第 32 卷第 9 期 控 制 理 论 与 应 用 Vol. 32 No. 9
2015 年 9 月 Control Theory & Applications Sep. 2015

DOI: 10.7641/CTA.2015.50249

混杂随机微分方程θ方法的几乎必然指数稳定性

莫浩艺[1,2], 邓飞其[1†], 彭云建[1]

(1. 华南理工大学 系统工程研究所, 广东 广州 510640; 2. 广东工业大学 应用数学学院, 广东 广州 510006)

摘要: 大部分的混杂随机微分方程很难得到解析解, 因此利用数值方法研究其数值解具有重要意义. 本文研究θ方法产生的数值解的几乎必然指数稳定性. 在单边Lipschitz条件和线性增长条件下, 首先给出方程的平凡解是几乎必然指数稳定的. 然后在相同条件下, 运用Chebyshev不等式和Borel-Cantelli引理, 证明了对$\theta \in [0,1]$, θ方法重现平凡解的几乎必然指数稳定性. θ方法是一种比现有的Euler-Maruyama方法和向后Euler-Maruyama方法更广的方法. 当θ等于1或0时, 它分别退化为上述两种方法之一. 本文的结论对上述两种方法同样适用. 最后, 数值例子和仿真说明了对不同的θ所提出方法的有效性和稳定性.

关键词: 布朗运动; θ方法; 马尔科夫链; 几乎必然指数稳定; 混杂系统

中图分类号: O211.63, O241.8 **文献标识码**: A

Almost sure exponential stability of θ-method for hybrid stochastic differential equations

MO Hao-yi[1,2], DENG Fei-qi[1†], PENG Yun-jian[1]

(1. Systems Engineering Institute, South China University of Technology, Guangzhou Guangdong 510640, China;
2. School of Applied Mathematics, Guangdong University of Technology, Guangzhou Guangdong 510006, China)

Abstract: It is difficult to obtain analytical solutions for most of the hybrid stochastic differential equations (SDEs), so the research on the numerical solutions by the use of numerical methods is of great significance. This paper focuses on the almost sure exponential stability of the numerical solutions produced by the θ-method. Under the one-sided Lipschitz condition and the linear growth condition, the almost sure exponential stability of the trivial solution for hybrid SDEs is first introduced. Then, by applying the Chebyshev inequality and the Borel-Cantelli lemma, we prove that the θ-method reproduces the corresponding stability of the trivial solution under the same conditions for $\theta \in [0,1]$. The θ-method is a more general method than the existing Euler-Maruyama method as well as the backward Euler-Maruyama method. When θ is equal to 1 or 0, it degenerates to one of the above two methods, respectively. The results of this paper are also applicable to these two methods. Finally, a numerical example and its simulations with different θ are given to illustrate the effectiveness and the stability of the proposed method.

Key words: Brownian motion; θ-method; Markov chains; almost sure exponential stability; hybrid systems

1 Introduction

Hybrid systems, described as the stochastic differential equations with Markovian switching, are derived from the stochastic differential equations (SDEs). For the reasons of environmental disturbances, the structures of the systems may change abruptly. Generally, one way to model such abrupt changes is to use the continuous-time Markov chains $r(t)$. The SDEs with Markovian switching are the specific forms of such systems. This type of equations has been considered as a convenient mathematical framework for the formulation of various design problems in different fields such as target tracking (evasive target tracking problem), fault tolerant control and manufacturing processes[1–3].

One of the important classes of the hybrid systems is governed by the n-dimensional nonlinear hybrid SDEs

$$\mathrm{d}x(t) = f(x(t), r(t))\mathrm{d}t + g(x(t), r(t))\mathrm{d}B(t) \quad (1)$$

on $t \geqslant 0$, given $x(0) = x_0 \neq 0$ in $\mathbb{R}^n$ and $r(0) = i_0 \in S$. As a standing hypothesis, we assume that f, g: $\mathbb{R}^n \times S \to \mathbb{R}^n$ are smooth enough for hybrid SDE (1) to have a unique global solution $x(t)$ on

Received 28 March 2015; accepted 25 September 2015.
†Corresponding author. E-mail: aufqdeng@scut.edu.cn; Tel.: +86 13922202550.
Supported by National Natural Science Foundation of China (61273126), Research Foundation for the Doctoral Program of Higher Education of China (20130172110027), Fundamental Research Funds for the Central Universities(2013ZZ0056, 2015ZM073) and Ph.D Start-up Fund of Natural Science Foundation of Guangdong Province(2014A030310388).

$[0,\infty)$.

Since hybrid SDE (1) does not have explicit solutions, it is important and necessary to know how to obtain the approximate solutions which can be computed numerically. In recent years, a number of methods to obtain the approximate solutions to SDEs have been proposed, and the problem of stability analysis of numerical methods for such equations has attracted a lot of attention [4–14]. Among these references, some try to answer the question "can a numerical method reproduce the almost sure exponential stability of the underlying hybrid SDEs?" For example, Pang et al.[13] showed that Euler-Maruyama (EM) method could reproduce the almost sure exponential stability of the tested hybrid SDEs under some sufficient condition. The key condition imposed in [13] was the global Lipschitz condition. Then without the global Lipschitz condition, Mao et al.[14] showed that backward Euler-Maruyama (BEM) method could capture the almost sure exponential stability of highly non-linear hybrid SDEs, but the EM method might not. It is well known that the θ-method is more general than these methods and may be specialized as the EM and the BEM by choosing $\theta=1$ and $\theta=0$. We wonder what is the answer to this question for the θ-method. Actually, for SDEs and stochastic delay differential equations, the stability analysis of θ-method have received a better research [15–20]. However, for hybrid SDEs, relatively little research is available on the exponential stability of θ-method, which is then chosen as the topic of this paper. Our effort is to show that the θ-method can also reproduce the almost sure exponential stability of the exact solution of hybrid SDEs under some conditions similar to those in [14]. To show the stability of the θ-scheme, for the first time, we will give the figure of the projective domain of numerical solutions in the simulations. Let us first state the conditions.

Assumption 1 f and g satisfy the linear growth condition. That is, there is an $h>0$ such that

$$|f(x,i)|\vee|g(x,i)|\leqslant h|x|,\ \forall(x,i)\in\mathbb{R}^n\times S. \quad (2)$$

Assumption 2 There are constants μ_i $(i\in S)$ such that

$$(x-y)^{\mathrm T}(f(x,i)-f(y,i))\leqslant\mu_i|x-y|^2,\ \forall x,y\in\mathbb{R}^n \quad (3)$$

and

$$\sigma_i=\sup_{x\in\mathbb{R}^n,x\neq0}\Big(\frac{|g(x,i)|^2}{|x|^2}-\frac{2|x^{\mathrm T}g(x,i)|^2}{|x|^4}\Big)<\infty. \quad (4)$$

2 Notations and lemmas

Throughout this paper, we let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant0},\mathrm{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant0}$ satisfying the usual conditions, that is increasing and right continuous, with $\mathcal{F}_0$ containing all P-null sets. $B(t)$ is assumed to be a scalar Brownian motion defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant0},\mathrm{P})$. Let $\mathcal{L}^b_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $\mathbb{R}^n$-valued random variables. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace (or Frobenius) norm in $\mathbb{R}^{n\times m}$. The inner product of x,y in $\mathbb{R}^n$ is denoted by $\langle x,y\rangle$ or $x^{\mathrm T}y$.

Let $r(t)$, $t\geqslant0$, be a right continuous Markov chain on the probability space taking values in a finite state space $S=\{1,2,\cdots,N\}$ with generator $\Gamma=[\gamma_{ij}]_{N\times N}$ given by

$$\mathrm{P}\{r(t+\delta)=j\,|r(t)=i\}=\begin{cases}\gamma_{ij}\delta+\circ(\delta), & \text{if } i\neq j,\\ 1+\gamma_{ii}\delta+\circ(\delta), & \text{if } i=j,\end{cases}$$

where $\delta>0$. Here $\gamma_{ij}\geqslant0$ is the transition rate of $r(t)$ from state i to state j if $i\neq j$ while $\gamma_{ii}=-\sum_{j\neq i}\gamma_{ij}$. We note that almost every sample path of $r(t)$ is a right continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+:=[0,\infty)$. As a standing hypothesis, we assume that the Markov chain is irreducible in this paper. That is to say, this condition is equivalent to that, for any $i,\ j\in S$, we can find finite numbers $i_1,\ i_2,\cdots,i_k\in S$ such that

$$\gamma_{i,i_1}\gamma_{i_1,i_2}\cdots\gamma_{i_k,j}>0.$$

Note that Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is rank$(\Gamma)=N-1$. Under this condition, the Markov chain has a unique stationary probability distribution $\pi=(\pi_1,\pi_2,\cdots,\pi_N)\in\mathbb{R}^{1\times N}$ which can be determined by solving

$$\pi\Gamma=0,$$
$$\text{s.t. }\sum_{j=1}^N\pi_j=1\text{ and }\pi_j>0\text{ for all }j\in S.$$

To learn more about the Markov chain, please see [3]. Now, we define the θ-method for hybrid SDE (1), which is a discrete approximations $X_k\approx x(t_k)$, with $t_k=k\Delta$, where $X_0=x(0),r_0^{\Delta}=i_0$ and mainly

$$X_{k+1}=X_k+(1-\theta)f(X_{k+1},r_k^{\Delta})\Delta+\theta f(X_k,r_k^{\Delta})\Delta+g(X_k,r_k^{\Delta})\Delta B_k, \quad (5)$$

$k=0,1,2,\cdots$, where $\Delta>0$ is the stepsize, $\theta\in[0,1]$ is a fixed parameter, and $\Delta B_k:=B((k+1)\Delta)-B(k\Delta)$ is the Brownian increment. This scheme admits a trade-off between the past state and the current state of the system. With the choice $\theta=0$ and $\theta=1$, (5) reduces to the BEM method and the EM method, respectively.

We note that Assumption 1 implies that

$$f(0,i)=0 \text{ and } g(0,i)=0, \text{ for all } i\in S. \quad (6)$$

It is easy to observe that the solution of Eq.(1) will remain zero if it starts from zero. The solution $x(t)\equiv 0$ is called a trivial solution to equation (1). Assumption 1 also ensures that any solution of Eq.(1) starting from a non-zero state will remain non-zero with probability 1 (see p.120 in [21]).

Let us explain that the θ-method (5) is well defined under the condition (3), which follows from the following lemma.

Lemma 2.1 Assume that f satisfies the one-sided Lipschitz condition (3). If $(1-\theta)\Delta\max\limits_{i\in S}\{\mu_i\}<1, \theta\in[0,1]$, then the θ-method (5) is well defined, (see [3, 8]).

Definition2.1[17,22] The trivial solution of Eq.(1) is said to be almost surely exponentially stable if

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{t}<0, \text{ a.s.} \quad (7)$$

for any initial data $x(0)\in\mathcal{L}^b_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)$ and $x(t)=x(t,t_0,x_0)$.

Definition 2.2[17] The approximate solution X_k of Eq.(5) is said to be almost surely exponentially stable if

$$\limsup_{k\to\infty}\frac{\log|X_k|}{k\Delta}<0, \text{ a.s.} \quad (8)$$

for any bounded variable X_0 and $X_k=X_k(0,X_0)$.

3 Stability of trivial solution and the θ-method approximation

In this section, we will show that the θ-method (5) can preserve the almost sure exponential stability of the trivial solution of the hybrid SDEs. The following theorem shows that the trivial solution of Eq.(1) is almost surely exponentially stable.

Theorem 3.1[14] Let Assumptions 1 and 2 hold. If $\sum\limits_{i\in S}\pi_i(\mu_i+0.5\sigma_i)<0$, then the trivial solution of Eq.(1) is almost surely exponentially stable.

Theorem 3.2 Let Assumptions 1 and 2 hold. If $\sum\limits_{i\in S}\pi_i(\mu_i+0.5\sigma_i)<0$, then for $\theta\in[0,1]$ and any $\varepsilon\in(0,\lambda)$, where $\lambda=|\sum\limits_{i\in S}\pi_i(\mu_i+0.5\sigma_i)|$, there is a $\Delta^*\in(0,1)$ with $2(1-\theta)\Delta^*(\max\limits_{i\in S}|\mu_i|)<1$ such that for any $\Delta<\Delta^*$, the θ-method (5) has the property that

$$\limsup_{k\to\infty}\frac{1}{k\Delta}\log|X_k|\leqslant\sum_{i\in S}\pi_i(2\mu_i+\sigma_i)+\varepsilon<0, \text{ a.s.} \quad (9)$$

Proof We divide the proof into three steps.

Step 1 For any $\theta\in[0,1]$, we rewrite (5) that

$$X_{k+1}+(\theta-1)\Delta f(X_{k+1},r_k^\Delta)= X_k+\theta\Delta f(X_k,r_k^\Delta)+g(X_k,r_k^\Delta)\Delta B_k$$

and

$$\begin{aligned}&|X_{k+1}|^2[1+2(\theta-1)\frac{\langle X_{k+1},f(X_{k+1},r_k^\Delta)\rangle\Delta}{|X_{k+1}|^2}+\\&\frac{(\theta-1)^2|f(X_{k+1},r_k^\Delta)|^2\Delta^2}{|X_{k+1}|^2}]=\\&|X_k|^2[1+\frac{1}{|X_k|^2}(2\langle X_k,\theta\Delta f(X_k,r_k^\Delta)+\\&g(X_k,r_k^\Delta)\Delta B_k\rangle+|\theta\Delta f(X_k,r_k^\Delta)+\\&g(X_k,r_k^\Delta)\Delta B_k|^2)].\end{aligned} \quad (10)$$

By Assumption 1, which implies $f(t,0)=0$, $\forall t>0$, we get from (3) that

$$\begin{aligned}&|X_{k+1}|^2[1+2(\theta-1)\frac{\langle X_{k+1},f(X_{k+1},r_k^\Delta)\rangle\Delta}{|X_{k+1}|^2}]\geqslant\\&|X_{k+1}|^2[1+2(\theta-1)\Delta\mu_{r_k^\Delta}],\\&|X_{k+1}|^2[1+2(\theta-1)\Delta\mu_{r_k^\Delta}]\leqslant\\&|X_k|^2[1+\frac{1}{|X_k|^2}(2\langle X_k,\theta\Delta f(X_k,r_k^\Delta)+\\&g(X_k,r_k^\Delta)\Delta B_k\rangle+|\theta\Delta f(X_k,r_k^\Delta)+\\&g(X_k,r_k^\Delta)\Delta B_k|^2)],\end{aligned} \quad (11)$$

where Δ is the stepsize, $\langle X_{k+1},f(X_{k+1},r_k^\Delta)\rangle\leqslant\mu_{r_k^\Delta}|X_{k+1}|^2$, $\Delta B_k:=B((k+1)\Delta)-B(k\Delta)$. Letting

$$\begin{aligned}&\xi_k(r_k^\Delta,\theta)=\\&\frac{1}{|X_k|^2}(2\langle X_k,\theta\Delta f(X_k,r_k^\Delta)+g(X_k,r_k^\Delta)\Delta B_k\rangle+\\&|\theta\Delta f(X_k,r_k^\Delta)+g(X_k,r_k^\Delta)\Delta B_k|^2)],\end{aligned} \quad (12)$$

then we have

$$|X_{k+1}|^2\leqslant\frac{|X_k|^2}{1+2(\theta-1)\Delta\mu_{r_k^\Delta}}(1+\xi_k(r_k^\Delta,\theta)), \quad (13)$$

where $X_k\neq0$, otherwise $\xi_k(r_k^\Delta,\theta)$ is set to -1. Clearly, $\xi_k(r_k^\Delta,\theta)\geqslant-1$. Let $\mathcal{G}_t=\sigma(\{r(u)\}_{u\geqslant0},\{B(s)\}_{0\leqslant s\leqslant t})$, namely the σ-algebra produced by $\{r(u)\}_{u\geqslant0}$ and $\{B(s)\}_{0\leqslant s\leqslant t}$, we take the conditional expectation on $|X_{k+1}|^p$,

$$\begin{aligned}&\mathrm{E}(|X_{k+1}|^p|_{\mathcal{G}_{k\Delta}})\leqslant\\&\frac{|X_k|^p}{[1+2(\theta-1)\Delta\mu_{r_k^\Delta}]^{\frac{p}{2}}}\mathbf{1}_{\{x_k\neq0\}}\mathrm{E}[1+\xi_k(r_k^\Delta,\theta)^{\frac{p}{2}}\Big|_{\mathcal{G}_{k\Delta}}].\end{aligned} \quad (14)$$

For any $p\in(0,1)$, from the following inequality

$$\begin{aligned}&(1+u)^{p/2}\leqslant\\&1+\frac{p}{2}u+\frac{p(p-2)}{2^2\cdot2!}u^2+\frac{p(p-2)(p-4)}{2^3\cdot3!}u^3,\end{aligned} \quad (15)$$

where $u\geqslant-1$, we can estimate that

$$\begin{aligned}&\mathrm{E}(|X_{k+1}|^p|_{\mathcal{G}_{k\Delta}})\leqslant\\&\frac{|X_k|^p}{[1+2(\theta-1)\Delta\mu_{r_k^\Delta}]^{\frac{p}{2}}}\mathbf{1}_{\{x_k\neq0\}}\mathrm{E}(1+\frac{p}{2}\xi_k(r_k^\Delta,\theta)+\end{aligned}$$

$$\frac{p(p-2)}{2^2\cdot 2!}{\xi_k}^2(r_k^{\Delta},\theta)+\frac{p(p-2)(p-4)}{2^3\cdot 3!}\cdot {\xi_k}^3(r_k^{\Delta},\theta)|\mathcal{G}_{k\Delta}), \tag{16}$$

where $\mathbf{1}_A$ denotes the indicator for A. Since

$$\begin{cases} \mathrm{E}({\Delta B_k}^{2n-1}|\mathcal{G}_{k\Delta})=\mathrm{E}({\Delta B_k}^{2n-1})=0, \\ \mathrm{E}({\Delta B_k}^{2n}|\mathcal{G}_{k\Delta})=\mathrm{E}({\Delta B_k}^{2n})= \\ \qquad (\Delta)^n\cdot(2n-1)!!, \end{cases} \tag{17}$$

we see that

$$\begin{aligned} &\mathbf{1}_{\{x_k\neq 0\}}\mathrm{E}(\xi_k(r_k^{\Delta},\theta)\,|\mathcal{G}_{k\Delta})= \\ &\mathbf{1}_{\{x_k\neq 0\}}\mathrm{E}\{\frac{1}{|X_k|^2}[2\left\langle X_k,\theta\Delta f(X_k,r_k^{\Delta})\right\rangle+ \\ &2\left\langle X_k,g(X_k,r_k^{\Delta})\Delta B_k\right\rangle+(\theta\Delta)^2\left|f(X_k,r_k^{\Delta})\right|^2+ \\ &\left|g(X_k,r_k^{\Delta})\right|^2(\Delta B_k)^2+ \\ &2\theta\Delta\left\langle f(X_k,r_k^{\Delta}),g(X_k,r_k^{\Delta})\right\rangle\Delta B_k]\,|\mathcal{G}_{k\Delta}\}= \\ &\mathbf{1}_{\{x_k\neq 0\}}\{\frac{1}{|X_k|^2}[2\left\langle X_k,\theta\Delta f(X_k,r_k^{\Delta})\right\rangle+ \\ &(\theta\Delta)^2\left|f(X_k,r_k^{\Delta})\right|^2+\left|g(X_k,r_k^{\Delta})\right|^2\Delta]\,|\mathcal{G}_{k\Delta}\}\leqslant \\ &\mathbf{1}_{\{x_k\neq 0\}}\{\frac{1}{|X_k|^2}[2\theta\Delta\mu_{r_k^{\Delta}}|X_k|^2+(\theta\Delta)^2h^2|X_k|^2+ \\ &\left|g(X_k,r_k^{\Delta})\right|^2\Delta]\}. \end{aligned} \tag{18}$$

Similarly, we can show that

$$\begin{aligned} &\mathbf{1}_{\{x_k\neq 0\}}\mathrm{E}(\xi_k^2(r_k^{\Delta},\theta)\,|\mathcal{G}_{k\Delta})= \\ &\mathbf{1}_{\{x_k\neq 0\}}\{\frac{1}{|X_k|^4}[4\left\langle X_k,\theta\Delta f(X_k,r_k^{\Delta})\right\rangle^2+ \\ &4\left\langle X_k,g(X_k,r_k^{\Delta})\right\rangle^2\Delta+(\theta\Delta)^4\left|f(X_k,r_k^{\Delta})\right|^4+ \\ &3\Delta^2\left|g(X_k,r_k^{\Delta})\right|^4+ \\ &4\theta^2\Delta^3\left|\left\langle f(X_k,r_k^{\Delta}),g(X_k,r_k^{\Delta})\right\rangle\right|^2+ \\ &4\left\langle X_k,\theta\Delta f(X_k,r_k^{\Delta})\right\rangle(\theta\Delta)^2\left|f(X_k,r_k^{\Delta})\right|^2+ \\ &4\left\langle X_k,\theta\Delta f(X_k,r_k^{\Delta})\right\rangle\left|g(X_k,r_k^{\Delta})\right|^2\Delta+ \\ &8\left\langle X_k,g(X_k,r_k^{\Delta})\right\rangle\left\langle f(X_k,r_k^{\Delta}),g(X_k,r_k^{\Delta})\right\rangle\theta\Delta^2+ \\ &2\theta^2\Delta^3\left|f(X_k,r_k^{\Delta})\right|^2\left|g(X_k,r_k^{\Delta})\right|^2]\}\geqslant \\ &\mathbf{1}_{\{x_k\neq 0\}}\frac{4\left\langle X_k,g(X_k,r_k^{\Delta})\right\rangle^2\Delta}{|X_k|^4}-C_{\theta,h}\Delta^2, \end{aligned} \tag{19}$$

$$\mathbf{1}_{\{x_k\neq 0\}}\mathrm{E}(\xi_k^3(r_k^{\Delta},\theta)\,|\mathcal{G}_{k\Delta})\leqslant C_{\theta,h}\Delta^2, \tag{20}$$

where $C_{\theta,h}$ is a constant dependent on θ and h. Substituting (18) – (20) into (16), and then from (4) and Assumption 1, we obtain

$$\begin{aligned} &\mathrm{E}(|X_{k+1}|^p\,|\mathcal{G}_{k\Delta})\leqslant \\ &\frac{|X_k|^p}{[1+2(\theta-1)\Delta\mu_{r_k^{\Delta}}]^{p/2}}\mathbf{1}_{\{x_k\neq 0\}}\{1+\frac{p}{2}[2\theta\Delta\mu_{r_k^{\Delta}}+ \\ &(\theta\Delta h)^2+\frac{\left|g(X_k,r_k^{\Delta})\right|^2\Delta}{|X_k|^2}]+\frac{p(p-2)}{8}\times \\ &\frac{4\left\langle X_k,g(X_k,r_k^{\Delta})\right\rangle^2\Delta}{|X_k|^4}+C_{p,\theta,h}\Delta^2\}\leqslant \\ &\frac{|X_k|^p}{[1+2(\theta-1)\Delta\mu_{r_k^{\Delta}}]^{p/2}}\mathbf{1}_{\{x_k\neq 0\}}\{1+\frac{p(p-2)}{2}\times \\ &\frac{\left\langle X_k,g(X_k,r_k^{\Delta})\right\rangle^2\Delta}{|X_k|^4}+\frac{p}{2}\frac{\left|g(X_k,r_k^{\Delta})\right|^2\Delta}{|X_k|^2}+ \\ &p\theta\Delta\mu_{r_k^{\Delta}}+\frac{p}{2}(\theta\Delta h)^2+C_{p,\theta,h}\Delta^2\}\leqslant \\ &\frac{|X_k|^p}{[1+2(\theta-1)\Delta\mu_{r_k^{\Delta}}]^{p/2}}\{1+\frac{p}{2}\Delta\sigma_{r_k^{\Delta}}+\frac{p^2}{2}h^2\Delta+ \\ &p\theta\Delta\mu_{r_k^{\Delta}}+\bar{C}_{p,\theta,h}\Delta^2\}, \end{aligned} \tag{21}$$

where $C_{p,\theta,h}$ is a constant dependent on p,θ and h. $\bar{C}_{p,\theta,h}=C_{p,\theta,h}\Delta^2+\frac{p}{2}(\theta\Delta h)^2$. $\sigma_{r_k^{\Delta}}$ means that we apply $g(X_k,r_k^{\Delta})$ to (4).

Step 2 For any $\varepsilon\in(0,\lambda)$, we choose p sufficiently small to confirm that $ph^2\leqslant\frac{1}{4}\varepsilon$. Then we have

$$\begin{aligned} &(1-2(1-\theta)\Delta\mu_{r_k^{\Delta}})^{p/2}\geqslant \\ &1-p(1-\theta)\Delta\mu_{r_k^{\Delta}}-C_3\Delta^2, \end{aligned} \tag{22}$$

where $C_3=C_3(p,\theta)>0$ for sufficiently small Δ. By further reducing Δ, we may ensure that the following inequalities hold

$$\begin{aligned} &\bar{C}_{p,\theta,h}\Delta<\frac{1}{8}p\varepsilon,\ C_3\Delta<\frac{1}{4}p\varepsilon, \\ &|p(1-\theta)\Delta\mu_{r_k^{\Delta}}+\frac{1}{4}p\varepsilon\Delta|\leqslant\frac{1}{2}. \end{aligned} \tag{23}$$

Using (22) and (23), (21) becomes

$$\begin{aligned} &\mathrm{E}(|X_{k+1}|^p|\mathcal{G}_{k\Delta})\leqslant \\ &\frac{|X_k|^p(1+\frac{p}{2}\Delta\sigma_{r_k^{\Delta}}+p\theta\Delta\mu_{r_k^{\Delta}}+\frac{1}{4}p\varepsilon\Delta)}{1-p(1-\theta)\Delta\mu_{r_k^{\Delta}}-C_3\Delta^2}\leqslant \\ &\frac{1+\frac{p}{2}\Delta(\sigma_{r_k^{\Delta}}+2\theta\mu_{r_k^{\Delta}}+\frac{1}{2}\varepsilon)}{1-p(1-\theta)\Delta\mu_{r_k^{\Delta}}-\frac{1}{4}p\varepsilon\Delta}|X_k|^p. \end{aligned} \tag{24}$$

Obviously, for any $u\in[-\frac{1}{2},\frac{1}{2}]$, we have

$$\begin{aligned} &\frac{1}{1-u}=1+u+u^2\sum_{i=0}^{\infty}u^i\leqslant \\ &1+u+u^2\sum_{i=0}^{\infty}(\frac{1}{2})^i=1+u+2u^2. \end{aligned} \tag{25}$$

By further reducing Δ to insure that

$$2p[(1-\theta)\mu_{r_k^{\Delta}}+\frac{1}{4}\varepsilon]^2\Delta+\frac{1}{2}(\sigma_{r_k^{\Delta}}+$$

$$2\theta\mu_{r_k^\Delta} + \frac{1}{2}\varepsilon) \times \{p[(1-\theta)\mu_{r_k^\Delta} + \frac{1}{4}\varepsilon]\Delta + 2p^2[(1-\theta)\mu_{r_k^\Delta} + \frac{1}{4}\varepsilon]^2\Delta^2\} \leqslant \frac{1}{4}\varepsilon. \quad (26)$$

Then using (25), (24) becomes

$$\begin{aligned}&\mathrm{E}(|X_{k+1}|^p|_{\mathcal{G}_{k\Delta}}) \leqslant \\ &|X_k|^p(1+\frac{p}{2}\Delta(\sigma_{r_k^\Delta} + 2\theta\mu_{r_k^\Delta} + \frac{1}{2}\varepsilon)) \times \\ &\{1 + p[(1-\theta)\mu_{r_k^\Delta} + \frac{1}{4}\varepsilon]\Delta + \\ &2p^2[(1-\theta)\mu_{r_k^\Delta} + \frac{1}{4}\varepsilon]^2\Delta^2\} \leqslant \\ &[1 + p(\frac{1}{2}\sigma_{r_k^\Delta} + \theta\mu_{r_k^\Delta} + (1-\theta)\mu_{r_k^\Delta} + \\ &\frac{2}{4}\varepsilon + \frac{1}{4}\varepsilon)\Delta] \times |X_k|^p = \\ &[1 + p(\frac{1}{2}\sigma_{r_k^\Delta} + \mu_{r_k^\Delta} + \frac{3}{4}\varepsilon)\Delta]|X_k|^p. \quad (27)\end{aligned}$$

Since this holds for all $k \geqslant 0$, we also have

$$\begin{aligned}&\mathrm{E}(|X_{k+1}|^p|_{\mathcal{G}_{(k-1)\Delta}}) \leqslant \\ &\mathrm{E}(|X_k|^p|_{\mathcal{G}_{(k-1)\Delta}})[1 + p(\frac{1}{2}\sigma_{r_k^\Delta} + \mu_{r_k^\Delta} + \frac{3}{4}\varepsilon)\Delta] \leqslant \\ &|X_{k-1}|^p \prod_{n=k-1}^{k}[1 + p(\frac{1}{2}\sigma_{r_n^\Delta} + \mu_{r_n^\Delta} + \frac{3}{4}\varepsilon)\Delta].\end{aligned}$$

Repeating this procedure yields

$$\begin{aligned}&\mathrm{E}(|X_{k+1}|^p|_{\mathcal{G}_0}) \leqslant \\ &|X_0|^p \prod_{n=0}^{k}[1 + p(\frac{1}{2}\sigma_{r_n^\Delta} + \mu_{r_n^\Delta} + \frac{3}{4}\varepsilon)\Delta]. \quad (28)\end{aligned}$$

Taking expectations on both sides, it reads

$$\mathrm{E}|X_{k+1}|^p \leqslant |X_0|^p \mathrm{E}\exp(\sum_{n=0}^{k}\log[1 + p(\frac{1}{2}\sigma_{r_n^\Delta} + \mu_{r_n^\Delta} + \frac{3}{4}\varepsilon)\Delta]). \quad (29)$$

For any $\varepsilon \in (0, \lambda)$, we therefore have

$$\begin{aligned}&\mathrm{e}^{p\Delta(\lambda-\varepsilon)(1+k)}\mathrm{E}(|X_{k+1}|^p) \leqslant \\ &|x_0|^p\mathrm{E}\exp\{p\Delta(\lambda-\varepsilon)(1+k) + \\ &\sum_{n=0}^{k}\log[1 + p(\frac{1}{2}\sigma_{r_n^\Delta} + \mu_{r_n^\Delta} + \frac{3}{4}\varepsilon)\Delta]\}. \quad (30)\end{aligned}$$

We further reduce Δ to ensure that

$$p(\frac{1}{2}\sigma_i + \mu_i + \frac{3}{4}\varepsilon)\Delta > -1,\ i \in S.$$

With the inequality

$$\log(1+x) \leqslant x,\ x > -1,$$

and by the ergodic property of the Markov chain, we derive that

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{1+k}\sum_{n=0}^{k}\log[1 + p(\frac{1}{2}\sigma_{r_n^\Delta} + \mu_{r_n^\Delta} + \frac{3}{4}\varepsilon)\Delta] = \\ &\sum_{i\in S}\pi_i\log[1 + p(\frac{1}{2}\sigma_i + \mu_i + \frac{3}{4}\varepsilon)\Delta] \leqslant \\ &p\Delta\sum_{i\in S}\pi_i(\frac{1}{2}\sigma_i + \mu_i + \frac{3}{4}\varepsilon) = \\ &p\Delta(-\lambda + \frac{3}{4}\varepsilon),\ \ \text{a.s.} \quad (31)\end{aligned}$$

It therefore follows that

$$\begin{aligned}&\lim_{k\to\infty}\{p\Delta(\lambda-\varepsilon)(1+k) + \\ &\sum_{n=0}^{k}\log[1 + p(\frac{1}{2}\sigma_{r_n^\Delta} + \mu_{r_n^\Delta} + \frac{3}{4}\varepsilon)\Delta]\} = -\infty,\ \ \text{a.s.} \quad (32)\end{aligned}$$

From (32) and by the Fatou lemma (see [23]), we obtain from (30) that

$$\lim_{k\to\infty}\mathrm{e}^{p\Delta(\lambda-\varepsilon)(1+k)}\mathrm{E}(|X_{k+1}|^p) = 0. \quad (33)$$

Step 3 Eq.(33) implies that there is an integer k_0 such that

$$\mathrm{E}(|X_k|^p) \leqslant \mathrm{e}^{-pk\Delta(\lambda-\varepsilon)},\ \forall k \geqslant k_0.$$

By the Chebyshev inequality, we get

$$\mathrm{P}\{|X_k|^p > k^2\mathrm{e}^{-pk\Delta(\lambda-\varepsilon)}\} \leqslant \frac{1}{k^2}.$$

Then applying the Borel-Cantelli lemma (see[21, p. 7]), we see that for almost all $\omega \in \Omega$,

$$|X_k|^p \leqslant k^2\mathrm{e}^{-pk\Delta(\lambda-\varepsilon)} \quad (34)$$

holds for all but finitely many $k \geqslant k_0$. That is to say, there exists a $k_1(\omega) \geqslant k_0$, for almost all $\omega \in \Omega$, when $k \geqslant k_1$, (34) holds. And this implies

$$\frac{1}{k\Delta}\log(|X_k|) \leqslant \frac{2\log k}{pk\Delta} - (\lambda - \varepsilon),$$

whenever $k \geqslant k_1$. We therefore obtain that

$$\limsup_{k\to\infty}\frac{1}{k\Delta}\log(|X_k|) \leqslant -\lambda + \varepsilon,\ \ \text{a.s.}$$

as required.

Remark 3.1 In Mao et al. [14] and Pang et al. [13], it was shown that the BEM method ($\theta = 0$) and the EM method ($\theta = 1$) may reproduce the almost sure exponential stability of the trivial solution of the hybrid SDE. This theorem extends the previous results and shows that θ-method can preserve the similar almost sure exponential stability of the underlying equation for $\theta \in [0, 1]$. This implies that the results of this paper is more general than that of Mao et al.[14] and Pang et al.[13].

Remark 3.2 For SDEs without Markov chains, using the semimartingale convergence theorem, Li et al.[8] have shown that the θ-method may reproduce the almost sure exponential stability of the trivial solution of the SDEs when $\theta \in (\frac{1}{2}, 1]$. But it is not the whole interval $[0, 1]$ for θ. Here, by the Chebyshev inequality, the Borel-Cantelli lemma and the inequality techniques, we show that for $\theta \in [0, 1]$, θ-method admits the corresponding stability of the trivial solution of the hybrid SDEs.

4 Generalization

In this section, we will generalize the results to the multi-dimensional Brownian motion case. Let the equation be

$$\mathrm{d}x(t)=f(x(t),r(t))\mathrm{d}t+\sum_{j=1}^{d} g_j(x(t),r(t))\mathrm{d}B_j(t) \tag{35}$$

on $t\geqslant 0$, given $x(0)=x_0\neq 0$ in $\mathbb{R}^n$ and $r(0)=i_0\in S$. $(B_1(t),\cdots,B_d(t))$ is assumed to be a d-dimensional Brownian motion defined on the complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\mathrm{P})$. As a standing hypothesis, we assume $f,g_1,\cdots,g_d$: $\mathbb{R}^n\times S\to\mathbb{R}^n$ are smooth enough for the hybrid SDE (35) to have a unique global solution $x(t)$ on $[0,\infty)$. Now, we impose the following assumptions.

Assumption 4.1 f and g_j satisfy the linear growth condition. That is, there is an $h>0$ such that

$$|f(x,i)|\vee|g_j(x,i)|\leqslant h|x|,$$
$$\forall(x,i)\in\mathbb{R}^n\times S,\ 1\leqslant j\leqslant d. \tag{36}$$

Assumption 4.2 There are constants μ_i $(i\in S)$ such that

$$(x-y)^{\mathrm{T}}(f(x,i)-f(y,i))\leqslant\mu_i|x-y|^2, \tag{37}$$

$\forall x,y\in\mathbb{R}^n$ and

$$\sigma_i:=\sup_{x\in\mathbb{R}^n,\,x\neq 0}\{\sum_{j=1}^{d}(\frac{|g_j(x,i)|^2}{|x|^2}-\frac{2|x^{\mathrm{T}}g_j(x,i)|^2}{|x|^4})\}<\infty. \tag{38}$$

The θ-method applied to hybrid SDE (35) produces approximations $X_k\approx x(t_k)$, with $t_k=k\Delta$, where $X_0=x(0), r_0^{\Delta}=i_0$ and

$$X_{k+1}=X_k+(1-\theta)f(X_{k+1},r_k^{\Delta})\Delta+\theta f(X_k,r_k^{\Delta})\Delta+\sum_{j=1}^{d} g_j(X_k,r_k^{\Delta})\Delta B_{jk}, \tag{39}$$

$k=0,1,2,\cdots$, where $\Delta>0$ is the stepsize, $\theta\in[0,1]$ is a fixed parameter, and $\Delta B_{jk}:=B_j((k+1)\Delta)-B_j(k\Delta)$.

The following theorems show that the almost sure exponential stability of the trivial solution of Eq.(35) and the θ-method (39), respectively.

Theorem 4.1[14] Let Assumptions 4.1 and 4.2 hold. If $\sum_{i\in S}\pi_i(\mu_i+0.5\sigma_i)<0$, then the trivial solution to Eq.(35) is almost surely exponentially stable for all $x_0\in\mathbb{R}^n$.

Theorem 4.2 Let Assumptions 4.1 and 4.2 hold. If $\sum_{i\in S}\pi_i(\mu_i+0.5\sigma_i)<0$, then for $\theta\in[0,1]$ and any $\varepsilon\in(0,\lambda)$, where $\lambda=|\sum_{i\in S}\pi_i(\mu_i+0.5\sigma_i)|$, there is a $\Delta^*\in(0,1)$ with $2(1-\theta)\Delta^*(\max_{i\in S}|\mu_i|)<1$ such that for any $\Delta<\Delta^*$, the θ-method (39) has the property that

$$\limsup_{k\to\infty}\frac{1}{k\Delta}\log|X_k|\leqslant\sum_{i\in S}\pi_i(2\mu_i+\sigma_i)+\varepsilon<0,\ \text{ a.s.} \tag{40}$$

Theorem 4.2 can be proved in a similar way as the scalar Brownian motion version for Theorem 3.2, so we omit the proof.

5 Example and simulations

In this section, we give a numerical example and its simulations with different θ to illustrate the almost sure exponential stability of the θ scheme for the hybrid SDE (1).

Example 5.1 Consider the following two-dimensional hybrid SDE:

$$\mathrm{d}x(t)=A(r(t))x(t)\mathrm{d}t+G(r(t))x(t)\mathrm{d}B(t) \tag{41}$$

on $t\geqslant 0$ with initial value $x(0)=x_0\in\mathbb{R}^2$, where $r(t)$ is a Markov chain with the state space $S=\{1,2\}$ and the generator

$$\Gamma=\begin{bmatrix}-1 & 1\\ 4 & -4\end{bmatrix}.$$

A and G are mappings from $S\to\mathbb{R}^{2\times 2}$. For convenience, we will write $A(i)=A_i$ and $G(i)=G_i$. Eq.(1) corresponds to

$$f(x,i)=A_ix \text{ and } g(x,i)=G_ix,\ (x,i)\in\mathbb{R}^2\times S,$$
$$A_1=\begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix},\ A_2=\begin{bmatrix}-1 & -2\\ 2 & 1\end{bmatrix},$$
$$G_1=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\ G_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

It is easy to see that its unique stationary distribution $\pi=(\pi_1,\pi_2)=(\frac{4}{5},\frac{1}{5})$. Clearly,

$$|f(x,i)|\vee|g(x,i)|\leqslant 3|x|,\ \forall(x,i)\in\mathbb{R}^2\times S,$$
$$(x-y)^{\mathrm{T}}(f(x,i)-f(y,i))\leqslant|x-y|^2,\ \forall(x,i)\in\mathbb{R}^2\times S.$$

It is also easy to compute that

$$\sigma_1:=\sup_{x\in\mathbb{R}^n,x\neq 0}(\frac{4x_1^2+4x_2^2}{x_1^2+x_2^2}-\frac{2(2x_1^2+2x_2^2)^2}{(x_1^2+x_2^2)^2})=-4,$$

$$\sigma_2:=\sup_{x\in\mathbb{R}^n,x\neq 0}(\frac{x_1^2+x_2^2}{x_1^2+x_2^2}-\frac{2(x_1^2+x_2^2)^2}{(x_1^2+x_2^2)^2})=-1,$$

$$\sum_{i\in S}\pi_i(2\mu_i+\sigma_i)=\pi_1(2\times 1-4)+\pi_2(2\times 1-1)=-\frac{7}{5}.$$

These show that the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, the exact solutions of Eq.(41) are almost surely exponentially stable. Applying the θ-scheme (5) and choosing $\Delta = 0.025$ and $\theta = 0.2,\ \theta = 0.6$ respectively, the simulations of Eq.(41) are as follows.

From Figs.1 and 2, we see that the numerical solution tends to zero. When $\Delta = 0.025,\ \theta = 0,\ \theta = 1$ and θ takes different values, the simulation of Eq.(41) is showed in Fig.3. The numerical solutions also tend to zero.

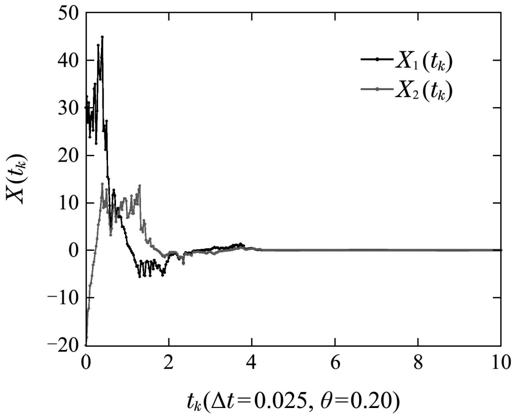

Fig. 1 Numerical solution of Eq.(41) with initial data $x_1(0) = 30,\ x_2(0) = -20\,(\theta = 0.2)$

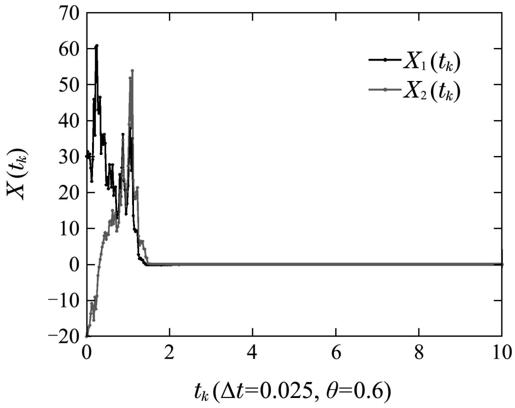

Fig. 2 Numerical solution of Eq.(41) with initial data $x_1(0) = 30,\ x_2(0) = -20\,(\theta = 0.6)$

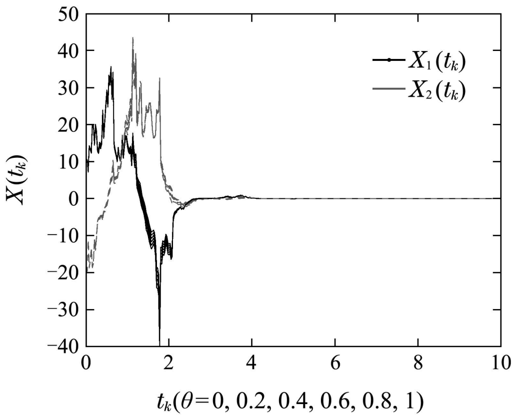

Fig. 3 Numerical solution of Eq.(41) with different θ under the same initial data

To our best knowledge, Fig.4 is the first time that appears to describe the stability of the θ-scheme. It shows the projective domain of numerical solution $X_1(t_k)$ and $X_2(t_k)$. Given the same Δ, x_0 and ΔB_k, we note that, horizontal axis represents the change of θ, vertical axis represents the projection of numerical solution $X_1(t_k)$ or $X_2(t_k)$. When θ takes a certain value, for example, $\theta = 0.2$, a list of points on the vertical axis show the projection of the same curve in different time. From Fig.4, we can see when θ takes different values in $[0, 1]$, the projection points on the vertical axis are always in a certain range, while they do not appear larger fluctuation. This shows that the θ-scheme is not so sensitive to the change of parameter θ. In other words, the θ-scheme is stable and reliable. The above simulations show that θ-scheme (5) can reproduce the stability of the trivial solutions of Eq.(41) for $\theta \in [0, 1]$.

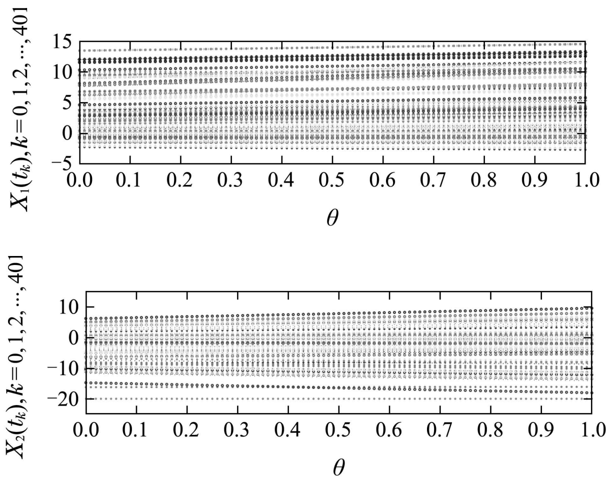

Fig. 4 The projective domain of the numerical solution $X(t_k)$ for Eq.(41) with different θ

6 Conclusions

In this paper, we discuss the θ-method can reproduce the almost sure exponential stability behavior of the trivial solution of the hybrid SDEs under the same conditions. The θ-method is a more general approach, which contains the existing EM method and the BEM method. And we show that for the whole interval $[0, 1]$ of θ, θ-method can reproduce the corresponding stability very well. This implies that the result of this paper is more general than the existing results in Mao et al.[14] and Pang et al.[13].

References:

[1] YUAN C, MAO X. Convergence of the Euler-Maruyama method for stochastic differential equations with Markovian switching [J]. *Mathematics and Computers in Simulation*, 2004, 64(2): 223 – 235.

[2] MARITON M. *Jump Linear Systems in Automatic Control* [M]. New York: Marcel Dekker, 1990.

[3] MAO X, YUAN C. *Stochastic Differential Equations with Markovian Switching* [M]. London: Imperial College Press, 2006.

[4] KLOEDEN P E, PLATEN E. *Numerical Solution of Stochastic Differential Equations* [M], Berlin: Springer-Verlag, 1992.

[5] BURRAGEA K, BURRAGE P, MITSUI T. Numerical solutions of stochastic differential equations-implementation and stability issues [J]. *Journal of Computational and Applied Mathematics*, 2000, 125(1): 171 – 182.

[6] SAITO Y, MITSUI T. Stability analysis of numerical schemes for stochastic differential equations [J]. *SIAM Journal on Numerical Analysis*, 1996, 33(6): 2254 – 2267.

[7] BURRAGE K, TIAN T. A note on the stability properties of the Euler methods for solving stochastic differential equations [J]. *New Zealand Journal of Mathematics*, 2000, 29(2): 115 – 127.

[8] CHEN L, WU F. Almost sure exponential stability of the θ-method for stochastic differential equations [J]. *Statistics and Probability Letters*, 2012, 82(9): 1669 – 1676.

[9] HIGHAM D J, MAO X, YUAN C. Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations [J]. *SIAM Journal on Numerical Analysis*, 2007, 45(2): 592 – 609.

[10] HIGHAM D J, MAO X, STUART A M. Exponential mean-square stability of numerical solutions to stochastic differential equations [J]. *LMS Journal of Computation and Mathematics*, 2003, 6: 297 – 313.

[11] HIGHAM D J. A-stability and stochastic mean-square stability [J]. *BIT Numerical Mathematics*, 2000, 40(2): 404 – 409.

[12] HIGHAM D J. Mean-square and asymptotic stability of the stochastic theta method [J]. *SIAM Journal on Numerical Analysis*, 2000, 38(3): 753 – 769.

[13] PANG S, DENG F, MAO X. Almost sure and moment exponential stability of Euler-Maruyama discretizations for hybrid stochastic differential equations [J]. *Journal of Computational and Applied Mathematics*, 2008, 213(1): 127 – 141.

[14] MAO X, SHEN Y, GRAY A. Almost sure exponential stability of backward Euler-Maruyama discretizations for hybrid stochastic differential equations [J]. *Journal of Computational and Applied Mathematics*, 2011, 235(5): 1213 – 1226.

[15] MAO X. Exponential stability of equidistant Euler-Maruyama approximations of stochastic differential delay equations [J]. *Journal of Computational and Applied Mathematics*, 2007, 200(1): 297 – 316.

[16] HUANG C. Mean square stability and dissipativity of two classes of theta methods for systems of stochastic delay differential equations [J]. *Journal of Computational and Applied Mathematics*, 2014, 259(Part A): 77 – 86.

[17] WU F, MAO X, SZPRUCH L. Almost sure exponential stability of numerical solutions for stochastic delay differential equations [J]. *Numerische Mathematik*, 2010, 115(4): 681 – 697.

[18] BAKER C T H, BUCKWAR E. Exponential stability in p-th mean of solutions and of convergent Euler-type solutions, of stochastic delay differential equations [J]. *Journal of Computational and Applied Mathematics*, 2005, 184(2): 404 – 427.

[19] LIU M, CAO W, FAN Z. Convergence and stability of the semi-implicit Euler method for a linear stochastic differential delay equation [J]. *Journal of Computational and Applied Mathematics*, 2004, 170(2): 255 – 268.

[20] CAO W, LIU M, FAN Z. MS-stability of the Euler-Maruyama method for stochastic differential delay equations [J]. *Applied Mathematics and Computation*, 2004, 159(1): 127 – 135.

[21] MAO X. *Stochastic Differential Equations and Applications* [M]. Horwood, Chichester, UK: Horwood Publishing, 1997.

[22] SHEN Yi, ZHANG Yumin, LIAO Xiaoxin. Exponential stability of nonlinear stochastic large-scale systems with delays [J]. *Control Theory & Applications*, 2002, 19(4): 571 – 574.
(沈轶, 张玉民, 廖晓昕. 时滞非线性随机大系统的指数稳定性[J]. 控制理论与应用, 2002, 19(4): 571 – 574.)

[23] LOÈVE M. *Probability Theory* [M]. London: D. Van Nostrand Company Inc., 1963.

作者简介:

莫浩艺 (1979–), 女, 博士研究生, 讲师, 主要研究方向为随机微分方程的数值解, E-mail: mhy04@163.com;

邓飞其 (1961–), 男, 博士生导师, 教授, 主要研究方向为复杂系统的稳定、镇定及鲁棒控制, E-mail: aufqdeng@scut.edu.cn;

彭云建 (1974–), 男, 硕士生导师, 副教授, 主要研究方向为随机电力系统的稳定及控制, E-mail: pengyunjian@sina.com.