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概率级联布尔网络的集镇定及其应用

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摘要: 随着系统生物学和医学的迅速发展, 基因调控网络已经成为一个热点研究领域. 布尔网络作为研究生物系统和基因调控网络的一种重要模型, 近年来引起了包括生物学家和系统科学家在内的很多学者的广泛关注. 本文利用代数状态空间方法, 研究了概率级联布尔网络的集镇定问题. 首先给出概率级联布尔网络集镇定的定义, 并利用矩阵的半张量积给出了概率级联布尔网络的代数表示. 其次基于该代数表示, 定义了一组合适的概率能达集, 并给出了概率级联布尔网络集镇定问题可解的充要条件. 最后将所得的理论结果应用于概率级联布尔网络的同步分析及n人随机级联演化布尔博弈的策略一致演化行为分析.

关键词: 概率级联布尔网络; 镇定; 矩阵半张量积; 随机演化布尔博弈

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Set stabilization of probabilistic cascading Boolean networks and its applications

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Abstract: With the rapid development of systems biology and medical science, gene regulatory networks have become a heated research field. As an important model for studying biological systems and gene regulatory networks, in the past few decades, Boolean networks have attracted extensive attention from many scholars including biologists and system scientists. This paper studies the set stabilization problem of probabilistic cascading Boolean networks (PCBNs). Firstly, the concept of set stabilization of PCBNs has been proposed, and the considered PCBN is converted to an equivalent algebraic form by using the semi-tensor product of matrices. Secondly, based on the equivalent algebraic form, a series of probabilistic reachable sets is defined and a necessary and sufficient condition is presented for the set stabilization of PCBNs. Finally, as applications of set stabilization of PCBNs, the synchronization of PCBNs and strategy consensus of n-person random cascading evolutionary Boolean games are investigated, respectively, and several necessary and sufficient conditions are presented.

Key words: probabilistic cascading Boolean network; stabilization; semi-tensor product of matrices; random evolutionary Boolean game

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1 引言

随着系统生物学和医学的迅速发展, 基因调控网络(gene regulatory networks, GRN)已经成为理论生物学的重要研究内容. 近年来, 人们提出了多种数学模型[1]来研究基因调控网络, 包括微分方程模型, 贝叶斯网络模型和布尔网络(Boolean networks, BN)等. 在这些模型中, 由于布尔网络不含参数, 可以对大规模的基因调控网络进行建模, 所以其应用最为广泛. 作为确定性布尔网络的推广, Shmulevich等[2]在2002年引入了概率布尔网络(probabilistic boolean networks, PBN). 概率布尔网络由一组布尔网络和一个控制不同布尔网络之间切换的概率结构组成. 在过去的20年里, 布尔网络和概率布尔网络的研究引起了系统科学家和生物学家的广泛关注[3–6].

目前对于布尔网络的研究已有多种方法, 包括计算机模拟, 计算代数等. 近年来, 程代展教授提出了一

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种新的矩阵乘积, 即矩阵的半张量积(semi-tensor product of matrices, STPM)[7–8], 用于布尔网络的分析和控制. 使用这种方法, 一个逻辑变量可以被表示为一个向量, 一个逻辑函数可以转换成一个等价的代数形式. 因此, 布尔网络的非线性动态系统可以转化为经典离散时间线性系统的形式. 到目前为止, 利用矩阵半张量积方法, 人们对布尔网络[9–22]和概率布尔网络[3,23–24]的分析和控制已经取得了许多研究结果[2,25–29].

作为布尔网络中一个有趣的研究课题, 如何更新网络动态引起了许多学者的研究兴趣[30–37]. Robert[34]提出了布尔网络的并行和级联更新规则. Aracena等[30]研究了布尔网络中确定性更新规则的等价类. Luo等[33]提出了一种异步随机更新规则. 文献[35]研究了异步切换布尔网络的同步问题. Yang等[33]考虑了异步随机布尔网络的吸引子. 文献[38]利用矩阵半张量积分析了确定性级联布尔网络的全局收敛性. 在这些更新规则中, 级联布尔网络是一种异步布尔网络, 其中所有节点都使用已经更新的节点的布尔值来更新当前的值[31,38].

本文利用矩阵半张量积方法研究概率级联布尔网络的集镇定问题, 并将所得结果应用于概率级联布尔网络的同步和n人随机级联演化布尔博弈的策略一致分析中. 相比确定性级联布尔网络, 概率级联布尔网络引入了随机性, 用来处理确定性级联布尔网络建模中的不确定性, 增加了系统的鲁棒性. 相比概率布尔网络, 概率级联布尔网络是一种异步的概率布尔网络, 其行为分析更复杂, 处理难度更大. 如文献[38]中所述, 布尔网络与级联布尔网络即使有相同逻辑函数, 也无法保证两个系统的全局收敛是等价的. 所以, 有必要提出新的方法来研究概率级联布尔网络. 本文的主要贡献如下: 一方面, 利用矩阵半张量积将概率级联布尔网络转化为等价的代数形式, 便于概率级联布尔网络的分析和控制; 另一方面, 通过构造一组概率能达集, 给出了概率级联布尔网络集镇定的一个充要条件, 并可利用MATLAB较为容易地验证这一条件.

本文的剩余部分安排如下: 第2部分列出了一些关于矩阵半张量积的必要的预备知识. 第3部分提出概率级联布尔网络的集镇定问题, 并给出本文的主要结果. 作为应用, 第4部分将所得的理论结果用于概率级联布尔网络的同步控制和n人随机级联演化布尔博弈的策略一致分析. 第5部分给出本文的结论.

下面列出本文中用到的符号: $\mathbb{R}$, $\mathbb{N}$和$\mathbb{Z}_+$分别表示实数、自然数和正整数.

$$\mathcal{D}:=\{0,1\}, \mathcal{D}^n:=\underbrace{\{\mathcal{D}\times\cdots\times\mathcal{D}\}}_{n}.$$

$$\varDelta_k:=\{\delta_k^i|i=1,2,\cdots,k\},$$

δ_k^i表示单位矩阵I_k的第i列. 一个$n\times t$维矩阵A称作逻辑矩阵, 如果$A=[\delta_n^{i_1}\ \ \delta_n^{i_2}\ \ \cdots\ \ \delta_n^{i_t}]$. 简记$A$为$A=\delta_n[i_1\ \ i_2\ \ \cdots\ \ i_t]$. 定义所有$n\times t$维逻辑矩阵的集合为$\mathcal{L}_{n\times t}$. $\mathrm{Col}_i(A)$表示矩阵A的第i列. 矩阵A的所有列组成的集合记为$\mathrm{Col}(A)$. 设$A,B\in\mathcal{L}_{n\times r}$, 则

$$A*B:=$$
$$[\mathrm{Col}_1(A)\otimes\mathrm{Col}_1(B)\ \ \cdots\ \ \mathrm{Col}_r(A)\otimes\mathrm{Col}_r(B)],$$

其中$\otimes$表示矩阵的Kronecker积.

2 预备知识

本文使用的主要数学工具为矩阵的半张量积, 定义如下:

定义 1 给定两个矩阵$A\in\mathbb{R}^{m\times n}$和$B\in\mathbb{R}^{p\times q}$. 令$\alpha=\mathrm{lcm}(n,p)$为$n$和$p$的最小公倍数, 则矩阵$A$和$B$的半张量积定义为

$$A\ltimes B=(A\otimes I_{\frac{\alpha}{n}})(B\otimes I_{\frac{\alpha}{p}}). \tag{1}$$

注 1 矩阵的半张量积是普通矩阵乘积的推广, 因此我们一般省略符号“$\ltimes$”.

下面给出矩阵半张量积的基本性质:

引理 1 i) 设$X\in\mathbb{R}^{t\times 1}$是一个列向量, A为任意维数矩阵, 则$X\ltimes A=(I_t\otimes A)\ltimes X$. ii) 设$X\in\mathcal{L}_{k\times 1}$. 则$X^2=M_{r,k}\ltimes X$, 其中: $M_{r,k}=\mathrm{diag}\{\delta_k^1,\delta_k^2,\cdots,\delta_k^k\}$称为降阶矩阵, 其用途是将逻辑变量$x$的幂次降为$x$.

为了将逻辑映射转化为代数形式, 本文将“1”和“0”分别表示成向量“δ_2^1”和“δ_2^2”, 则$\mathcal{D}\sim\varDelta$, 其中“$\sim$”表示同一事物的2种不同表示形式.

引理 2 设$f(x_1,x_2,\cdots,x_n):\mathcal{D}^n\rightarrow\mathcal{D}$. 则存在唯一的逻辑矩阵$M_{\mathrm{f}}\in\mathcal{L}_{2\times 2^n}$使得

$$f(x_1,x_2,\cdots,x_n)=M_{\mathrm{f}}\ltimes_{i=1}^n x_i,\ x_i\in\mathcal{D}, \tag{2}$$

这里M_{f}称作f的结构矩阵.

3 主要结果

考虑如下带有控制的概率级联布尔网络:

$$\begin{cases}z_1(t+1)=f_1(U(t),z_1(t),z_2(t),\cdots,z_n(t)),\\ z_2(t+1)=f_2(U(t),z_1(t+1),z_2(t),\cdots,z_n(t)),\\ \qquad\vdots\\ z_n(t+1)=f_n(U(t),z_1(t+1),\cdots,\\ \qquad\qquad z_{n-1}(t+1),z_n(t)),\\ Y(t)=h(Z(t)),\end{cases} \tag{3}$$

其中:

$$Z(t)=(z_1(t),\cdots,z_n(t))\in\mathcal{D}^n,$$
$$U(t)=(u_1(t),\cdots,u_m(t))\in\mathcal{D}^m$$

和$Y(t)=(y_1(t),\cdots,y_p(t))\in\mathcal{D}^p$分别表示系统(3)的状态、控制输入及输出. 逻辑函数$f_i:\mathcal{D}^{n+m}\rightarrow\mathcal{D}$, $i=$

$1,2,\cdots,n$在每一时刻可在一有限集中选取, 即$f_i\in\{f_i^1,f_i^2,\cdots,f_i^{l_i}\}$, 并且$f_i=f_i^{j_i}$的概率为$\mathrm{P}\{f_i=f_i^{j_i}\}=p_{ij_i}>0, i=1,\cdots,n, j_i=1,\cdots,l_i$, 显然$\sum\limits_{j_i=1}^{l_i}p_{ij_i}=1$. $h:\mathcal{D}^n\mapsto\mathcal{D}^p$为给定的逻辑函数.

首先给出系统(3)能够以概率1集镇定的定义.

定义 2　给定一非空集合$S\subseteq\mathcal{D}^n$. 系统(3)称为以概率1镇定到集合S, 如果存在一控制序列$\{U(t):t\in\mathbb{N}\}$和一正整数T使得

$$\mathrm{P}\{Z(t)\in S\mid Z(0)=Z_0,U(t)\}=1$$

对任意初始状态$Z_0\in\mathcal{D}^n$和任意整数$t\geqslant T$成立.

从而, 由定义2可知, 给定一非空集合$S\subseteq\mathcal{D}^n$, 状态反馈集镇定问题可以转化为设计如下形式的状态反馈控制:

$$\begin{cases}u_1(t)=g_1(Z(t)),\\ \quad\vdots\\ u_m(t)=g_m(Z(t)),\end{cases}\tag{4}$$

使得由式(3)和式(4)构成的闭环系统能够以概率1全局稳定到集合S, 其中$g_i:\mathcal{D}^n\mapsto\mathcal{D}, i=1,\cdots,m$为待定的布尔函数.

现在利用矩阵半张量积将概率级联布尔网络(3)和状态反馈控制(4)转化为等价的代数形式. 为此, 我们首先给出下述引理.

引理 3　令$\ltimes_{i=1}^n z_i(t)=z(t)$, 则

$$z(t)\ltimes_{i=l+1}^n z(t)=J_l z(t)$$

对任意的$l=1,\cdots,n-1$均成立, 其中

$$J_l=I_{2^l}\otimes M_{r,2^{n-l}}.$$

证　事实上, 对任意的$l=1,\cdots,n-1$, 有

$$\begin{aligned}&z(t)\ltimes_{i=l+1}^n z_i(t)=\\&z_1(t)\cdots z_l(t)(\ltimes_{i=l+1}^n z_i(t))^2=\\&z_1(t)\cdots z_l(t)M_{r,2^{n-l}}\ltimes_{i=l+1}^n z_i(t)=\\&(I_{2^l}\otimes M_{r,2^{n-l}})z(t),\end{aligned}$$

即可得$J_l=I_{2^l}\otimes M_{r,2^{n-l}}$.

令$u(t)=\ltimes_{i=1}^m u_i(t)$, 由引理2可得$f_i^{j_i}$, h和g_j的结构矩阵分别为$M_i^{j_i}\in\mathcal{L}_{2\times 2^{m+n}}$, $H\in\mathcal{L}_{2^p\times 2^n}$和$G_j\in\mathcal{L}_{2\times 2^n}$. 则基于引理3有

$$\begin{aligned}&z_1(t+1)=M_1^{j_1}u(t)z(t):=F_1^{j_1}u(t)z(t),\\&z_i(t+1)=\\&M_i^{j_i}u(t)\ltimes_{k=1}^{i-1}z_k(t+1)\ltimes_{k=i}^n z_k(t)=\\&M_i^{j_i}u(t)(*_{k=1}^{i-1}F_k^{j_k})u(t)z(t)\ltimes_{k=i}^n z_k(t)=\\&M_i^{j_i}u(t)(*_{k=1}^{i-1}F_k^{j_k})u(t)J_{i-1}z(t)=\\&M_i^{j_i}[I_{2^m}\otimes(*_{k=1}^{i-1}F_k^{j_k})]M_{r,2^m}u(t)\ltimes J_{i-1}z(t)=\\&M_i^{j_i}[I_{2^m}\otimes(*_{k=1}^{i-1}F_k^{j_k})]M_{r,2^m}\ltimes\\&(I_{2^m}\otimes J_{i-1})u(t)z(t):=F_i^{j_i}u(t)z(t),\ i=2,\cdots,n,\end{aligned}$$

并有

$$u_j(t)=G_jz(t),\ j=1,\cdots,m.$$

于是可得系统(3)的代数形式为

$$\begin{cases}z(t+1)=Qu(t)z(t),\\ y(t)=Hz(t),\end{cases}\tag{5}$$

状态反馈控制(4)的代数形式为

$$u(t)=Gz(t),\tag{6}$$

这里$Q\in\mathcal{L}_{2^n\times 2^{n+m}}$在每一时刻均以概率$\mathrm{P}\{Q=Q_i\}=p_i$从集合$\{Q_1,Q_2,\cdots,Q_r\}$中取值, 其中:

$$\begin{aligned}&Q_i=F_1^{j_1}*F_2^{j_2}*\cdots*F_n^{j_n},\\&j_k\in\{1,\cdots,l_k\},\ k=1,\cdots,n,\\&p_i=\prod_{k=1}^n p_{kj_k},\ i=1,\cdots,r,\ r=\prod_{k=1}^n l_k.\end{aligned}$$

$G=G_1*G_2*\cdots*G_m$称为状态反馈增益矩阵.

接下来, 基于等价形式(5), 研究状态反馈集镇定问题.

给定初始状态$z_0\in\varDelta_{2^n}$和状态反馈增益矩阵$G\in\mathcal{L}_{2^m\times 2^n}$, 则通过计算可以得到

$$\begin{aligned}&\mathrm{P}\{z(t)\in S\mid z(0)=z_0,u(t)=Gz(t)\}=\\&\sum_{a\in S}\mathrm{P}\{z(t)=a\mid z(0)=z_0,u(t)=Gz(t)\}.\end{aligned}\tag{7}$$

给定一非空集合$S\subseteq\varDelta_{2^n}$, 通过递推, 定义如下一组集合$\{\varPhi_k(S):k\in\mathbb{Z}_+\}$:

$$\begin{aligned}\varPhi_1(S)=\{&\delta_{2^n}^\alpha\in\varDelta_{2^n}:\exists\,\mu\in\{1,2,\cdots,2^m\},\\&\text{使得}\sum_{a\in S}\mathrm{P}\{z(t+1)=a\mid z(t)=\delta_{2^n}^\alpha,\\&u(t)=\delta_{2^m}^\mu\}=1\},\end{aligned}\tag{8}$$

$$\begin{aligned}\varPhi_{k+1}(S)=\{&\delta_{2^n}^\alpha\in\varDelta_{2^n}:\exists\,\mu\in\{1,2,\cdots,2^m\},\\&\text{使得}\sum_{a\in\varPhi_k(S)}\mathrm{P}\{z(t+1)=a\mid z(t)=\delta_{2^n}^\alpha,\\&u(t)=\delta_{2^m}^\mu\}=1\}.\end{aligned}\tag{9}$$

通过如上集合的构造, 可得下述引理.

引理 4　i) 如果$S\subseteq\varPhi_1(S)$, 则对任意正整数$k\geqslant 1$, $\varPhi_k(S)\subseteq\varPhi_{k+1}(S)$均成立.

ii) 如果$\varPhi_1(S)=S$, 则对任意正整数$k\geqslant 1$, $\varPhi_k(S)=S$均成立.

iii) 如果存在一正整数$j\geqslant 1$使得$\varPhi_j(S)=\varPhi_{j+1}(S)$成立, 则对任意正整数$k\geqslant j$, $\varPhi_k(S)=\varPhi_j(S)$均成立.

现在, 给出本文的主要结论.

定理 1　系统(3)的状态反馈集镇定问题是可解

的, 当且仅当存在一个正整数$T \leqslant 2^n$使得

$$\begin{cases} S \subseteq \varPhi_1(S), \\ \varPhi_T(S) = \varDelta_{2^n}. \end{cases} \tag{10}$$

证 必要性. 假设系统(3)的状态反馈集镇定问题是可解的, 则由定义2可知, 存在一个状态反馈增益矩阵$G \in \mathcal{L}_{2^m \times 2^n}$和正整数$T \in \mathbb{Z}_+$使得对任意的初始状态$z_0 \in \varDelta_{2^n}$和正整数$t \geqslant T$有

$$\begin{aligned} 1 = & \mathrm{P}\{z(t) \in S \mid z(0) = z_0, u(t) = Gz(t)\} = \\ & \sum_{a \in S} \mathrm{P}\{z(t) = a \mid z(0) = z_0, u(t) = Gz(t)\} = \\ & \sum_{a \in S} \sum_{\delta_{2^n}^{a_1}, \cdots, \delta_{2^n}^{a_{t-1}} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} \mid z(0) = z_0, \\ & u(0) = Gz_0\} \times \cdots \times \mathrm{P}\{z(t) = a | z(t-1) = \\ & \delta_{2^n}^{a_{t-1}}, u(t-1) = G\delta_{2^n}^{a_{t-1}}\} \end{aligned} \tag{11}$$

和

$$\begin{aligned} 1 = & \mathrm{P}\{z(t+1) \in S \mid z(0) = z_0, u(t) = Gz(t)\} = \\ & \sum_{a \in S} \sum_{\delta_{2^n}^{a_t} \in \varDelta_{2^n}} \mathrm{P}\{z(t+1) = a \mid z(t) = \delta_{2^n}^{a_t}, \\ & u(t) = G\delta_{2^n}^{a_t}\}[\sum_{\delta_{2^n}^{a_1}, \cdots, \delta_{2^n}^{a_{t-1}} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \\ & \delta_{2^n}^{a_1} | z(0) = z_0, u(0) = Gz_0\} \times \cdots \times \mathrm{P}\{z(t) = \\ & \delta_{2^n}^{a_t} | z(t-1) = \delta_{2^n}^{a_{t-1}}, u(t-1) = G\delta_{2^n}^{a_{t-1}}\}] \end{aligned} \tag{12}$$

成立.

当$\delta_{2^n}^{a_t} \in \varDelta_{2^n} \backslash S$时, 由式(11)可知,

$$\begin{aligned} & \sum_{\delta_{2^n}^{a_1}, \cdots, \delta_{2^n}^{a_{t-1}} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} | z(0) = z_0, \\ & u(0) = Gz_0\} \times \cdots \times \mathrm{P}\{z(t) = \delta_{2^n}^{a_t} | z(t-1) = \\ & \delta_{2^n}^{a_{t-1}}, u(t-1) = G\delta_{2^n}^{a_{t-1}}\} = 0. \end{aligned}$$

因此,

$$\begin{aligned} 1 = & \sum_{\delta_{2^n}^{a_t} \in S} \sum_{a \in S} \mathrm{P}\{z(t+1) = a \mid z(t) = \delta_{2^n}^{a_t}, \\ & u(t) = G\delta_{2^n}^{a_t}\}[\sum_{\delta_{2^n}^{a_1}, \cdots, \delta_{2^n}^{a_{t-1}} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \\ & \delta_{2^n}^{a_1} | z(0) = z_0, u(0) = Gz_0\} \times \cdots \times \mathrm{P}\{z(t) = \\ & \delta_{2^n}^{a_t} | z(t-1) = \delta_{2^n}^{a_{t-1}}, u(t-1) = G\delta_{2^n}^{a_{t-1}}\}]. \end{aligned} \tag{13}$$

接下来, 证明对任意的$\delta_{2^n}^{a_t} \in S$有

$$\sum_{a \in S} \mathrm{P}\{z(t+1) = a | z(t) = \delta_{2^n}^{a_t}, u(t) = G\delta_{2^n}^{a_t}\} = 1 \tag{14}$$

成立.

事实上, 如果式(14)不成立, 则可找到一个$\delta_{2^n}^{a'_t} \in S$使得

$$0 \leqslant \sum_{a \in S} \mathrm{P}\{z(t+1) = a | z(t) = \delta_{2^n}^{a'_t}, u(t) = G\delta_{2^n}^{a'_t}\} < 1.$$

于是有

$$\begin{aligned} 1 = & \sum_{\delta_{2^n}^{a_t} \in S} \sum_{a \in S} \mathrm{P}\{z(t+1) = a | z(t) = \delta_{2^n}^{a_t}, u(t) = \\ & G\delta_{2^n}^{a_t}\}[\sum_{\delta_{2^n}^{a_1}, \cdots, \delta_{2^n}^{a_{t-1}} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} | z(0) = z_0, \\ & u(0) = Gz_0\} \times \cdots \times \mathrm{P}\{z(t) = \delta_{2^n}^{a_t} | \\ & z(t-1) = \delta_{2^n}^{a_{t-1}}, u(t-1) = G\delta_{2^n}^{a_{t-1}}\}] < \\ & \sum_{\delta_{2^n}^{a_t} \in S} \sum_{\delta_{2^n}^{a_1}, \cdots, \delta_{2^n}^{a_{t-1}} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} | z(0) = \\ & z_0, u(0) = Gz_0\} \times \cdots \times \mathrm{P}\{z(t) = \delta_{2^n}^{a_t} | \\ & z(t-1) = \delta_{2^n}^{a_{t-1}}, u(t-1) = G\delta_{2^n}^{a_{t-1}}\} = 1. \end{aligned}$$

显然, 上式是矛盾的. 因此, 式(14)成立, 且由式(8)可得$S \subseteq \varPhi_1(S)$.

为了证明$\varPhi_T(S) = \varDelta_{2^n}$, 只需证明, 如果对任意的初始状态$z_0 \in \varDelta_{2^n}$和正整数$t \geqslant T$, 式(11)成立, 则$z_0 \in \varPhi_t(S)$成立即可.

假设式(11)对$t = 2$和任意初始策略$z_0 \in \varDelta_{2^n}$都成立. 那么, 由

$$\sum_{\delta_{2^n}^{a_1} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} \mid z(0) = z_0, u(0) = Gz_0\} = 1,$$

$$\mathrm{P}\{z(1) = \delta_{2^n}^{a_1} \mid z(0) = z_0, u(0) = Gz_0\} > 0,$$

可得$\sum_{a \in S} \mathrm{P}\{z(2) = a \mid z(1) = \delta_{2^n}^{a_1}, u(1) = G\delta_{2^n}^{a_1}\} = 1$. 因此, $\delta_{2^n}^{a_1} \in \varPhi_1(S)$, 并且

$$\sum_{\delta_{2^n}^{a_1} \in \varPhi_1(S)} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} \mid z(0) = z_0, u(0) = Gz_0\} = 1.$$

于是, 由式(9)可得$z_0 \in \varPhi_2(S)$.

假设$t (t \geqslant 2)$时刻结论成立. 考虑$t+1$时刻, 有

$$\begin{aligned} & \sum_{a \in S} \sum_{\delta_{2^n}^{a_1}, \cdots, \delta_{2^n}^{a_t} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} | z(0) = z_0, u(0) = \\ & Gz_0\} \times \mathrm{P}\{z(2) = \delta_{2^n}^{a_2} | z(1) = \delta_{2^n}^{a_1}, u(1) = G\delta_{2^n}^{a_1}\} \times \\ & \cdots \times \mathrm{P}\{z(t+1) = a \mid z(t) = \delta_{2^n}^{a_t}, u(t) = G\delta_{2^n}^{a_t}\} = \\ & \sum_{\delta_{2^n}^{a_1} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} | z(0) = z_0, u(0) = Gz_0\} \times \\ & [\sum_{a \in S} \sum_{\delta_{2^n}^{a_2}, \cdots, \delta_{2^n}^{a_t} \in \varDelta_{2^n}} \mathrm{P}\{z(2) = \delta_{2^n}^{a_2} | z(1) = \delta_{2^n}^{a_1}, \\ & u(1) = G\delta_{2^n}^{a_1}\} \times \cdots \times \mathrm{P}\{z(t+1) = \\ & a \mid z(t) = \delta_{2^n}^{a_t}, u(t) = G\delta_{2^n}^{a_t}\}] = 1. \end{aligned}$$

因为

$$\sum_{\delta_{2^n}^{a_1} \in \varDelta_{2^n}} \mathrm{P}\{z(1) = \delta_{2^n}^{a_1} | z(0) = z_0, u(0) = Gz_0\} = 1,$$

$$\mathrm{P}\{z(1) = \delta_{2^n}^{a_1} | z(0) = z_0, u(0) = Gz_0\} > 0,$$

于是有

$$\begin{aligned} & \sum_{a \in S} \sum_{\delta_{2^n}^{a_2}, \cdots, \delta_{2^n}^{a_t} \in \varDelta_{2^n}} \mathrm{P}\{z(2) = \delta_{2^n}^{a_2} | z(1) = \delta_{2^n}^{a_1}, \\ & u(1) = G\delta_{2^n}^{a_1}\} \times \cdots \times \mathbb{P}\{z(t+1) = \end{aligned}$$

$a \mid z(t)=\delta_{2^n}^{a_t}, u(t)=G\delta_{2^n}^{a_t}\}=1.$

因此, $\delta_{2^n}^{a_1}\in\Phi_t(S)$且

$$\sum_{\delta_{2^n}^{a_1}\in\Phi_t(S)} \mathrm{P}\{z(1)=\delta_{2^n}^{a_1}\mid z(0)=z_0, u(0)=Gz_0\}=1,$$

由式(9)可知$z_0\in\Phi_{t+1}(S)$.

综上所述, 由归纳假设知, 对$\forall t\geqslant T, \Phi_t(S)=\Delta_{2^n}$都成立. 令$T$为满足$\Phi_t(S)=\Delta_{2^n}$的最小正整数. 由引理5可知, 对任意正整数$1\leqslant t\leqslant T$, 都有$|\Phi_t(S)|\geqslant t$. 因此, $T\leqslant|\Phi_T(S)|=2^n$.

充分性. 假设存在一个正整数$1\leqslant T\leqslant 2^n$使得式(10)成立. 因为$S\subseteq\Phi_1(S)$, 则由引理5可知$\Phi_1(S)\subseteq\cdots\subseteq\Phi_T(S)=\Delta_{2^n}$. 设

$$\widehat{\Phi}_i(S)=\Phi_i(S)\setminus\Phi_{i-1}(S),\ i=1,\cdots,T, \tag{15}$$

其中: $\Phi_0(S):=\varnothing$. 显然, $\widehat{\Phi}_i(S)\bigcap\widehat{\Phi}_j(S)=\varnothing, \forall i,j\in\{1,\cdots,T\}, i\neq j$, 并且$\bigcup\limits_{i=1}^{T}\widehat{\Phi}_i(S)=\Delta_{2^n}$. 因此, 对任意的$1\leqslant i\leqslant 2^n$, 都存在唯一的正整数$1\leqslant k_i\leqslant T$使得$\delta_{2^n}^i\in\widehat{\Phi}_{k_i}(S)$成立. 当$k_i=1$时, 由式(8)可知, 存在一正整数$1\leqslant\mu_i\leqslant 2^m$使得

$$\sum_{a\in S}\mathrm{P}\{z(t+1)=a\mid z(t)=\delta_{2^n}^i, u(t)=\delta_{2^m}^{\mu_i}\}=1 \tag{16}$$

成立. 当$2\leqslant k_i\leqslant T$时, 由式(9)可知, 存在一正整数$1\leqslant\mu_i\leqslant 2^m$使得

$$\sum_{a\in\Phi_{k_i-1}(S)}\mathrm{P}\{z(t+1)=a|z(t)=\delta_{2^n}^i, u(t)=\delta_{2^m}^{\mu_i}\}=1 \tag{17}$$

成立.

设$G=\delta_{2^m}[\mu_1\ \ \mu_2\ \ \cdots\ \ \mu_{2^n}]\in\mathcal{L}_{2^m\times 2^n}$. 经计算可得, 系统(5)对于任意的正整数$t\geqslant T$和任意初始状态$z_0\in\Delta_{2^n}$都有

$$\mathrm{P}\{z(t)\in S|\ z(0)=z_0, u(t)=Gz(t)\}=1.$$

因此, 状态反馈集镇定问题是可解的.　　证毕.

事实上, 在实际计算中$\Phi_k(S), k=1,\cdots,T$的计算难度较大. 现在, 基于代数形式, 利用矩阵方法构造$\Phi_k(S), k=1,\cdots,T$. 令$Q'=[Q_1\ \ Q_2\ \ \cdots\ \ Q_r], Q_i\in\mathcal{L}_{2^n\times 2^{n+m}}$, 定义矩阵

$$M=\sum_{i=1}^{r}p_iQ_i\in\mathbb{R}^{2^n\times 2^{n+m}},$$

并将其分解成2^m块, 即$M=[M_1\ \ M_2\ \ \cdots\ \ M_{2^m}]$, 其中: $M_i\in\mathcal{L}_{2^n\times 2^n}, i=1,\cdots,2^m$. 设$z(t+1)=\delta_{2^n}^j$, $z(t)=\delta_{2^n}^i, u(t)=\delta_{2^m}^\mu$, 可以得到

$$\mathrm{P}\{z(t+1)=\delta_{2^n}^j\mid z(t)=\delta_{2^n}^i, u(t)=\delta_{2^m}^\mu\}=(M_\mu)_{j,i}. \tag{18}$$

接下来, 给出系统(5)一致一步可达的定义.

定义 3　给定一个非空集合$S=\{\delta_{2^n}^{\alpha_1},\cdots,\delta_{2^n}^{\alpha_{|S|}}\}$和$z=\delta_{2^n}^i\in\Delta_{2^n}$. 集合$S$称为从$z$出发以概率1一致一步可达的, 如果存在一个整数$\mu\in\{1,\cdots,2^m\}$使得

$$\sum_{\delta_{2^n}^\alpha\in S}\mathrm{P}\{z(t+1)=\delta_{2^n}^\alpha\mid z(t)=\delta_{2^n}^i, u(t)=\delta_{2^m}^\mu\}=1$$

成立.

引理 5　给定集合$S=\{\delta_{2^n}^{\alpha_1},\cdots,\delta_{2^n}^{\alpha_{|S|}}\}$和$z=\delta_{2^n}^i\in\Delta_{2^n}$. 集合$S$为从$z$以概率1一致一步可达的, 当且仅当存在一整数$\mu\in\{1,\cdots,2^m\}$使得

$$\sum_{j\in\{\alpha_1,\cdots,\alpha_{|S|}\}}(M_\mu)_{j,i}=1 \tag{19}$$

成立.

基于引理5通过归纳假设, 得到下述结论:

定理 2　假设$\Phi_k(S)=\{\delta_{2^n}^{\alpha_1^k},\delta_{2^n}^{\alpha_2^k},\cdots,\delta_{2^n}^{\alpha_{|\Phi_k(S)|}^k}\}$, $k=0,\cdots,T-1$, 其中$\Phi_0(S):=S$, 则

$$\Phi_{k+1}(S)=\{\delta_{2^n}^\alpha:\exists\mu\in\mathbb{N}, 1\leqslant\mu\leqslant 2^m, \text{使得}\sum_{j\in\{\alpha_1^k,\cdots,\alpha_{|\Phi_k(S)|}^k\}}(M_\mu)_{j,\alpha}=1\}. \tag{20}$$

4　应用

4.1　概率级联布尔网络的同步

本小节考虑两个耦合布尔网络的同步问题, 形式如下:

$$\begin{cases} z_1(t+1)=f_1(U(t),z_1(t),z_2(t),\cdots,z_n(t)),\\ z_2(t+1)=f_2(U(t),z_1(t+1),z_2(t),\cdots,z_n(t)),\\ \quad\vdots\\ z_n(t+1)=f_n(U(t),z_1(t+1),\cdots,\\ \qquad z_{n-1}(t+1),z_n(t)),\end{cases} \tag{21}$$

$$\begin{cases} \hat{z}_1(t+1)=\hat{f}_1(Z(t),\hat{Z}(t)),\\ \hat{z}_2(t+1)=\hat{f}_2(Z(t),\hat{Z}(t)),\\ \quad\vdots\\ \hat{z}_n(t+1)=\hat{f}_n(Z(t),\hat{Z}(t)),\end{cases} \tag{22}$$

这里: $Z(t)=(z_1(t),\cdots,z_n(t))\in\mathcal{D}^n$和$\hat{Z}(t)=(\hat{z}_1(t),\cdots,\hat{z}_n(t))\in\mathcal{D}^n$分别表示系统(21)–(22)的状态变量, $U(t)=(u_1(t),\cdots,u_m(t))\in\mathcal{D}^m$为控制输入; $f_i:\mathcal{D}^{n+m}\rightarrow\mathcal{D}$和$\hat{f}_j:\mathcal{D}^{2n}\rightarrow\mathcal{D}, i,j=1,\cdots,n$为布尔函数, 其中: $f_i(i=1,\cdots,n)$在每一时刻均以概率$\mathrm{P}\{f_i=f_i^{j_i}\}=p_{ij_i}>0$从集合$\{f_i^1,f_i^2,\cdots,f_i^{l_i}\}$中选取, $i=1,\cdots,n, j_i=1,\cdots,l_i$, 显然$\sum\limits_{j_i=1}^{l_i}p_{ij_i}=1$.

接下来, 作者将耦合系统(21)–(22)转化为代数形式. 令

$$z(t)=\ltimes_{i=1}^n z_i(t),\ \hat{z}(t)=\ltimes_{i=1}^n\hat{z}_i(t),$$
$$u(t)=\ltimes_{i=1}^m u_i(t),$$

由引理1可得代数形式如下:

$$\begin{cases} z(t+1) = Qu(t)z(t), \\ \hat{z}(t+1) = \hat{Q}z(t)\hat{z}(t), \end{cases} \tag{23}$$

这里$Q \in \mathcal{L}_{2^n \times 2^{n+m}}$在每一时刻以概率$\mathrm{P}\{Q = Q_i\} = p_i$从集合$\{Q_1, Q_2, \cdots, Q_r\}$中选取, 其中

$$p_i = \prod_{k=1}^{n} p_{kj_k},\ i = 1, \cdots, r,\ \hat{Q} \in \mathcal{L}_{2^n \times 2^{2n}}.$$

现在给出耦合系统(21)–(22)同步的定义.

定义 4 耦合系统(21)–(22)称为以概率1同步的, 如果存在一个控制序列$\{U(t) : t \in \mathbb{N}\}$和一个正整数$T$使得对任意初始状态$z(0) \in \Delta_{2^n}$, $\hat{z}(0) \in \Delta_{2^n}$和任意整数$t \geqslant T$都有

$$\mathrm{P}\{z(t; z(0), u(t)) = \hat{z}(t; z(0), \hat{z}(0))\} = 1$$

成立.

由上述定义可知, 耦合系统(21)–(22)的状态反馈同步问题即可转化为设计形如

$$u(t) = \hat{G}z(t)\hat{z}(t) \tag{24}$$

的状态反馈控制, 使得存在一正整数T, 对$\forall t \geqslant T$, $\forall\, z(0) \in \Delta_{2^n}$和$\forall\, \hat{z}(0) \in \Delta_{2^n}$有

$$\mathrm{P}\{z(t; z(0), u(t)) = \hat{z}(t; z(0), \hat{z}(0))\} = 1.$$

令$\zeta(t) = z(t)\hat{z}(t)$, 可得如下扩维系统:

$$\begin{aligned} &\zeta(t+1) = \\ &Qu(t)z(t) \ltimes \hat{Q}z(t)\hat{z}(t) = \\ &Q(I_{2^{n+m}} \otimes \hat{Q})u(t)M_{r,2^n}z(t)\hat{z}(t) = \\ &Q(I_{2^{n+m}} \otimes \hat{Q})(I_{2^m} \otimes M_{r,2^n})u(t)\zeta(t) := \\ &\tilde{Q}u(t)\zeta(t), \end{aligned} \tag{25}$$

这里$\tilde{Q} = Q(I_{2^{n+m}} \otimes \hat{Q})(I_{2^m} \otimes M_{r,2^n}) \in \mathcal{L}_{k^{2n} \times k^{2n+m}}$在每一时刻以概率

$$\mathrm{P}\{\tilde{Q} = \tilde{Q}_i = Q_i(I_{2^{n+m}} \otimes \hat{Q})(I_{2^m} \otimes M_{r,2^n})\} = p_i$$

从集合$\{\tilde{Q}_1, \tilde{Q}_2, \cdots, \tilde{Q}_r\}$中选取, 其中:

$$p_i = \prod_{k=1}^{n} p_{kj_k},\ i = 1, \cdots, r.$$

定义集合

$$\Theta = \{\delta_{2^n}^i \ltimes \delta_{2^n}^i : i = 1, 2, \cdots, 2^n\}. \tag{26}$$

得到下述结论.

定理 3 耦合系统(21)和(22)的状态反馈同步问题是可解的, 当且仅当在状态反馈控制(24)下, 系统(25)以概率1镇定到非空集合$S \subseteq \Theta$.

基于定理1和定理3得到下述结论:

定理 4 耦合系统(21)和(22)的状态反馈同步问题是可解的, 当且仅当存在正整数$T \leqslant 2^n$和非空集合$S \subseteq \Theta$使得$S \subseteq \Phi_1(S)$且$\Phi_T(S) = \Delta_{2^n}$.

4.2 随机级联演化布尔博弈的策略一致

考虑下述带有控制的的n人随机演化布尔博弈:

$$\begin{cases} z_1(t+1) = & f_1(U(t), z_1(t), \cdots, z_n(t), \\ & \omega_1(t, p_1), y(t)), \\ z_2(t+1) = & f_2(U(t), z_1(t+1), z_2(t), \cdots, z_n(t), \\ & \omega_2(t, p_2), y(t)), \\ & \vdots \\ z_n(t+1) = & f_n(U(t), z_1(t+1), \cdots, \\ & z_{n-1}(t+1), z_n(t), \omega_n(t, p_n), y(t)), \\ y(t) = h(Z(t)), & \end{cases} \tag{27}$$

这里: $Z(t) = (z_1(t), \cdots, z_n(t)) \in \mathcal{D}^n$表示$t$时刻所有玩家的策略局势, $U(t) = (u_1(t), \cdots, u_m(t)) \in \mathcal{D}^m$表示$t$时刻所有伪玩家(即控制输入)的策略局势, $\omega_i(t, p_i) \in \mathcal{D}$表示每一玩家以概率$\mathrm{P}\{\omega_i(t, p_i) = 1\} = p_i$ $(0 \leqslant p_i \leqslant 1)$做正确选择, $y(t) \in \mathcal{D}$表示博弈的结果, $f_i : \mathcal{D}^{m+n+2} \to \mathcal{D}$, $i = 1, \cdots, n$和$h : \mathcal{D}^n \to \mathcal{D}$为布尔函数.

首先, 给出带有控制的n人随机演化布尔博弈策略一致的定义.

定义 5 n人随机演化布尔博弈称为以概率1策略一致, 如果存在控制(4)和正整数T, 对任意初始策略局势$Z_0 \in \mathcal{D}^n$和任意整数$t \geqslant T$有

$$\mathrm{P}\{Z(t; Z_0, U(t)) \in \{(\underbrace{1, \cdots, 1}_{n}), (\underbrace{0, \cdots, 0}_{n})\}\} = 1$$

成立.

为了研究n人随机演化布尔博弈策略一致的问题, 将系统(27)转化为如下代数形式:

$$\begin{cases} z(t+1) = \bar{Q}\omega(t)u(t)z(t), \\ y(t) = \bar{H}z(t), \end{cases} \tag{28}$$

这里$\omega(t) = \ltimes_{i=1}^{n} \omega_i(t, p_i)$, $\bar{Q} \in \mathcal{L}_{2^n \times 2^{2n+m}}$且$\bar{H} \in \mathcal{L}_{2 \times 2^n}$.

将$\bar{Q}$分成2^n块, 即$\bar{Q} = [\bar{Q}_1\ \ \cdots\ \ \bar{Q}_{2^n}]$, 其中$\bar{Q}_i \in \mathcal{L}_{2^n \times 2^{n+m}}$. 对$\forall \omega(t) = \delta_{2^n}^j$, $j \in \{1, \cdots, 2^n\}$, 都存在唯一的$j_1, \cdots, j_n \in \{1, 2\}$使得$\omega(t) = \delta_2^{j_1} \ltimes \cdots \ltimes \delta_2^{j_n}$. 事实上,

$$j = \sum_{k=1}^{n-1} (j_k - 1)2^{n-j} + j_n.$$

则

$$\mathrm{P}\{\omega_q(t, p_q) = \delta_2^{j_q}\} = \begin{cases} p_q, & j_q = 1, \\ 1 - p_q, & j_q = 2, \end{cases} \tag{29}$$

这里$q = 1, \cdots, n$, 且

$$\mathrm{P}\{\omega(t) = \delta_{2^n}^j\} = \prod_{q=1}^{n} \mathrm{P}\{\omega_q(t, p_q) = \delta_2^{j_q}\}. \tag{30}$$

于是, 系统(27)可转化为下述等价的随机控制系

统:

$$z(t+1)=Fu(t)z(t), \tag{31}$$

其中:

$$\mathrm{P}\{F=\bar{Q}_j\}=\mathrm{P}\{\omega(t)=\delta_{2^n}^j\},\ j=1,\cdots,2^n.$$

接下来, 基于等价的随机控制系统(31)研究n人随机演化布尔博弈策略一致问题.

从定义5可以看出, 策略一致问题可转化为系统(31)以概率1镇定到集合$S\subseteq\Lambda:=\{\delta_{2^n}^1,\delta_{2^n}^{2^n}\}$. 因此, 由定理1可得下述结论.

定理 5 n人随机演化布尔博弈策略一致问题是可解的, 当且仅当存在正整数$T\leqslant 2^n$和非空集合$S\subseteq\Lambda$使得$S\subseteq\varPhi_1(S)$且$\varPhi_T(S)=\varDelta_{2^n}$.

5 结论

本文研究了概率级联布尔网络的集镇定问题. 基于矩阵的半张量积, 给出了概率级联布尔网络的代数表示. 基于该代数表示, 定义了一组适当的概率能达集. 利用概率能达集的性质, 给出了概率级联布尔网络的集镇定问题可解的充要条件. 此外, 我们还将所得的理论结果应用于概率级联布尔网络的同步控制及n人随机级联演化布尔博弈的策略一致分析中.

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