i|[|B_iy + g_i(y) - A_ix - f_i(x)| + \\
&\sigma_i + |D^{\alpha-1}(\frac{\lambda}{\mu}e_i + (e^{\mathrm{T}}e)^{-\mu}e_i + \\
&D^{\alpha-1}e_i\frac{t^{-(1-\alpha)-1}}{\Gamma(\alpha-1)})| + k_i] = \\
&\sum_{i=1}^{n} |s_i|(\theta_i||x|| + \psi_i||y|| + \delta_i + \rho_i + L_i^{\mathrm{m}} + \\
&L_i^{\mathrm{s}} - \sigma_i) - \sum_{i=1}^{n} k_i|s_i| \leqslant -\sum_{i=1}^{n} k_i|s_i|.
\end{aligned}$$

According to reference [13], we can deduce that

$$\begin{aligned}
\dot{V}_2(t) &\leqslant -\sum_{i=1}^{n} k_i|s_i| \leqslant -k\sum_{i=1}^{n}|s_i| \leqslant \\
&-\sqrt{2}k(\frac{1}{2}\sum_{i=1}^{n} s_i^2)^{1/2} = -\sqrt{2}kV_2^{1/2}(t).
\end{aligned}$$

where $k = \min\{k_i, i = 1, 2, \cdots, n\}$. Then according to Lemma 3, we can include that the trajectories of error system (10) will converge to the sliding surface $s_i = 0$ in the finite time $T_2 \leqslant \dfrac{\sqrt{\sum_{i=1}^{n} s_i^2(0)}}{k}$. Therefore, the proof is completed.

4 Numerical simulation

In this section, an example is provided to verify the effectiveness and feasibility of the proposed method.

We take the uncertain fractional-order Chen system as master system, described by

$$\underbrace{\begin{pmatrix} D^\alpha x_1 \\ D^\alpha x_2 \\ D^\alpha x_3 \end{pmatrix}}_{D^\alpha x} = [\underbrace{\begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} -0.3\sin t & 0 & 0 \\ 0 & 0.2\sin t & 0 \\ 0 & 0 & 0.15\sin(3t) \end{pmatrix}}_{\Delta A}] \times \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} +$$

$$\underbrace{\begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} -0.5\sin(\pi x_1) \\ -0.5\sin(2\pi x_2) \\ -0.5\sin(3\pi x_3) \end{pmatrix}}_{\Delta f(x)} + \underbrace{\begin{pmatrix} -0.1\cos t \\ -0.1\cos t \\ -0.1\cos t \end{pmatrix}}_{d^{\mathrm{m}}(t)}. \tag{23}$$

Take the uncertain fractional-order Lorenz system as slave system, written as

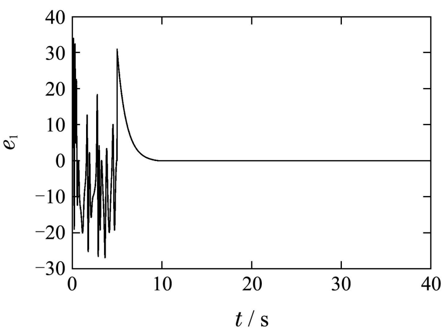
$$\underbrace{\begin{pmatrix} D^\alpha y_1 \\ D^\alpha y_2 \\ D^\alpha y_3 \end{pmatrix}}_{D^\alpha y} = [\underbrace{\begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}}_{B} + \underbrace{\begin{pmatrix} -0.1\sin t & 0 & 0 \\ 0 & 0.2\sin t & 0 \\ 0 & 0 & 0.3\sin(3t) \end{pmatrix}}_{\Delta B}] \times \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{y} +$$

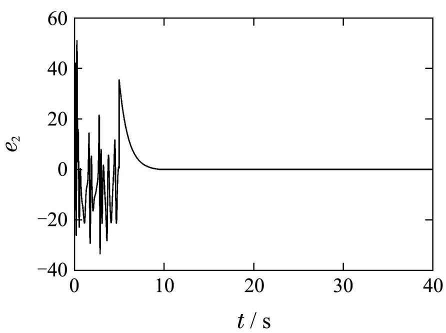
$$\underbrace{\begin{pmatrix} 0 \\ -y_1y_3 \\ y_1y_2 \end{pmatrix}}_{g(y)} + \underbrace{\begin{pmatrix} 0.1\sin(ty_1) \\ 0.2\sin(ty_2) \\ 0.3\sin(ty_3) \end{pmatrix}}_{\Delta g(y)} + \underbrace{\begin{pmatrix} 0.1\cos t \\ 0.1\cos t \\ 0.1\cos t \end{pmatrix}}_{d^{\mathrm{s}}(t)} + \underbrace{\begin{pmatrix} h_1(u_1(t)) \\ h_2(u_2(t)) \\ h_3(u_3(t)) \end{pmatrix}}_{h(u(t))}. \tag{24}$$

The dead-zone nonlinear inputs are given by

$$h_i(u_i(t)) = \begin{cases} (u_i(t)-1)(0.8-0.4\cos(u_i(t))), & u_i(t) > 1, \\ 0, & -1 \leqslant u_i(t) \leqslant 1, \\ (u_i(t)+1)(1-0.5\cos(u_i(t))), & u_i(t) < -1. \end{cases}$$

It is obvious that $\beta_{+i} = 0.4,\ \beta_{-i} = 0.5,\ \beta_i = 0.4,$ $\gamma_i = \frac{5}{2}$. In this simulation, letting $\alpha = 0.998$, the initial states are randomly chosen as $x(0) = (1\ \ -2\ \ -2)^{\mathrm{T}}$, $y(0) = (0\ \ 1\ \ -1)^{\mathrm{T}}$. Setting the control parameters as $\lambda = 1,\ \mu = 1/2$, when the controller is activated at $t = 5\,\mathrm{s}$, we can obtain the desired state trajectories of error system, displayed in Fig. 1.





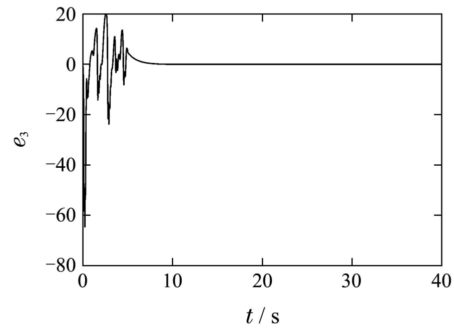
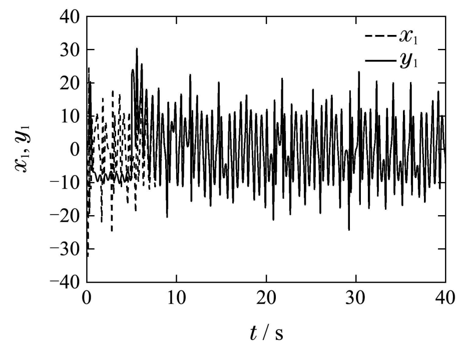
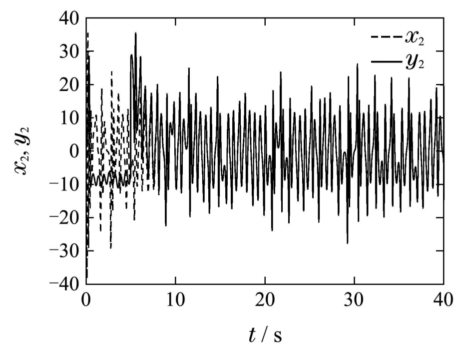


Fig. 1 Time response of synchronization error system with control





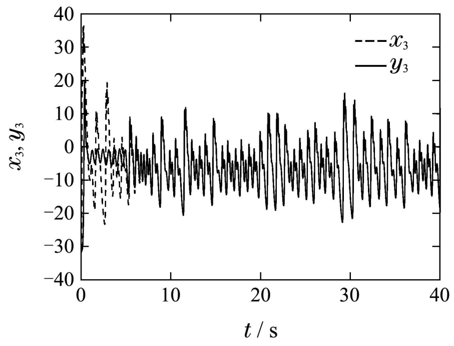


Fig. 2 Time evolution of systems (23) and (24) with control

From Fig. 1, it is obvious that the error trajectories converge to zero after a period time, which implies that the synchronization between two different fractional order chaotic systems is achieved. Furthermore, the time evolutions of master and slave systems are shown in Fig. 2.

All simulation results have demonstrated the effectiveness and applicability of the proposed approach[14–23].

5 Conclusions

In this paper, the finite-time synchronization between two different fractional-order chaotic systems with dead-zone nonlinear inputs is investigated. Both master and slave systems are perturbed by parameter uncertainties, model uncertainties and external disturbances. On the basis of Lyapunov stability theory and finite-time control technique, a robust control law is designed to ensure the finite-time convergence of reaching and sliding phases. Effectiveness and feasibility of the proposed method are demonstrated by a simulation example.

References:

[1] BAGLEY R L, CALICO R A. Fractional order state equations for the control of viscoelastically damped structure [J]. *Journal of Guidance, Control, Dynamics*, 1991, 14(2): 304 – 482.

[2] SUN H H, ABDELWAHAD A A, OHARAL B. Linear approximation of transfer function with a pole of fractional power [J]. *IEEE Transaction on Automation Control*, 1984, 29(5): 441 – 444.

[3] ICHISE M, NAGAYANAGI Y, KOJIMA T. An analog simulation of non-integer order transfer functions for analysis of electrode process [J]. *Journal of Electroanalytical Chemistry and Interfacial Electrochemistry*, 1971, 33(2): 253 – 265.

[4] HEAVISIDE O. *Electromagnetic Theory* [M]. New York: Chelsea, 1971.

[5] ODIBAT Z M. Adaptive feedback control and synchronization of non-identical chaotic fractional order systems [J]. *Nonlinear Dynamics*, 2010, 60(4): 479 – 487.

[6] TARGHVAFARD H, ERJAEE G H. Phase and anti-phase synchronization of fractional order chaotic via active control [J]. *Communications in Nonlinear Science and Numerical Simulation*, 2011, 16(10): 4079 – 4088.

[7] ZHANG R X, YANG S P. Adaptive synchronization of fractional-order chaotic systems via a single driving variable [J]. *Nonlinear Dynamics*, 2011, 66(4): 831 – 837.

[8] LU J G. Nonlinear observer design to synchronize fractional-order chaotic system via a scalar transmitted signal [J]. *Physica A*, 2006, 359: 107 – 118.

[9] CHEN L P, QU J F, CHAI Y, et al. Synchronization of a class of fractional-order chaotic neural networks [J]. *Entropy*, 2013, 15: 3265 – 3276.

[10] CHEN L P, CHAI Y, WU R C, et al. Cluster synchronization in fractional-order complex dynamical networks [J]. *Physics Letters A*, 2012, 376(35): 2381 – 2388.

[11] PODLUBNY I. *Fractional Differential Equations* [M]. New York: Academic Press, 1999.

[12] BHAT S P, BERNSTEIN D S. Finite-time stability of continuous autonomous systems [J]. *SIAM Journal on Control and Optimization*, 2000, 38(3): 751 – 766.

[13] HARDY G H, LITTLEWOOD J E, POLYA G. *Inequalities* [M]. Cambridge: Cambridge University Press, 1952.

[14] LUO R Z, WANG Y L. Finite-time modified projective synchronization between two different chaotic systems with parameter and model uncertainties and external disturbances via sliding control [J]. *Indian Journal of Physics*, 2014, 88(3): 301 – 309.

[15] CHEN D Y, LIU Y X, MA X Y, et al. Control of a class of fractional-order chaotic systems via sliding mode [J]. *Nonlinear Dynamics*, 2012, 67(1): 893 – 901.

[16] SENEJOHNNY D M, DELAVARI H. Active sliding observer scheme based fractional chaos synchronization [J]. *Communications in Nonlinear Science and Numerical Simulation*, 2012, 17(11): 4373 – 4383.

[17] LIU P. Robust finite-time stabilization of unified chaotic complex system with certain and uncertain parameters [J]. *Chinese Physics B*, 2013, 22(7): 070501.

[18] DIETHELM K, FORD N. A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 2002, 29(1): 3 – 22.

[19] YIN C, DADRAS S, ZHONG S M, et al. Control of a novel class of fractional-order chaotic systems via adaptive sliding mode control approach [J]. *Applied Mathematical Modelling*, 2013, 37(4): 2469 – 2483.

[20] FAIEGHI M R, DELAVARI H, BALEANU D. A note on stability of sliding mode dynamics in suppression of fractional-order chaotic systems [J]. *Computers and Mathematics with Application*, 2013, 66(5): 832 – 837.

[21] AGHABABA M P. Finite-time chaos control and synchronization of fractional-order nonautonomous chaotic (hyperchaotic) systems using fractional nonsingular terminal sliding mode technique [J]. *Nonlinear Dynamics*, 2012, 69(1): 247 – 267.

[22] YANG C C. Synchronization of rotating pendulum via self-learning terminal sliding-mode control subject to input nonlinearity [J]. *Nonlinear Dynamics*, 2013, 72(3): 695 – 705.

[23] ABOOEE A, HAERI M. Stabilisation of commensurate fractional-order polytopic non-linear differential inclusion subject to input nonlinearity and unknown disturbances [J]. *IET Control Theory and Applications*, 2013, 7(12): 1624 – 1633.

作者简介:

田小敏 (1987–), 女, 分别于2009年和2012年在安徽理工大学获得学士和硕士学位, 自2012年起在东南大学攻读博士学位, 研究方向为分数阶系统的稳定性分析和控制器设计等, E-mail: tianxiaomin100@163.com;

费树岷 (1961–), 男, 于1995年在北京航空航天大学获得博士学位, 从1995–1997在东南大学自动化学院做博士后研究工作, 目前为东南大学自动化学院教授, 研究方向为非线性系统、时滞系统、混杂系统等, E-mail: smfei@seu.edu.cn;

柴 琳 (1978–), 女, 于2007年在东南大学获得博士学位, 目前为东南大学副研究员, 研究方向为非线性系统、时滞系统、自适应控制等, E-mail: chailin_1@163.com.