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多阶段均值–方差资产负债管理的随机控制*

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摘要: 资产负债管理研究如何合理分配资产以到达最小化风险同时确保期望剩余财富(财富减去负债)达到一定水平. 本文在均值–方差投资组合理论的框架下研究两类资产负债管理模型, 包括带有跨期均值–方差投资目标和带有非破产约束的模型. 由于在动态规划意义下, 方差不具有可分性质, 传统的随机最优控制方法难以直接应用. 如采用处理动态均值–方差优化问题的嵌入法来解决以上问题会带来计算上的困难. 本文借鉴平均场控制的思想对以上两类问题加以研究. 本文假设了非常宽泛的市场模型: 所有的资产都是风险资产; 债务和风险资产之间存在相关性. 在此市场假设模型下, 本文给出了最优投资策略(控制率)的解析表达式和均值–方差有效前沿的表达形式. 本研究成果为投资者提供了新的投资策略, 可应用于更复杂的资产负债管理中.

关键词: 多期投资组合; 随机控制; 资产负债管理; 平均场方法; 金融应用

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Stochastic control for multiperiod mean-variance asset-liability management

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Abstract: The objective of asset and liability management (ALM) is to seek an optimal portfolio policy such that a risk measure (variance of the surplus) is minimized while achieving a certain threshold level for the expected value of the surplus. This paper studies two multiperiod mean-variance-based ALM models including the one with intertemporal risk control and the other one with no bankruptcy restriction. Due to the nonseparability of the variance, it is hard to solve this problem by stochastic control approach directly. Instead of adopting the widely used embedding method which may encounter computational difficulty in solving these problems, we develop a novel stochastic control approach of a mean-field type. Under a general market assumption, the analytical portfolio policies and mean-variance efficient frontiers are derived for these two ALM problems. The new result developed in this paper provides investors with efficient ways in characterizing their optimal portfolio and liability management strategies for these sophisticated mean-variance-based ALM models.

Key words: multiperiod portfolio optimization; stochastic control systems; asset-liability management; mean-field formulation; finance applications

1 Introduction

Portfolio selection attempts to find the best allocation of the initial investment among a basket of assets in order to minimize a risk measure while achieving a certain level of the expected return. Such a core concept to strike a balance between the mean and risk is fundamental and essential in modern investment theory. The mean-risk framework in portfolio selection was initiated by the seminal work of the mean-variance (MV) portfolio selection theory developed by Markowitz[1] half century ago. Since then, significant efforts have been witnessed in extending the static MV portfolio selection theory to dynamic MV portfolio selection. However, due to the nonseparability of the variance term in the sense of dynamic programming, such an extension was blocked until 2000 when Li and Ng[2] and Zhou and Li[3] finally made breakthrough in solving, respectively, the discrete-time and continuous-time MV formulations analytically. The critical technique used in [2] and [3] is the so called embedding method, in which

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the original dynamic MV portfolio optimization problem is embedded into an auxiliary stochastic control problem, which is essentially of a linear-quadratic type of stochastic control. Following the similar embedding idea in [2], multiperiod dynamic MV portfolio selection models have been extended in various directions, see, for example, [4−7]. Readers may refer [6−7] and references therein for more complete surveys. Although the embedding technique succeeds in solving the conventional dynamic MV portfolio optimization problems, it encounters some computational difficulties or even fails when additional practical constraints are present in the multiperiod MV portfolio selection model, e.g., the no-bankruptcy restrictions [5] or the constraints for intertemporal risk control[4]. Recently, Cui et al. [8] showed that all these problems can be solved efficiently by using the so called mean-field stochastic control approach, and their research work motivates us to revisit the asset-liability management problem by using the mean-field formulation method as a solution methodology.

Besides investing in the security market, these investment institutions, such as insurance companies, pension fund and banks, also have to pay great attention to their liabilities. The asset-liability management (ALM) model considers the best decision on both the investment and the liabilities, i.e., the portfolio positions under the effects of the liability. The subject of ALM has been attracting a great deal of attentions from both academic community and financial industry, see, for example, [9−11]. As for the MV type of the ALM problems, Sharpe and Tint[9] proposed a static (single-period) MV formulation and showed the impact of liability on the investment performance. Using the embedding technique introduced for MV portfolio selection in [2], Leippold et al.[12] and Chui and Li[13] extended the mean-variance ALM model to dynamic settings of discrete-time and continuous-time, respectively. The past years have also seen several further extensions of dynamic mean-variance ALM in the literature, e.g., Yi et al.[14] considered the uncertain investment horizon in ALM, Zeng and Li[15] used the jump diffusion process to model the asset evolutions, and Chen and Yang [16] adopted the region switching model for the asset process. Recently, Yi et al.[17] adopted the mean-field formulation and solve the portfolio optimization problem with uncertain exit time. Cui et al.[18] extended such a result to ALM problem with an uncertain exit time in a market with one riskless asset, multiple risky assets and one liability.

In this work, we consider the mean-variance ALM problems with the intertemporal risk control and no bankruptcy restriction in an incomplete market with only risky assets and one liability. Instead of treating these problems separately, we consider a unified ALM formulation, under which the ALM problems with the intertemporal risk control and no bankruptcy restriction become its special cases. To avoid the computational difficulties arising from the embedding method[4−5], we adopt the mean-field control approach introduced by Cui et al.[8] for the mean-variance portfolio selection problem. We would like to point out several contributions of our work in this paper. We derive the analytical portfolio policy for the mean-variance ALM problem with a general market setting with all assets being risky. We show that the optimal portfolio policy is a linear feedback policy with respect to both the current wealth and liability. Our new result provides investors efficient ways in characterizing their optimal portfolio strategies for these sophisticated mean-variance-based ALM models. From technical point of view, due to the liability and our general market setting with only multiple risky assets, the state space in our model becomes two dimensional. Furthermore, since we assume that all assets are risky, the resulted mean-field type of dynamics are more general than the one studied in [8–18] and [17], where the state space is scalar and a risk-free asset exists. The most related work to our model is Cui et al.[18], which studied the ALM problem with uncertain exit time by using the mean-field control approach. To tackle the uncertain exiting time, an auxiliary problem is introduced in [18]. This auxiliary problem shares a similar formulation of problem $(\mathcal{P}_1)$ in our paper. However, compared to our model in which all assets are risky, the resulted mean-field dynamics of wealth in [18] is simpler since a risk-free asset is included. Thus, the auxiliary problem studied in [18] can be regarded as a special case of our result. In our paper, we also study the ALM model with no-bankruptcy restriction in problem $(\mathcal{P}_2)$, which is not covered in [18].

This paper is organized as follows. After we present two mean-variance ALM models in Section 2, we develop the optimal portfolio policies for these problems in Section 3. We demonstrate our solution procedure by an illustrative example in Section 4. In this paper, we use A' to denote the transpose of matrix A.

2 Problem formulation

We consider a capital market consisting of $n+1$ risky assets whose random returns evolve in total T periods labeled by $t=0,\cdots,T-1$. Let $\hat{\boldsymbol{r}}_t\in\mathbb{R}^n$ be the random return vector of the first n risky assets and $\bar{r}_t$ be the random return of the $(n+1)$-th asset in period t. We denote the random liability rate in period t as p_t. An investor enters the market with initial wealth x_0 and initial liability l_0 at period $t=0$, and he can allocate his wealth in the $n+1$ assets at the beginning of each period t, for all $t=0,\cdots,T-1$. We assume that the investor has the information of the first and second order moments and the correlation of the return vector of these risk assets and the liability rate, respectively. All the randomness are modeled by a complete probability space $\{\Omega, \mathcal{F}, P\}$, where Ω is the sample space, $\mathcal{F}$ is the complete filtration, and P is the probability measure. The information set (filtration) at the beginning of period t, $t = 1, \cdots, T-1$, is denoted as $\mathcal{F}_t$ which is the sigma algebra generated by the realization of random variables $\{\hat{\boldsymbol{r}}_k, \bar{r}_k, p_k\}$ for $k = 0, \cdots, t-1$. Note that the filtration at time $t = 0$ reduces to $\mathcal{F}_0 = \{\Omega, \varnothing\}$. We denote the conditional expectation under the filtration $\mathcal{F}_t$ as $\mathrm{E}[\cdot|\mathcal{F}_t]$. To simplify the notation, we use $\mathrm{E}[Y]$ for $\mathrm{E}[Y|\mathcal{F}_0]$, which is just the unconditional expectation. Furthermore, we use $\mathrm{Cov}[Y] = \mathrm{E}[YY'] - \mathrm{E}[Y]\mathrm{E}[Y']$ to denote the covariance matrix of a random vector Y and $\mathrm{Var}[y] = \mathrm{E}[y^2] - \mathrm{E}^2[y]$ for the variance of random variable y.

Assumption 1 Let $Y_t = [\hat{\boldsymbol{r}}_t, \bar{r}_t, p_t]'$ be the generalized return vector for $t = 0, \cdots, T-1$. We assume that Y_t, $t = 0, \cdots, T-1$, are statistically independent for different time periods, and the expected value $\mathrm{E}[Y_t]$ and the covariance matrix $\mathrm{Cov}[Y_t]$ are known, $t = 0, \cdots, T-1$.

Note that Assumption 1 means that we can compute the function of the moments related to $\{\hat{\boldsymbol{r}}_t, \bar{r}_t, p_t\}$ up to the order 2. Without loss of generality, we also assume a nondegenerate condition that matrix $\mathrm{Cov}[Y_t]$ is positive definite[2]. Note that, when we set the return of the $(n+1)$-th asset, $\bar{r}_t$, deterministic, we can regard the $(n+1)$-th asset as the risk-free asset, and our model thus covers the case where a risk-free asset exists in the market.

Let x_t be the wealth level at period t and $\boldsymbol{u}_t = (u_t^1, u_t^2, \cdots, u_t^n)'$ be the portfolio vector representing the dollar amounts invested in the first n risky asset. Then, at period t, the dollar amount allocated to the $(n+1)$-th asset is $x_t - \sum\limits_{i=1}^{n} u_t^i$, which further leads to the following stochastic difference equation which governs the wealth dynamics,

$$x_{t+1} = \bar{r}_t(x_t - \boldsymbol{1}'\boldsymbol{u}_t) + \hat{\boldsymbol{r}}_t'\boldsymbol{u}_t = \bar{r}_t x_t + \boldsymbol{r}_t'\boldsymbol{u}_t, \quad (1)$$
$$t = 0, \cdots, T-1,$$

where $\boldsymbol{1}$ is the all-one vector and $\boldsymbol{r}_t = \hat{\boldsymbol{r}}_t - \boldsymbol{1}\bar{r}_t$ is the excess return vector of the first n risky assets with respect to the $(n+1)$-th asset. Furthermore, we denote the liability at period t by l_t , $t = 0, \cdots, T-1$, which is governed by the following stochastic difference equation,

$$l_{t+1} = p_t l_t, \quad t = 0, \cdots, T-1, \quad (2)$$

where p_t is the random liability rate. As the same as in the literature (see, for example, [9] and [12]), the investor's portfolio policy does not affect the stochastic process of the liability, although the liability rate p_t and the excess return vector $\boldsymbol{r}_t$ are correlated. Finally, we denote the wealth surplus, the difference between the wealth and the liability at period t as $s_t = x_t - l_t$.

In this work, we consider two mean-variance-based ALM models. The first problem under consideration is the following mean-variance ALM model with the intertemporal risk control,

$$\begin{aligned} (\mathcal{P}_1): \quad & \max\ (\mathrm{E}[s_\mathrm{T}] - \omega_\mathrm{T}\mathrm{Var}[s_\mathrm{T}]) + \\ & \sum_{t\in\mathcal{I}} \alpha_t(\mathrm{E}[s_t] - \gamma_t\mathrm{Var}[s_t]), \\ & \text{s.t.} \begin{cases} x_{t+1} = \bar{r}_t x_t + \boldsymbol{r}_t'\boldsymbol{u}_t, \\ l_{t+1} = p_t l_t, \\ s_t = x_t - l_t,\ t = 1, \cdots, T, \end{cases} \end{aligned} \quad (3)$$

where $\mathcal{I} = \{\tau_1, \cdots, \tau_h\} \subset \{1, \cdots, T-1\}$ is the set of periods in which the investor needs to evaluate the mean-variance pair of the surplus, $\alpha_t \geqslant 0$ is the weighting coefficient measuring the importance of the mean-variance pair of time period t, $\gamma_t > 0$ and ω_T are the trade-off coefficients between the return and variance of s_t at period t and period T, respectively. Note that problem $(\mathcal{P}_1)$ covers several well studied models in the literature. If we let $\mathcal{I} = \emptyset$, then problem $(\mathcal{P}_1)$ degenerates to the conventional mean-variance ALM model studied in [12] and [13]. If we let $p_t = 0$ for $t = 0, \cdots, T-1$, then problem $(\mathcal{P}_1)$ becomes the conventional mean-variance portfolio optimization problem with intertemporal restriction investigated in [4] and [19].

Besides problem $(\mathcal{P}_1)$, we are also interested in the mean-variance ALM model with no bankruptcy control, i.e., we want to control the bankruptcy probability under a threshold level for all time periods, $P(s_t \leqslant \eta_t) \leqslant \beta_t$, $t = 1, \cdots, T-1$, where η_t is the disaster level and β_t is the tolerance probability of the bankruptcy event, for $t = 1, \cdots, T-1$. To integrate such constraints into the ALM problem, we use the Tchebycheff inequality as proposed in [5] to relax the problem into the following more tractable formulation,

$$\begin{aligned} (\mathcal{P}_2): \quad & \max\ \mathrm{E}[s_\mathrm{T}] - \omega_\mathrm{T}\mathrm{Var}[s_\mathrm{T}], \\ & \text{s.t. } \{x_t, \boldsymbol{u}_t\} \text{ satisfies (3)}, \\ & \quad \mathrm{Var}[s_t] \leqslant \beta_t(\mathrm{E}[s_t] - \eta_t)^2, \\ & \quad t = 1, \cdots, T-1. \end{aligned} \quad (4)$$

To solve both problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, we first define the following vectors:

$$\boldsymbol{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \boldsymbol{B}_t = \begin{pmatrix} \bar{r}_t & 0 \\ 0 & p_t \end{pmatrix},\ \boldsymbol{h}_t = \begin{pmatrix} \boldsymbol{r}_t \\ \boldsymbol{0} \end{pmatrix}, \quad (5)$$
$$t = 0, \cdots, T-1.$$

Note that both $\boldsymbol{B}_t \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{h}_t \in \mathbb{R}^{2\times n}$ are random matrices. We also define vector $\boldsymbol{z}_t$ to include both the wealth, x_t, and the liability, l_t, as its components as follows:

$$\boldsymbol{z}_t = \begin{pmatrix} x_t \\ l_t \end{pmatrix},\ t = 0, \cdots, T-1.$$

Both problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are closely related to the following stochastic control problem:

$$\begin{aligned}(\mathcal{P}_3(\Lambda)) : \max_{\{\boldsymbol{u}_0,\cdots,\boldsymbol{u}_{T-1}\}} & \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_\mathrm{T}] - \omega_\mathrm{T} \times \\ & \mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_\mathrm{T} - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_\mathrm{T}])^2] - \\ & \sum_{t=1}^{T-1} \lambda_t \Big(\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_t])^2] - \\ & a_t \mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_t)^2] + b_t \mathrm{E}(\boldsymbol{e}'\boldsymbol{z}_t) - c_t\Big), \quad (6) \\ & \text{s.t. } \boldsymbol{z}_{t+1} = \boldsymbol{B}_t \boldsymbol{z}_t + \boldsymbol{h}_t \boldsymbol{u}_t, \quad (7) \\ & \quad t = 0, \cdots, T-1,\end{aligned}$$

where $\boldsymbol{z}_0 = (x_0, l_0)'$, $\Lambda = (\lambda_1, \cdots, \lambda_{\mathrm{T}-1})$ is given with $\lambda_k \geqslant 0$ for $k = 1, \cdots, T-1$, and $a_t \geqslant 0$, b_t, c_t are given parameters, $t = 1, \cdots, T-1$. While the state variables x_t and l_t are governed by the stochastic difference equations in (1) and (2), respectively, they are both observable at time t. Note that problem $(\mathcal{P}_3(\Lambda))$ is different from problem (GMV) studied in [8], since the liability is involved, the state vector $\boldsymbol{z}_t$ in problem $(\mathcal{P}_3(\Lambda))$ is two dimensional and the matrix $\boldsymbol{B}_t$ is a random matrix. A major difficulty which blocks a direct application of the traditional stochastic control approaches is due to the nonseparability in the sense of dynamic programming caused by the variance term. We use the mean-field type approach in this paper to conquer such a problem. In the following section, we first develop a solution method for problem $(\mathcal{P}_3(\Lambda))$ and then apply such a solution scheme to problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$.

3 Main results

In this section, we first develop the analytical solution for problem $(\mathcal{P}_3(\Lambda))$ by adopting a similar solution idea of the mean-field formulation given in [8]. Under Assumption 1, $\boldsymbol{h}_t$ is $\mathcal{F}_{t+1}$ measurable, and x_t and $\boldsymbol{u}_t$ are $\mathcal{F}_t$ measurable. Thus, we have $\mathrm{E}[\boldsymbol{h}_t'\boldsymbol{u}_t] = \mathrm{E}\big[\mathrm{E}[\boldsymbol{h}_t'\boldsymbol{u}_t|\mathcal{F}_t]\big] = \mathrm{E}\big[\mathrm{E}[\boldsymbol{h}_t'|\mathcal{F}_t]\boldsymbol{u}_t\big] = \mathrm{E}[\boldsymbol{h}_t']\mathrm{E}[\boldsymbol{u}_t]$. Furthermore, taking expectation on both sides of (7) yields $\mathrm{E}[\boldsymbol{z}_{t+1}]{=}\mathrm{E}[\boldsymbol{B}_t]\mathrm{E}[\boldsymbol{z}_t] + \mathrm{E}[\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{u}_t]$, $t = 0, \cdots, T-1$. Problem $(\mathcal{P}_3(\Lambda))$ can be now reformulated as follows:

$$\begin{aligned}&(\bar{\mathcal{P}}_3(\Lambda)) : \\ &\max\ \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_\mathrm{T}] - \omega_\mathrm{T}\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_\mathrm{T} - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_\mathrm{T}])^2] - \\ &\sum_{t=1}^{T-1} \lambda_t (\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_t])^2] - \\ &a_t\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_t)^2] + b_t\mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_t] - c_t), \\ &\text{s.t.} \begin{cases} \mathrm{E}[\boldsymbol{z}_{t+1}] = \mathrm{E}[\boldsymbol{B}_t]\mathrm{E}[\boldsymbol{z}_t] + \mathrm{E}[\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{u}_t], \\ t = 0, \cdots, T-1, \\ \boldsymbol{z}_{t+1} - \mathrm{E}[\boldsymbol{z}_{t+1}] = \\ \boldsymbol{B}_t\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{B}_t]\mathrm{E}[\boldsymbol{z}_t] + \boldsymbol{h}_t\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{u}_t] = \\ \boldsymbol{B}_t(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t]) + (\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])\mathrm{E}[\boldsymbol{z}_t] + \\ \boldsymbol{h}_t(\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t]) + (\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])\mathrm{E}[\boldsymbol{u}_t]. \end{cases}\end{aligned} \quad (8)$$

Before we give the main result, we introduce the following sequences of matrices and vectors, which are defined by backward recursions for $k = T-1, \cdots, 0$,

$$\begin{aligned}&\boldsymbol{M}_k = \\ &\lambda_k\boldsymbol{e}\boldsymbol{e}' - \mathrm{E}[\boldsymbol{B}_k'\boldsymbol{M}_{k+1}\boldsymbol{h}_k]\mathrm{E}^{-1}[\boldsymbol{h}_k'\boldsymbol{M}_{k+1}\boldsymbol{h}_k] \times \\ &\mathrm{E}[\boldsymbol{B}_k'\boldsymbol{M}_{k+1}\boldsymbol{h}_k]' + \mathrm{E}[\boldsymbol{B}_k'\boldsymbol{M}_{k+1}\boldsymbol{B}_k], \quad (9) \\ &\boldsymbol{Q}_k = \mathrm{E}[(\boldsymbol{h}_k - \mathrm{E}[\boldsymbol{h}_k])'\boldsymbol{M}_{k+1}(\boldsymbol{h}_k - \mathrm{E}[\boldsymbol{h}_k])] + \\ &\qquad \mathrm{E}[\boldsymbol{h}_k]'\boldsymbol{G}_{k+1}\mathrm{E}[\boldsymbol{h}_k], \quad (10) \\ &\boldsymbol{G}_k = \\ &-\lambda_k a_k\boldsymbol{e}\boldsymbol{e}' - \boldsymbol{W}_k'\boldsymbol{Q}_k^{-1}\boldsymbol{W}_k + \mathrm{E}[\boldsymbol{B}_k]'\boldsymbol{G}_{k+1}\mathrm{E}[\boldsymbol{B}_k] + \\ &\mathrm{E}[(\boldsymbol{B}_k - \mathrm{E}[\boldsymbol{B}_k])'\boldsymbol{M}_{k+1}(\boldsymbol{B}_k - \mathrm{E}[\boldsymbol{B}_k])], \quad (11) \\ &\boldsymbol{F}_k = \\ &-\lambda_k b_k\boldsymbol{e}' - \boldsymbol{F}_{k+1}\mathrm{E}[\boldsymbol{h}_k]\boldsymbol{Q}_k^{-1}\boldsymbol{W}_k + \boldsymbol{F}_{k+1}\mathrm{E}[\boldsymbol{B}_k], \quad (12) \\ &C_k = \\ &\lambda_k c_k + \frac{1}{4}\boldsymbol{F}_{k+1}\mathrm{E}[\boldsymbol{h}_k]\boldsymbol{Q}_k^{-1}\mathrm{E}[\boldsymbol{h}_k]'\boldsymbol{F}_{k+1}' + C_{k+1}, \quad (13) \\ &\boldsymbol{W}_k' = \\ &(\mathrm{E}[\boldsymbol{B}_k]'\boldsymbol{G}_{k+1}\mathrm{E}[\boldsymbol{h}_k]) + \mathrm{E}[(\boldsymbol{B}_k - \mathrm{E}[\boldsymbol{B}_k])' \times \\ &\boldsymbol{M}_{k+1}(\boldsymbol{h}_k - \mathrm{E}[\boldsymbol{h}_k])] \quad (14)\end{aligned}$$

with the boundary conditions being given as

$$\boldsymbol{M}_\mathrm{T} = \omega_\mathrm{T}\boldsymbol{e}\boldsymbol{e}',\ \boldsymbol{G}_\mathrm{T} = \boldsymbol{0},\ \boldsymbol{F}_\mathrm{T} = \boldsymbol{e}',\ C_\mathrm{T} = 0.$$

It is not difficult to verify from above definitions that matrices $\boldsymbol{M}_k$, $\boldsymbol{G}_k$ and $\boldsymbol{Q}_k$ are all positive semidefinite, for $t = 0, \cdots, T$. Furthermore, under Assumption 1, all the above matrices can be computed off-line.

Proposition 1 The optimal portfolio policy of problem $(\mathcal{P}_3(\Lambda))$ is given as

$$\begin{aligned}\boldsymbol{u}_t = \mathrm{E}[\boldsymbol{u}_t] - \mathrm{E}^{-1}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]' \times \\ \big(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t]\big), \quad (15)\end{aligned}$$

$$\mathrm{E}[\boldsymbol{u}_t] = \frac{1}{2}\boldsymbol{Q}_t^{-1}\mathrm{E}[\boldsymbol{h}_t]'\boldsymbol{F}_{t+1}' - \boldsymbol{Q}_t^{-1}\boldsymbol{W}_t\mathrm{E}[\boldsymbol{z}_t], \quad (16)$$

where the optimal expected value of $\boldsymbol{z}_t$ is

$$\begin{aligned}\mathrm{E}[\boldsymbol{z}_t] = \prod_{s=0}^{t-1}[\mathrm{E}[\boldsymbol{B}_s]z_0 - \mathrm{E}[\boldsymbol{h}_s]\boldsymbol{Q}_s^{-1}\lambda_s] + \\ \sum_{j=0}^{t-1}\tfrac{1}{2}\mathrm{E}[\boldsymbol{h}_j]\boldsymbol{Q}_j^{-1}\mathrm{E}[\boldsymbol{h}_j]'\boldsymbol{F}_{j+1}' \times \\ \prod_{\ell=j+1}^{t-1}[\mathrm{E}[\boldsymbol{B}_\ell] - \mathrm{E}[\boldsymbol{h}_\ell]\boldsymbol{Q}_\ell^{-1}\lambda_\ell]. \quad (17)\end{aligned}$$

Proof We define the value function of problem $(\mathcal{P}_3(\Lambda))$ at time t as

$$\begin{aligned}&J_t(\boldsymbol{z}_t, \mathrm{E}[\boldsymbol{z}_t]) = \\ &\max_{\boldsymbol{u}_t,\cdots,\boldsymbol{u}_{T-1}} \{\sum_{k=t}^{T-1} -\lambda_k\Big(\mathrm{E}\big[(\boldsymbol{e}'\boldsymbol{z}_k - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_k])^2\big] - \\ &\quad a_k\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_k)^2] + b_k\mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_k] - c_k\Big) + \\ &\quad (\mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_\mathrm{T}] - \omega_\mathrm{T}\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_\mathrm{T} - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_\mathrm{T}])^2])\},\end{aligned}$$

where the state variables are $\boldsymbol{z}_t$ and $\mathrm{E}[\boldsymbol{z}_t]$ which are both $\mathcal{F}_t$ measurable. Due to the principle of optimality and the Markov property of the problem formulation, the value function $J_t(\boldsymbol{z}_t, \mathrm{E}[\boldsymbol{z}_t])$ satisfies the following recursion:

$$J_t(\boldsymbol{z}_t, \mathrm{E}[\boldsymbol{z}_t]) = -\lambda_t(\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_t])^2] - a_t\mathrm{E}[(\boldsymbol{e}'\boldsymbol{z}_t)^2] + b_t\mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_t] - c_t) + \mathrm{E}[J_{t+1}(\boldsymbol{z}_{t+1}, \mathrm{E}[\boldsymbol{z}_{t+1}]) \mid \mathcal{F}_t] \tag{18}$$

with boundary condition $J_{\mathrm{T}}(\boldsymbol{z}_{\mathrm{T}}, \mathrm{E}[\boldsymbol{z}_{\mathrm{T}}]) = \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_{\mathrm{T}}] - \omega_{\mathrm{T}}\mathrm{E}^2\big[\boldsymbol{e}'\boldsymbol{z}_{\mathrm{T}} - \mathrm{E}[\boldsymbol{e}'\boldsymbol{z}_{\mathrm{T}}]\big]$. We claim that the value function takes the following form:

$$J_t(\boldsymbol{z}_t, \mathrm{E}[\boldsymbol{z}_t]) = -(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'\boldsymbol{M}_t(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t]) - \mathrm{E}[\boldsymbol{z}_t]'\boldsymbol{G}_t\mathrm{E}[\boldsymbol{z}_t] + \boldsymbol{F}_t\mathrm{E}[\boldsymbol{z}_t] + C_t. \tag{19}$$

At stage $t = T$, the claim is obviously true. After we assume that the claim (19) is true at stage $t+1$, we now check the value function $J_t(\boldsymbol{z}_t, \mathrm{E}[\boldsymbol{z}_t])$ at stage t. We first compute the expectation $\mathrm{E}[J_{t+1}(\boldsymbol{z}_{t+1}, \mathrm{E}[\boldsymbol{z}_{t+1}])]$ by using (8) as follows:

$$\begin{aligned}&\mathrm{E}[J_{t+1}(\boldsymbol{z}_{t+1}, \mathrm{E}[\boldsymbol{z}_{t+1}])\mid \mathcal{F}_t] = \\ &\mathrm{E}[-(\boldsymbol{z}_{t+1} - \mathrm{E}[\boldsymbol{z}_{t+1}])'\boldsymbol{M}_{t+1}(\boldsymbol{z}_{t+1} - \mathrm{E}[\boldsymbol{z}_{t+1}]) - \\ &\mathrm{E}[\boldsymbol{z}_{t+1}]'\boldsymbol{G}_{t+1}\mathrm{E}[\boldsymbol{z}_{t+1}] + \boldsymbol{F}_{t+1}\mathrm{E}[\boldsymbol{z}_{t+1}] + C_{t+1}]|\mathcal{F}_t] = \\ &\mathcal{G}_1 + 2\mathcal{G}_2,\end{aligned} \tag{20}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are given as follows:

$$\begin{aligned}&\mathcal{G}_1 = \\ &(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{B}_t](\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t]) + \\ &\mathrm{E}[\boldsymbol{z}_t]'\mathrm{E}[(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])'\boldsymbol{M}_{t+1}(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])]\mathrm{E}[\boldsymbol{z}_t] + \\ &(\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t])'\mathrm{E}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t](\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t]) + \\ &\mathrm{E}[\boldsymbol{u}_t]\mathrm{E}[(\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])'\boldsymbol{M}_{t+1}(\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])]\mathrm{E}[\boldsymbol{u}_t] + \\ &2(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t](\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t]) + \\ &2\mathrm{E}[\mathbf{z}_t']\mathrm{E}[(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])'\boldsymbol{M}_{t+1}(\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])]\mathrm{E}[\boldsymbol{u}_t] - \\ &\mathrm{E}[\boldsymbol{z}_t]'\mathrm{E}[\boldsymbol{B}_t]'\boldsymbol{G}_{t+1}\mathrm{E}[\boldsymbol{B}_t]\mathrm{E}[\boldsymbol{z}_t] - \\ &\mathrm{E}[\boldsymbol{u}_t]'\mathrm{E}[\boldsymbol{h}_t]'\boldsymbol{G}_{t+1}\mathrm{E}[\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{u}_t] - \\ &2\mathrm{E}[\boldsymbol{z}_t]'\mathrm{E}[\boldsymbol{B}_t]'\boldsymbol{G}_{t+1}\mathrm{E}[\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{u}_t] + \\ &\boldsymbol{F}_{t+1}\mathrm{E}[\boldsymbol{B}_t]\mathrm{E}[\boldsymbol{z}_t] + \boldsymbol{F}_{t+1}\mathrm{E}[\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{u}_t] + C_{t+1},\end{aligned} \tag{21}$$

$$\begin{aligned}&\mathcal{G}_2 = \\ &(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])]\mathrm{E}[\boldsymbol{z}_t] + \\ &(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}(\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])]\mathrm{E}[\boldsymbol{u}_t] + \\ &\mathrm{E}[\mathbf{z}_t']\mathrm{E}[(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])'\boldsymbol{M}_{t+1}\boldsymbol{h}_t](\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t]) + \\ &(\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t])'\mathrm{E}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}(\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])]\mathrm{E}[\boldsymbol{u}_t].\end{aligned} \tag{22}$$

The purpose of partitioning $\mathrm{E}[J_{t+1}(\boldsymbol{z}_{t+1}, \mathrm{E}[\boldsymbol{z}_{t+1}])]$ into $\mathcal{G}_1$ and $\mathcal{G}_2$ is that all the terms in $\mathcal{G}_2$ actually do not affect the entire value function with respect to $\mathcal{F}_0$. This fact has been pin-pointed in Lemma 3 of [8]. Actually, it is not too difficult to verify that all the terms in (22) will be vanished when we take expectation with respect to $\mathcal{F}_0$,

$$\mathrm{E}[(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])]\mathrm{E}[\boldsymbol{z}_t] \mid \mathcal{F}_0] = 0, \tag{23}$$

$$\mathrm{E}[(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}(\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])]\mathrm{E}[\boldsymbol{u}_t] \mid \mathcal{F}_0] = 0, \tag{24}$$

$$\mathrm{E}[\mathrm{E}[\mathbf{z}_t']\mathrm{E}[(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])'\boldsymbol{M}_{t+1}\boldsymbol{h}_t](\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t]) \mid \mathcal{F}_0] = 0, \tag{25}$$

$$\mathrm{E}[(\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t])'\mathrm{E}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}(\boldsymbol{h}_t - \mathrm{E}[\boldsymbol{h}_t])]\mathrm{E}[\boldsymbol{u}_t] \mid \mathcal{F}_0] = 0. \tag{26}$$

Thus, to maximize $\mathrm{E}\big[J_{t+1}(\boldsymbol{z}_{t+1}, \mathrm{E}[\boldsymbol{z}_{t+1}])|\mathcal{F}_t\big]$, we only need to focus on $\mathcal{G}_1$. Rearranging the terms in (21) by completing the square gives rise to

$$\begin{aligned}&\max_{(\mathrm{E}[\boldsymbol{u}_t], \boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t])} \mathcal{G}_1 = \\ &-((\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t]) + \mathrm{E}^{-1}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]' \times \\ &(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t]))'\mathrm{E}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]((\boldsymbol{u}_t - \mathrm{E}[\boldsymbol{u}_t]) + \\ &\mathrm{E}^{-1}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]'(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])) - \\ &(\mathrm{E}[\boldsymbol{u}_t] - \frac{1}{2}\boldsymbol{Q}_t^{-1}\mathrm{E}[\boldsymbol{h}_t]'\boldsymbol{F}_{t+1}' + \boldsymbol{Q}_t^{-1}\boldsymbol{W}_t\mathrm{E}[\boldsymbol{z}_t])' \times \\ &\boldsymbol{Q}_t(\mathrm{E}[\boldsymbol{u}_t] - \frac{1}{2}\boldsymbol{Q}_t^{-1}\mathrm{E}[\boldsymbol{h}_t]'\boldsymbol{F}_{t+1}' + \boldsymbol{Q}_t^{-1}\boldsymbol{W}_t\mathrm{E}[\boldsymbol{z}_t]) + \\ &(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])'(\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]\mathrm{E}^{-1}[\boldsymbol{h}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t] \times \\ &\mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{h}_t]' - \mathrm{E}[\boldsymbol{B}_t'\boldsymbol{M}_{t+1}\boldsymbol{B}_t]) \times \\ &(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t]) - \mathrm{E}[\boldsymbol{z}_t]'(-\boldsymbol{W}_t'\boldsymbol{Q}_t^{-1}\boldsymbol{W}_t + \\ &\mathrm{E}[\boldsymbol{B}_t]'\boldsymbol{G}_{t+1}\mathrm{E}[\boldsymbol{B}_t] + \mathrm{E}[(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])' \times \\ &\boldsymbol{M}_{t+1}(\boldsymbol{B}_t - \mathrm{E}[\boldsymbol{B}_t])])\mathrm{E}[\boldsymbol{z}_t] + (-\boldsymbol{F}_{t+1}\mathrm{E}[\boldsymbol{h}_t]\boldsymbol{Q}_t^{-1} \times \\ &\boldsymbol{W}_t + \boldsymbol{F}_{t+1}\mathrm{E}[\boldsymbol{B}_t])\mathrm{E}[\boldsymbol{z}_t] + \\ &\frac{1}{4}\boldsymbol{F}_{t+1}\mathrm{E}[\boldsymbol{h}_t]\boldsymbol{Q}_t^{-1}\mathrm{E}[\boldsymbol{h}_t]'\boldsymbol{F}_{t+1}' + C_{t+1}.\end{aligned}$$

Maximizing $\mathcal{G}_1$ yields the optimal policy specified in (15) and (16). Substituting the optimal policy back to the value function gives rise to the result in (19), where $\boldsymbol{M}_k$, $\boldsymbol{Q}_k$, $\boldsymbol{G}_k$, $\boldsymbol{F}_k$, C_k and $\boldsymbol{W}_k$ are defined in (9)–(14), respectively. Thus, we complete the proof of the claim in (19). Substituting the optimal $\boldsymbol{u}_t^*$ back to (8) yields the recursion for $\mathrm{E}[\boldsymbol{z}_t]$,

$$\begin{aligned}&\mathrm{E}[\boldsymbol{z}_{t+1}] = \\ &(\mathrm{E}[\boldsymbol{B}_t] - \mathrm{E}[\boldsymbol{h}_t]\boldsymbol{Q}_t^{-1}\boldsymbol{W}_t)\mathrm{E}[\boldsymbol{z}_t] + \\ &\frac{1}{2}\mathrm{E}[\boldsymbol{h}_t]\boldsymbol{Q}_t^{-1}\mathrm{E}[\boldsymbol{h}_t']\boldsymbol{F}_{t+1}',\ t = 0, \cdots, T,\end{aligned}$$

which further leads to the expression in (17).

Based on Proposition 1, we can compute the expected value and the variance of the surplus of the terminal wealth. From (17), we have

$$\begin{aligned}&\mathrm{E}[s_{\mathrm{T}}] = \boldsymbol{e}'\mathrm{E}[\boldsymbol{z}_{\mathrm{T}}] = \\ &\boldsymbol{e}'\prod_{s=0}^{t-1}[\mathrm{E}[\boldsymbol{B}_s]z_0 - \mathrm{E}[\boldsymbol{h}_s]\boldsymbol{Q}_s^{-1}\lambda_s] + \\ &\boldsymbol{e}'\sum_{j=0}^{t-1}\frac{1}{2}\mathrm{E}[\boldsymbol{h}_j]\boldsymbol{Q}_j^{-1}\mathrm{E}[\boldsymbol{h}_j]'\boldsymbol{F}_{j+1}' \times \\ &\prod_{\ell=j+1}^{t-1}[\mathrm{E}[\boldsymbol{B}_\ell] - \mathrm{E}[\boldsymbol{h}_\ell]\boldsymbol{Q}_\ell^{-1}\lambda_\ell].\end{aligned} \tag{27}$$

The variance of the terminal surplus can be computed by the following formula:

$$\mathrm{Var}[s_{\mathrm{T}}] = \mathrm{E}[e' \boldsymbol{z}_{\mathrm{T}} \boldsymbol{z}'_{\mathrm{T}} \boldsymbol{e}] - \mathrm{E}^2[s_{\mathrm{T}}] = \boldsymbol{e}'\mathrm{E}[\boldsymbol{z}_{\mathrm{T}} \boldsymbol{z}'_{\mathrm{T}}]\boldsymbol{e} - (\boldsymbol{e}'\mathrm{E}[\boldsymbol{z}_{\mathrm{T}}])^2.$$

In order to compute the term $\mathrm{E}[\boldsymbol{z}_{\mathrm{T}} \boldsymbol{z}'_{\mathrm{T}}]$, we first substitute the optimal policy $\mathbf{u}_t^*$ back to (7) to get the dynamics of $\boldsymbol{z}_k$ under the optimal policy,

$$\boldsymbol{z}_{k+1} = \boldsymbol{B}_k \boldsymbol{z}_k + \boldsymbol{h}_t(\frac{1}{2}\boldsymbol{Q}_k^{-1}\mathrm{E}[\boldsymbol{h}'_t]\boldsymbol{F}'_{k+1} - \boldsymbol{Q}_k^{-1}\boldsymbol{W}_k\mathrm{E}[\boldsymbol{z}_k] - \mathrm{E}^{-1}[\boldsymbol{h}'_t \boldsymbol{M}_{k+1}\boldsymbol{h}_k]\mathrm{E}[\boldsymbol{B}'_k \boldsymbol{M}_{k+1}\boldsymbol{h}_k]'(\boldsymbol{z}_t - \mathrm{E}[\boldsymbol{z}_t])).$$

We then compute $\boldsymbol{z}_k \boldsymbol{z}'_k$, take its expectation and obtain the dynamics of $\mathrm{E}[\boldsymbol{z}_t \boldsymbol{z}'_t]$.

Once we know how to compute the pair $(\mathrm{E}[s_{\mathrm{T}}], \mathrm{Var}[s_{\mathrm{T}}])$, we can plot the efficient frontier by varying ω_{T}.

Based on the optimal solution of problem $(\mathcal{P}_3(\Lambda))$, we are able to construct the optimal solution of problem $(\mathcal{P}_1)$.

Propositions 2 Setting $a_t = 0,\ c_t = 0$ and

$$\lambda_k = \begin{cases} \alpha_k \gamma_k, & \text{if } k \in \mathcal{I}, \\ 0, & \text{if } k \notin \mathcal{I}, \end{cases}$$

$$b_k = \begin{cases} -\alpha_k/\lambda_k, & \text{if } k \in \mathcal{I}, \\ 0, & \text{if } k \notin \mathcal{I}, \end{cases}$$

in the problem formulation of $(\mathcal{P}_3(\Lambda))$, the optimal policy given in (15) solves problem $(\mathcal{P}_1)$.

To solve problem $(\mathcal{P}_2)$, we introduce $(T-1)$ Lagrange multipliers, $\Lambda = (\lambda_1, \cdots, \lambda_{T-1})$ with $\lambda_t \geqslant 0$ for the constraint $\mathrm{Var}[s_t] \leqslant \beta_t(\mathrm{E}[s_t] - \eta_t)^2$, for $t = 1, \cdots, T-1$, and consider the following relaxation of problem $(\mathcal{P}_2)$,

$$(\bar{\mathcal{P}}_2(\Lambda)) : \max\ \mathrm{E}[s_{\mathrm{T}}] - \omega_{\mathrm{T}}\mathrm{Var}[s_{\mathrm{T}}] - \sum_{k=1}^{T-1} \lambda_k(\mathrm{Var}[s_k] - \beta_k(\mathrm{E}[s_k] - \eta_k)^2),$$
$$\text{s.t. } \{x_t, \boldsymbol{u}_t\} \text{ satisfies (3).}$$

Note that problem $(\bar{\mathcal{P}}_2(\Lambda))$ is the same as problem $(\mathcal{P}_3(\Lambda))$, once we fix the parameters a_k, b_k and c_k at some particular values.

Propositions 3 Setting a_k, b_k and c_k at

$$a_k = \beta_k,\ b_k = 2\beta_k \eta_k,\ c_k = \beta_k \eta_k^2 \tag{28}$$

for $k = 1, \cdots, T-1$, and setting $\Lambda^* = (\lambda_1^*, \cdots, \lambda_{T-1}^*)'$ as the minimizer of

$$(\lambda_1^*, \cdots, \lambda_{T-1}^*)' = \arg \min_{\lambda_1 \geqslant 0, \cdots, \lambda_{T-1} \geqslant 0} \boldsymbol{z}'_0 \boldsymbol{G}_0 \boldsymbol{z}_0 + \boldsymbol{F}_0 \boldsymbol{z}_0 + C_0 \tag{29}$$

in problem formulation of $(\mathcal{P}_3(\Lambda))$. Then, the portfolio policy in (15) for problem $(\mathcal{P}_3(\Lambda^*))$ solves problem $(\mathcal{P}_2)$.

Proof It is not hard to see that problem $(\mathcal{P}_2)$ is a convex optimization problem. Thus, the strong duality relationship holds. On the other hand, once we fixed parameters a_k, b_k and c_k as given in (28), problem $(\mathcal{P}_3(\Lambda))$ is equivalent to problem $(\bar{\mathcal{P}}_2(\Lambda))$, which is the Lagrangian relaxation of problem $(\mathcal{P}_2)$. If we use $v(\cdot)$ to denote the optimal objective value of problem $(\cdot)$, we then have

$$v(\mathcal{P}_2) = \min_{\lambda_1 \geqslant 0, \cdots, \lambda_{T-1} \geqslant 0} v(\bar{\mathcal{P}}_2(\Lambda)). \tag{30}$$

From Proposition 1, we know that

$$v(\bar{\mathcal{P}}_2(\Lambda)) = v(\mathcal{P}_3(\Lambda)) = \boldsymbol{z}'_0 \boldsymbol{G}_0 \boldsymbol{z}_0 + \boldsymbol{F}_0 \boldsymbol{z}_0 + C_0,$$

where $\boldsymbol{G}_0$ and $\boldsymbol{F}_0$ are defined in (11) and (12), respectively.

Remark 1 For real implementation, we can use the following simple gradient-projection method to find the optimal Λ^* that solves (29). Let Λ^* and J^* be the current incumbent solution and the objective value of problem (29) under this incumbent solution. Let us set the step size as $\delta > 0$, the reduction rate as $0 < \rho < 1$ and the stopping criteria as $\epsilon > 0$, which is a small positive number.

Step 1 For given Λ, compute $\boldsymbol{F}_0$, $\boldsymbol{G}_0$ and $J(\Lambda) := \boldsymbol{z}'_0 \boldsymbol{G}_0 \boldsymbol{z}_0 + \boldsymbol{F}_0 \boldsymbol{z}_0 + C_0$. If $J(\Lambda) < J^*$, let $\Lambda^* = \Lambda$ and $J^* = J(\Lambda)$.

Step 2 If the stopping criterion, $|J^* - J(\Lambda)| \leqslant \epsilon$, is satisfied, stop. Otherwise, go to Step 3.

Step 3 Compute the numerical gradient of $J(\Lambda)$ as $\nabla J(\Lambda)$ and update Λ as $\Lambda = \Lambda - \delta \nabla J(\Lambda)$. Compute the projection, $\Lambda = \max(\mathbf{0}, \Lambda)$, update the step size $\delta = \rho\delta$ and go to Step 1. Please refer to [20] for more details in designing gradient project algorithms.

4 Examples

In this section, we use a simple example to illustrate the solution procedure for problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. We use the same market parameters given in the example of [14]. We denote the random returns of the two risky assets (A and B) in the portfolio as $\bar{r}_t$ (for asset A) and $\hat{r}_t$ (for asset B), respectively, and the liability rate as p_t. All these random variables are assumed to be independent and identically distributed with respect to time. More specifically, the expected value and the covariance matrix of random vector $Y = (\hat{r}_t, \bar{r}_t, p_t)$ are given by

$$\begin{cases} \mathrm{E}[\bar{r}_t] = 1.259,\ \mathrm{E}[\hat{r}_t] = 1.243,\ \mathrm{E}[p_t] = 1.224, \\ t = 0, \cdots, T-1, \end{cases} \tag{31}$$

$$\begin{cases} \mathrm{Cov}[Y_t] = \begin{pmatrix} 0.0148 & 0.0185 & 0.0146 \\ 0.0185 & 0.0855 & 0.0105 \\ 0.0146 & 0.0105 & 0.0288 \end{pmatrix}, \\ t = 0, \cdots, T-1. \end{cases} \quad (32)$$

The investor's initial wealth and liability are $x_0 = 10$ and $l_0 = 5$, respectively, and his investment horizon is set as $T = 6$. Let the investor first consider model $(\mathcal{P}_1)$ with $\alpha_k = 0.5$ and $\gamma_k = 0.2$ for $k = 1, \cdots, 5$, and $\mathcal{I} = \{1, \cdots, 5\}$. By Proposition 2, the optimal portfolio level of asset B can be written as $\boldsymbol{u}_t = \mathrm{E}[\boldsymbol{u}_t] - \boldsymbol{K}_t(\boldsymbol{z}_t - \mathrm{E}[z_t])$, for $t = 0, \cdots, 5$, where

$$\mathrm{E}[\boldsymbol{u}_0] = -0.264,\ \mathrm{E}[\boldsymbol{u}_1] = -0.404,\ \mathrm{E}[\boldsymbol{u}_2] = -0.576,$$
$$\mathrm{E}[\boldsymbol{u}_3] = -0.798,\ \mathrm{E}[\boldsymbol{u}_4] = -1.075,\ \mathrm{E}[\boldsymbol{u}_5] = -1.430,$$
$$\boldsymbol{K}_0 = (1.436, -1.856),\ \boldsymbol{K}_1 = (1.436, -1.756),$$
$$\boldsymbol{K}_2 = (1.436, -1.662),\ \boldsymbol{K}_3 = (1.436, -1.573),$$
$$\boldsymbol{K}_4 = (1.436, -1.487),\ \boldsymbol{K}_5 = (1.436, -1.403),$$
$$\mathrm{E}[\boldsymbol{z}_0] = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ \mathrm{E}[\boldsymbol{z}_1] = \begin{pmatrix} 11.568 \\ 6.120 \end{pmatrix},$$
$$\mathrm{E}[\boldsymbol{z}_2] = \begin{pmatrix} 13.373 \\ 7.491 \end{pmatrix},\ \mathrm{E}[\boldsymbol{z}_3] = \begin{pmatrix} 15.451 \\ 9.169 \end{pmatrix},$$
$$\mathrm{E}[\boldsymbol{z}_4] = \begin{pmatrix} 17.841 \\ 11.222 \end{pmatrix},\ \mathrm{E}[\boldsymbol{z}_5] = \begin{pmatrix} 20.587 \\ 13.737 \end{pmatrix}.$$

Figures 1 and 2 show the impact of the intertemporal risk restriction. Figure 1 plots $\mathrm{Var}[s_t]$, the variance of s_t, for $t = 0, \cdots, T$, when we set α_k at different values (with $\omega_{\mathrm{T}} = 0.2$). We can observe that the intertemporal risk level is decreasing when we increase α_k. Fig. 2 plots the efficient frontier of $(\mathrm{E}[s_{\mathrm{T}}], \mathrm{Var}[s_{\mathrm{T}}])$ for different α_k, $k = 1, \cdots, T-1$, when we vary ω_{T}. The efficient frontier generated from a larger α_k is dominated by the one generated by a smaller α_k. Thus, there is a tradeoff when choosing α_k, i.e., larger α_k may help reduce the intertemporal risk of s_t but worsen the efficiency of the terminal surplus, s_{T}.

Now let us assume that the investor considers model $(\mathcal{P}_2)$ with $\eta_k = 0$, $k = 1, \cdots, 5$, $\beta_1 = 0.2$, $\beta_2 = 0.2$, $\beta_3 = 0.2$, $\beta_4 = 0.25$ and $\beta_5 = 0.25$. From Proposition 3 and the simple searching algorithm mentioned in Remark 1, we can compute $\lambda_k^* = 0$ for $k = 1, \cdots, T-2$ and $\lambda_{T-1}^* = 0.001$. The statistics of $\mathrm{E}[s_t]$, $\mathrm{Var}[s_t]$ and $\mathrm{E}[\boldsymbol{u}_t]$ are listed in Table 1. That is to say, all the inequalities constraints, $\mathrm{Var}[s_t] \leqslant \beta_t(\mathrm{E}[s_t] - \eta_t)^2$, in problem $(\mathcal{P}_2)$ hold strictly. To verify the effect of model $(\mathcal{P}_2)$ in preventing the surplus falling into bankruptcy, we generate 2000 sample paths of the returns of the risky assets and liability which are assumed to be jointly normally distributed according to the expected vector and covariance matrix given in (31) and (32). Then we implement the optimal portfolio policies for these 2000 samples. The column 'num' in Table 1 records the number of bankruptcy occurrence with $s_t < 0$ in 2000 rounds.

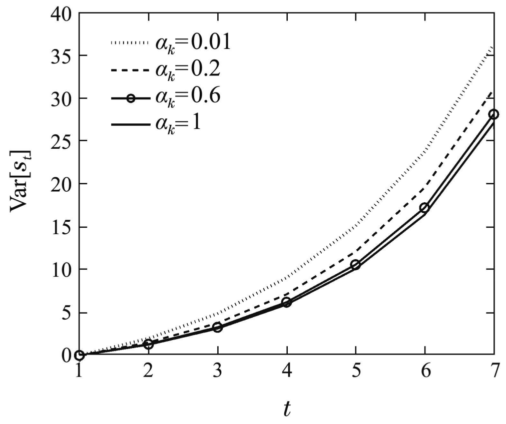

Fig. 1 Intertemporal variance $\mathrm{Var}[s_t]$

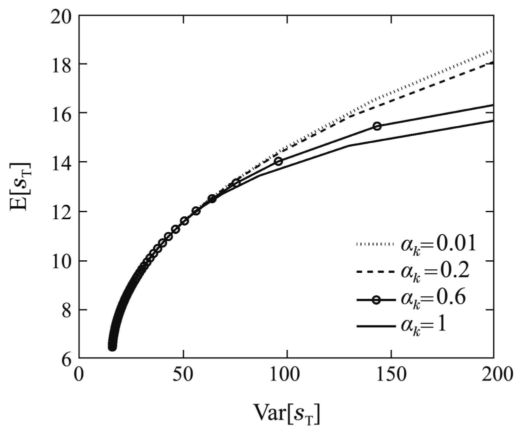

Fig. 2 $(\mathrm{Var}[s_{\mathrm{T}}], \mathrm{E}[s_{\mathrm{T}}])$ pair of model $(\mathcal{P}_1)$

Table 1 The statistics from simulation test for model $(\mathcal{P}_2)$

t	$\mathrm{E}[s_t]$	$\mathrm{Var}[s_t]$	$\mathrm{E}[u_t]$	num
0	5	0	–0.854	0
1	5.398	0.688	–0.942	0
2	5.780	1.764	–1.040	0
3	6.124	3.542	–1.150	7
4	6.406	5.958	–1.274	35
5	6.588	9.865	–1.414	61
6	6.623	15.940	—	73

Remark 2 In our model, we assume that all the returns of the risky assets and liability rate are statistically independent. However, in real applications, the assets returns always exhibit certain degree of correlation among different time periods. Thus, investigating the correspondent ALM problems by using the mean-field control approach when assets returns are correlated is a challenge and meaningful future research direction.

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