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单参数离散时间非线性系统的可镇定性定理

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摘要: 本文对基本的离散时间非线性单参数随机系统建立了可镇定性定理. 该定理推进了文献[1]的结果, 进一步完善了关于离散时间自适应控制的反馈能力刻画. 离散时间单参数系统可镇定的一个重要非线性临界常数是4, 用以刻画关于幂函数类系统的反馈能力. 而作为本文定理的应用, 本文对一类典型的单参数离散时间非线性随机系统发现了新的可镇定临界常数2.

关键词: 反馈极限; 自适应控制; 最小二乘法; 可镇定性; 非线性系统; 离散时间

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Stabilizability theorem of discrete-time nonlinear systems with scalar parameters

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Abstract: This paper advances [1] by deducing a stabilizability theorem for discrete-time nonlinear systems with scalar parameters, which takes a step forward to the complete characterization of feedback limitations in discrete-time adaptive nonlinear control. It is well-known that exponent 4 is an important critical number to characterize the feedback capability for the basic discrete-time scalar-parameter systems, which are governed by power functions. As an application of our theorem, a new critical number 2 is derived for a typical class of discrete-time nonlinear stochastic systems with scalar parameters.

Key words: feedback limitations; adaptive control; least squares; stabilizability; nonlinear systems; discrete time

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1 Introduction

Most works on nonlinear adaptive control in the literature are focused on continuous-time systems[2–4]. But adaptive control between continuous- and discrete-time systems are rather different. As a matter of fact, a large class of continuous-time nonlinear systems can be globally stabilized by applying nonlinear damping or back-stepping techniques, no matter how fast their growth rates are[5–6]. However, the situation in the discrete-time case is different.

A heuristic result derived by [7] is that feedback limitations exist for discrete-time adaptive nonlinear control. [7] studied a basic discrete-time nonlinear random system with a scalar parameter:

$$y_{t+1} = \theta y_t^b + u_t + w_{t+1},$$

and demonstrated that $b = 4$ is the critical exponent for the stabilizability. Soon afterwards [8] established an "impossibility theorem" for the multi-parameter system

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_n y_t^{b_n} + u_t + w_{t+1}. \quad (1)$$

A polynomial rule on $b_1, \cdots, b_n$ was introduced in the theorem to describe the nonlinear growth rates that fail all feedback control laws in stabilizing system (1). Lately, [9] proved that the polynomial rule in fact serves as the necessary and sufficient condition of the stabilizability of system (1). Besides, some initial research

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on nonlinear parameterized systems with scalar parameters indicates that $b = 4$ is also an important exponent for the stabilizability[10]. Meanwhile, a parallel theory in the deterministic framework on feedback limitations has been developed accordingly (see [11–15]).

The aforementioned systems are all in polynomial forms. For the following relatively general system

$$y_{t+1} = \theta^{\tau} f(y_t) + u_t + w_{t+1},\ \theta \in \mathbb{R}^n, \qquad (2)$$

one may wonder if $|f(x)| = \mathrm{O}(|x|^4)$ is still the limit of nonlinear growth rates for discrete-time stabilizable systems? The first answer appeared in [1]. This work showed that for $n = 1$, system (2) is possible to be stabilized by a discrete-time feedback controller, even if it grows exponentially fast. The density of a regular set is defined in [1] to determine the stabilizability of system (2). As a matter of a fact, [1] provided a quantitative characterization on the densities of the concerned regular sets for both stabilizable systems and unstabilizable systems in discrete time. But there is still a gap between the two densities for stabilizable systems and unstabilizable systems. A theorem established here, which together with Theorem 2.6 in [1], takes a step further to the critical criterion of the stabilizability of system (2) for $n = 1$. A direct application of the two theorems illustrate that $b = 4$ is not a critical description in determining the stabilizability in general. By constructing a class of discrete-time nonlinear stochastic systems in an example below, a new critical number $\beta = 2$ is produced instead.

The rest of the paper is built up as follows. Section 2 presents the main results, while the corresponding proofs are given in Section 3. The argument is finally summarized in Section 4.

2 Main results

Consider the following system

$$y_{t+1} = \theta f(y_t) + u_t + w_{t+1},\ t \geqslant 0, \qquad (3)$$

where $\theta \in \mathbb{R}$ is an unknown parameter, $y_t, u_t, w_t \in \mathbb{R}$ are the system output, input and noise signals, respectively. In addition, let $f: \mathbb{R} \to \mathbb{R}$ be a known piecewise continuous function. Assume the initial value y_0 is independent of θ and $\{w_t\}$. Moreover,

A1) The noise $\{w_t\}$ is an i.i.d random sequence with $w_1 \sim N(0,1)$.

A2) Parameter $\theta \sim N(\theta_0, P_0)$ is independent of $\{w_t\}$.

Assumption A1)–A2) are called as Bayesian framework, which are widely used in the analysis of Kalman filter model. Here θ_0, P_0 are supposed to be known and as the initial values in the filter in theory. In the application, we can choose other initial values, it will not affect the filtering performance.

Definition 1 System (3) is said to be almost surely globally stabilizable, if there exits a feedback control law

$$u_t \in \mathcal{F}_t^y \triangleq \sigma\{y_i, 0 \leqslant i \leqslant t\},\ t = 0, 1, \cdots$$

such that for any initial conditions $y_0 \in \mathbb{R}$,

$$\sup_{t \geqslant 1} \frac{1}{t} \sum_{i=1}^{t} y_i^2 < +\infty,\ \text{ a.s..}$$

For years, it had been conjectured that $b = 4$ might provide a limit in describing the nonlinear growth rate of system (3) that is stabilizable. The fact is, however, although $b = 4$ is good enough to approximate the critical stabilizability condition for most common types of discrete-time nonlinear systems, it is not an exact critical number in general as we used to expect. The example below throws light on this issue and present a new critical phenomenon about stabilizability.

Example 1 Consider system (3) with

$$f(x) = \begin{cases} x^3, & x \in [-\mathrm{e}^e, \mathrm{e}^e], \\ |x|^{4 - 1/(\log(\log|x|))^{\beta}}, & x > \mathrm{e}^e \text{ or } x < -\mathrm{e}^e, \end{cases} \qquad (4)$$

where $\beta > 0$. The system is globally stabilizable whenever $\beta \in (0, 2]$ and unstabilizable if $\beta > 2$. Obviously, $\beta = 2$ is a new critical number here. Note that for any $b < 4$,

$$\lim_{x \to +\infty} \frac{|f(x)|}{x^b} = +\infty,\ \lim_{x \to +\infty} \frac{|f(x)|}{x^4} = 0.$$

It confirms the fact that x^4 cannot serve as the critical growth rate for system (3). As a matter of fact, if $\beta \in (0, 2]$, systems (3)–(4) can be stabilized by the least-squares based self-tuning regulator (LS-STR), which is defined later by (6)–(7). Fig. 1 simulates the stability of the closed-loop systems (3) (4) (6) and (7) for $\beta = 2$, $y_0 = 0$, $\theta_0 = 0$ and $P_0 = 1$. Of course, it just a simple simulation of trajectory and can not provide more information about the stability with probability 1, we need strictly analysis to confirm the criticality of $\beta = 2$.

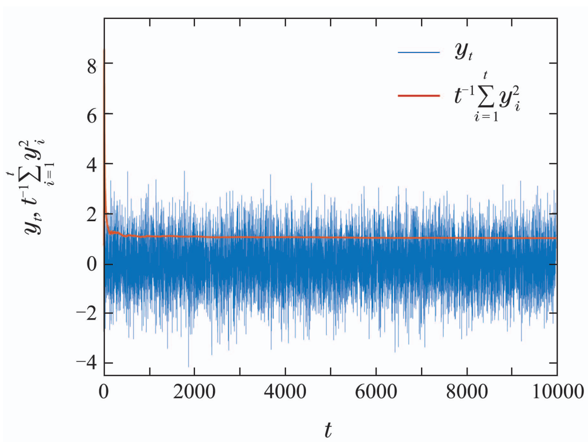

Fig. 1 The trajectory of the closed-loop system in Example 1 for $\beta = 2$

The critical number $\beta = 2$ in Example 1 cannot be deduced directly from the existing works. It originates from two theorems stated below. For this, assume $h : [0,+\infty) \to [0,+\infty)$ is a nonnegative monotone increasing piecewise continuous function and satisfies $h(|x|) = \mathrm{O}(x^4) + \mathrm{O}(1)$. Let $g(x) \triangleq |x|^{-1/4}h^{-1}(|x|)$, where h^{-1} denotes the inverse function of h.

Theorem 1 Under Assumptions A1)–A2), system (3) is globally stabilizable if $|f(x)| = \mathrm{O}(h(|x|))$, where h is chosen so that for some $\mu > \frac{1}{16}$,

$$\liminf_{t\to+\infty} \inf_{x\in[r_1^{2^t}, r_2^{t2^t}]} x^{-\mu/t^2}\frac{g(x)}{\log t} > 0,\ \forall r_2 > r_1 > \mathrm{e}^2. \tag{5}$$

Remark 1 Example 1 with $\beta \in (0,2]$ follows from the fact that if we let $h(x) = f(x)$, $x \geqslant 0$, then system (3) is globally stabilizable, according to Theorem 1 with $g(x) = |x|^{-1/4}h^{-1}(|x|)$. In this case, (5) holds. The proof is contained in Appendix.

On the other hand, Example 1 with $\beta > 2$ is unstabilizable due to

Theorem 2[1] Under Assumptions A1)–A2), system (3) is unstabilizable if there is a $\delta > 0$ such that

$$\sup_{x\in\mathbb{R}} \frac{\ell(S_h \cap [x-l, x+l])}{l} = \mathrm{O}\Big(\frac{1}{(\log(\log l))^{1+\delta}}\Big),$$

where $S_h \triangleq \{x : |f(x)| \leqslant h(|x|)\}$ with h satisfying

$$\sum_{t=1}^{+\infty} \sup_{x\in[\mathrm{e}^{2^t},+\infty)} x^{-1/16t^2} g(x) < +\infty.$$

Remark 2 The unstabilizability part of Example 1 is a direct consequence of Theorem 2, by taking

$$g(x) = x^{1/12(\log(\log|x|))^\beta}$$

with $\beta > 2$ (see Appendix for details).

3 Proof

3.1 Technique Lemmas

The feedback control law in this paper is designed based on the least-squares (LS) algorithm, which can be recursively defined by

$$\begin{cases} \theta_{t+1} = \theta_t + a_t P_t\phi_t(y_{t+1} - u_t - \phi_t^{\mathrm{T}}\theta_t), \\ P_{t+1} = P_t - a_t P_t\phi_t\phi_t^{\mathrm{T}}P_t,\ \ P_0 > 0, \\ \phi_t \triangleq f(y_t),\ \ t \geqslant 0, \end{cases} \tag{6}$$

where $a_t \triangleq (1 + \phi_t^{\mathrm{T}}P_t\phi_t)^{-1}$ and (θ_0, P_0) denotes a deterministic initial value. Let

$$u_t = -\theta_t f(y_t),\ t \geqslant 0. \tag{7}$$

By the closed-loop system (3)(6)–(7),

$$\tilde\theta_t = \frac{1}{r_{t-1}}\{\tilde\theta_0 - \sum_{i=0}^{t-1}\phi_i w_{i+1}\},$$

$$y_{t+1} = \tilde\theta_t f(y_t) + w_{t+1},$$

where $\tilde\theta_t \triangleq \theta - \theta_t$, $r_{-1} \triangleq P_0^{-1}$, $r_t \triangleq P_{t+1}^{-1} = P_0^{-1} + \sum_{i=0}^{t}\phi_i^2$, $t \geqslant 0$. Notice that the LS algorithm (6) is exactly the standard Kalman filter for $\theta \sim N(\theta_0, P_0)$, then

$$\theta_t = \mathrm{E}[\theta|\mathcal{F}_t^y],\ P_t = \mathrm{E}[(\tilde\theta_t)^2|\mathcal{F}_t^y].$$

So, y_{t+1} is conditionally Gaussian distributed given $\mathcal{F}_t^y$. For each $t \geqslant 0$, the conditional mean and variance satisfy

$$m_t \triangleq \mathrm{E}[y_{t+1}|\mathcal{F}_t^y] = u_t + \theta_t\phi_t = 0,\ \text{a.s.}$$

$$\sigma_t^2 \triangleq \mathrm{Var}(y_{t+1}|\mathcal{F}_t^y) = 1 + \phi_t P_t\phi_t = \frac{\phi_t^2}{r_{t-1}} + 1 = \frac{r_t}{r_{t-1}},\ \text{a.s.}$$

We first present several technique lemmas under Assumptions A1)–A2).

Lemma 1[1] Let $\{c_t\}_{t\geqslant1}$ be a sequence satisfying $\liminf\limits_{t\to+\infty}\frac{c_t}{\log t} > 0$, then

$$\sum_{t=1}^{+\infty}\int_{|x|\geqslant c_t}\mathrm{e}^{-x^2/2}\,\mathrm{d}x < +\infty.$$

Lemma 2[1] If $\ell(\{x : |f(x)| > 0\}) > 0$, then

$$\liminf_{t\to+\infty}\frac{r_t}{t} > 0,\ \text{a.s.}.$$

Lemma 3[1] Let $f(x) = \mathrm{O}(|x|^a) + \mathrm{O}(1)$ for some $a \geqslant 4$ and let $x_{\min} \leqslant x_{\max}$ denote the two solutions of equation $x^2 - (a-2)x + 1 = 0$. If $\ell(\{x : |f(x)| > 0\}) > 0$, then the following two statements hold:

i) $D_1 = D_2$ with $D_1 \triangleq \{\sup_t \sigma_t = +\infty\}$ and

$$D_2 \triangleq \{\liminf_{t\to+\infty}\frac{\log r_t}{\log r_{t-1}} \geqslant 1 + x_{\min}\};$$

ii) $P(D_3) = 0$ with

$$D_3 \triangleq \{\limsup_{t\to+\infty}\frac{\log r_t}{\log r_{t-1}} > 1 + x_{\max}\}.$$

Lemma 4 If (5) holds, then

$$\liminf_{x\to+\infty}\frac{g(x)}{\log x} > 0.$$

Proof Suppose $\liminf\limits_{x\to+\infty}\frac{g(x)}{\log x} \leqslant 0$. Then, there exists an infinite sequence $\{x_n\}_{n\geqslant1}$ satisfying $\lim\limits_{n\to+\infty} x_n = +\infty$ and

$$g(x_n) < \frac{1}{n}\log x_n. \tag{8}$$

Observe that for any $r_2 > r_1 > \mathrm{e}^2$,

$$r_1^{2^{t+1}} < r_2^{t2^t},\ \ t \geqslant 2$$

and hence

$$\bigcup_{t\geqslant2}[r_1^{2^t}, r_2^{t2^t}] = [r_1^4, +\infty).$$

Therefore, for any sufficiently large n, there is a positive integer k_n with $\lim\limits_{n\to+\infty} k_n = +\infty$ such that

$$x_n \in [r_1^{2^{k_n}}, r_2^{k_n 2^{k_n}}]. \tag{9}$$

This together with (8) and (9) yields

$$\limsup_{n\to+\infty} \inf_{x\in[r_1^{2^{k_n}}, r_2^{k_n 2^{k_n}}]} x^{-\mu/k_n^2} \frac{g(x)}{\log k_n} \leqslant$$
$$\limsup_{n\to+\infty} x_n^{-\mu/k_n^2} \frac{g(x_n)}{\log k_n} \leqslant \limsup_{n\to+\infty} x_n^{-\mu/k_n^2} \frac{\log x_n}{n\log k_n} \leqslant$$
$$\limsup_{n\to+\infty} r_1^{-\mu 2^{k_n}/k_n^2} \frac{k_n 2^{k_n}\log r_2}{n\log k_n} = 0,$$

and consequently,

$$\liminf_{t\to+\infty} \inf_{x\in[r_1^{2^t}, r_2^{t2^t}]} x^{-\mu/t^2} \frac{g(x)}{\log t} \leqslant 0.$$

We thus draws a contradiction of (5). QED.

3.2 Proof of Theorem 1

As already claimed in the proof of [1, Theorem 2.2], it suffices to show the stabilization for $\ell(\{x : |f(x)| > 0\}) > 0$. Under this condition, $\liminf_{t\to+\infty} \frac{r_t}{t} > 0$ almost surely due to Lemma 2. Denote

$$s_m \triangleq \frac{\log f^2(y_m)}{\log r_{m-1}} - 2,$$
$$S \triangleq \{\lim_{t\to+\infty} \frac{\log r_t}{\log r_{t-1}} = 2\} = \{\lim_{t\to+\infty} s_t = 0\},$$
$$U_m \triangleq \{s_{m-1} \leqslant 0,\ s_m \geqslant 0\},$$
$$V_m^C \triangleq \{s_m \geqslant \frac{2s_{m-1}}{2+s_{m-1}} - \frac{C}{m^2}\},\ C \geqslant 0.$$

Since the proof of [1, Theorem 2.2] indicates

$$\{\sup_t \sigma_t < +\infty\} \subseteq \{\frac{1}{t}\sum_{i=0}^{t} y_i^2 = \mathrm{O}(1)\},$$

taking account of Lemma 3 with $a = 4$, the remainder of the proof is sufficient to verify

$$P(S) = 0. \tag{10}$$

Without loss of generality, suppose $|f(x)| \leqslant h(|x|)$ for all $x \in \mathbb{R}$. Therefore, as long as m is sufficiently large,

$$P(U_{m+1}|\mathcal{F}_m^y) =$$
$$E\{I_{\{f^2(y_{m+1})\geqslant r_m^2\}} \cdot I_{\{f^2(y_m)\leqslant r_{m-1}^2\}}|\mathcal{F}_m^y\} \leqslant$$
$$I_{\{r_m\leqslant r_{m-1}+r_{m-1}^2\}} \cdot P(f^2(y_{m+1}) \geqslant r_m^2|\mathcal{F}_m^y) \leqslant$$
$$I_{\{r_m\leqslant r_{m-1}+r_{m-1}^2\}} \cdot P(h(|y_{m+1}|) \geqslant r_m|\mathcal{F}_m^y) =$$
$$I_{\{r_m\leqslant r_{m-1}+r_{m-1}^2\}} \cdot P(|y_{m+1}| \geqslant r_m^{1/4} g(r_m)|\mathcal{F}_m^y) =$$
$$I_{\{\sigma_m^2\leqslant\sqrt{(1+1/r_{m-1})r_m}\}} \cdot$$
$$\frac{1}{\sqrt{2\pi}}\int_{|x\cdot\sigma_m|\geqslant r_m^{1/4}g(r_m)} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \leqslant$$
$$I_{\{\sigma_m^2\leqslant\sqrt{(1+P_0)r_m}\}} \cdot$$
$$\frac{1}{\sqrt{2\pi}}\int_{|x|\geqslant(1+P_0)^{-1/4}g(r_m)} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \leqslant$$
$$\frac{1}{\sqrt{2\pi}}\int_{|x|\geqslant(1+P_0)^{-1/4}g(r_m)} \mathrm{e}^{-x^2/2}\,\mathrm{d}x, \tag{11}$$

and

$$P(V_{m+1}^C|\mathcal{F}_m^y) =$$
$$P(f^2(y_{m+1}) \geqslant r_m^{2+2s_m/(2+s_m)-C/m^2}|\mathcal{F}_m^y) \leqslant$$
$$P(h(|y_{m+1}|) \geqslant r_m^{1+s_m/(2+s_m)-C/2m^2}|\mathcal{F}_m^y) =$$
$$P(|y_{m+1}| \geqslant r_m^{1/4(1+s_m/(2+s_m)-C/2m^2)} \cdot$$
$$g(r_m^{1+s_m/(2+s_m)-C/2m^2})|\mathcal{F}_m^y) =$$
$$\frac{1}{\sqrt{2\pi}}\int_{|x|\geqslant R_m^C} \mathrm{e}^{-x^2/2}\,\mathrm{d}x, \tag{12}$$

where

$$R_m^C \triangleq r_m^{-C/8m^2} g(r_m^{1+s_m/(2+s_m)-C/2m^2}) \cdot (1 + r_{m-1}^{-1-s_m})^{-1/2(2+s_m)}.$$

Observe that $\liminf_{x\to+\infty} \frac{g(x)}{\log x} > 0$ in view of Lemma 4. Furthermore, according to $\liminf_{m\to+\infty} \frac{r_m}{m} > 0$ and $S = \{\lim_{t\to+\infty} s_t = 0\}$, we have

$$\liminf_{m\to+\infty} \frac{g(r_m)}{\log m} > 0 \tag{13}$$

and

$$\liminf_{m\to+\infty} \frac{1}{\log m} g(r_m^{1+s_m/2+s_m}) \cdot (1 + r_{m-1}^{-1-s_m})^{-1/2(2+s_m)} > 0 \text{ on } S. \tag{14}$$

Let $C = 0$. By virtue of (11)–(14) and Lemma 1, it deduces that

$$\sum_{m=1}^{+\infty} \frac{1}{\sqrt{2\pi}}\int_{|x|\geqslant R_m^0} \mathrm{e}^{-x^2/2}\,\mathrm{d}x < +\infty \text{ on } S,$$
$$\sum_{m=1}^{+\infty} \frac{1}{\sqrt{2\pi}}\int_{|x|\geqslant(1+P_0)^{-1/4}g(r_m)} \mathrm{e}^{-x^2/2}\,\mathrm{d}x < +\infty,$$

and

$$\sum_{m=1}^{+\infty} P(V_{m+1}^0|\mathcal{F}_m^y) < +\infty \text{ on } S,$$
$$\sum_{m=1}^{+\infty} P(U_{m+1}|\mathcal{F}_m^y) < +\infty.$$

Using Borel-Cantelli-Levy theorem, one has

$$\begin{cases} \sum\limits_{m=1}^{+\infty} I_{V_m^0} < +\infty \text{ on } S, \\ \sum\limits_{m=1}^{+\infty} I_{U_m} < +\infty. \end{cases} \text{a.s..} \tag{15}$$

Now, according to (15), on S, $\{s_t\}_{t\geqslant1}$ either satisfies

$$0 < s_t < \frac{2s_{t-1}}{2+s_{t-1}} \tag{16}$$

or

$$s_t < \frac{2s_{t-1}}{2+s_{t-1}} < 0, \tag{17}$$

where t is sufficiently large. However, if (17) holds, $|s_t| > \frac{2|s_{t-1}|}{2+s_{t-1}} > |s_{t-1}|$, and then $\lim_{t\to+\infty} s_t \neq 0$, which contradicts to the definition of S. Thus, $\{s_t\}_{t\geqslant 1}$ satisfies (16) on S, which means

$$S\subseteq \bigcup_{n\geqslant 1}\bigcap_{t\geqslant n+1}\{0<s_t<\frac{2s_{t-1}}{2+s_{t-1}}\}\subseteq \bigcup_{n\geqslant 1}\bigcup_{r\in Q^+}\bigcap_{t\geqslant n}\{0<s_t<\frac{2}{2/r+t-n}\}. \quad (18)$$

On the other hand, since $\liminf\limits_{t\to\infty}\frac{r_t}{t}>0$ almost surely,

$$S\subseteq \bigcup_{n\geqslant 1}\{r_n>\mathrm{e}^2\}= \bigcup_{n\geqslant 1}\bigcup_{s\in Q^+,s>\mathrm{e}^2}\{r_n\in(s,s+1)\}. \quad (19)$$

Denote

$$W_n^r\triangleq\bigcap_{t\geqslant n}\{0<s_t<\frac{2}{2/r+t-n}\},$$
$$T_n^s\triangleq\{r_n\in(s,s+1)\},$$

(18) and (19) leads to

$$S\subseteq\bigcup_{n\geqslant 1}((\bigcup_{r\in Q^+}W_n^r)\bigcap(\bigcup_{s\in Q^+,s>\mathrm{e}^2}T_n^s))\subseteq \bigcup_{n\geqslant 1}\bigcup_{r\in Q^+}\bigcup_{s\in Q^+,s>\mathrm{e}^2}(T_n^s\cap W_n^r).$$

To show (10), we only need to prove that for any $r,s\in Q^+$, $s>\mathrm{e}^2$ and $n\geqslant 1$,

$$P(T_n^s\cap W_n^r)=0. \quad (20)$$

Now, fix $r,s\in Q^+$, $s>\mathrm{e}^2$ and $n\geqslant 1$. Assume $W\triangleq T_n^s\cap W_n^r$ satisfies $P(W)>0$, we next show that on W,

$$r_m\in[s^{2^{m-n}},(C_1\cdot(s+1))^{(m-n+1)2^{m-n}}],\ \ m\geqslant n, \quad (21)$$

where $C_1=1+\frac{1}{s}$.

In fact, $r_m\geqslant s^{2^{m-n}}$ clearly holds. Moreover, for $m>n$,

$$\begin{aligned}&C_1r_m\leqslant\\&C_1r_{m-1}+C_1r_{m-1}^{2+2/(2/r+m-n)}=\\&C_1r_{m-1}^{2+2/(2/r+m-n)}(1+r_{m-1}^{-1-2/(2/r+m-n)})<\\&r_{m-1}^{2+2/(2/r+m-n)}C_1^2<(C_1r_{m-1})^{2+2/(m-n)},\end{aligned}$$

then

$$\begin{aligned}r_m<{}&C_1^{-1}(C_1r_n)^{\prod\limits_{i=n+1}^{m}(2+2/(i-n))}=\\&C_1^{-1}(C_1r_n)^{(m-n+1)2^{m-n}}<\\&C_1^{-1}(C_1(1+s))^{(m-n+1)2^{m-n}}<\\&(C_1(1+s))^{(m-n+1)2^{m-n}}\end{aligned}$$

and hence (21) follows.

Now, take some $\nu_1\in(\mathrm{e}^2,s)$, $\nu_2>C_1(s+1)$ and $C\in(\frac{1}{2},8\mu)$. On W, one has

$$r_m^{1+s_m/(2+s_m)-C/2m^2}\geqslant s^{2^{m-n}(1-C/2m^2)}>\nu_1^{2^{m-n}} \quad (22)$$

and

$$\begin{aligned}&r_m^{1+s_m/(2+s_m)-C/2m^2}<\\&(C_1\cdot(s+1))^{(m-n+1)2^{m-n}(1+s_m/(2+s_m))}<\\&(C_1\cdot(s+1))^{(m-n+1)2^{m-n}((2/r+m-n+3)/(2/r+m-n+1))}<\\&(C_1\cdot(s+1))^{(m-n+3)2^{m-n}}<\nu_2^{(m-n)2^{m-n}},\end{aligned} \quad (23)$$

where m is sufficiently large. Denote

$$Y_t\triangleq\inf_{[\nu_1^{2^t},\nu_2^{t2^t}]}x^{-\mu/t^2}g(x),$$

then by (21)–(23) and $\liminf\limits_{t\to+\infty}\frac{r_t}{t}>0$, for any sufficiently large m,

$$\begin{aligned}&R_m^C=\\&r_m^{-C/8m^2}g(r_m^{1+s_m/(2+s_m)-C/2m^2})\cdot\\&(1+r_{m-1}^{-1-s_m})^{-1/(4+4s_m)}\geqslant\\&r_m^{-C/8m^2}g(r_m^{1+s_m/(2+s_m)-C/2m^2})(1+r_{m-1}^{-1})^{-1/4}\geqslant\\&(r_m^{1+s_m/(2+s_m)-C/2m^2})^{-C/(8m^2-4C)}\cdot\\&g(r_m^{1+s_m/(2+s_m)-C/2m^2})\cdot\frac{1}{2}\geqslant\\&(r_m^{1+s_m/(2+s_m)-C/2m^2})^{-\mu/(m-n)^2}\cdot\\&g(r_m^{1+s_m/(2+s_m)-C/2m^2})\cdot\frac{1}{2}\geqslant\\&\inf_{[\nu_1^{2^{m-n}},\nu_2^{(m-n)2^{m-n}}]}x^{-\mu/(m-n)^2}g(x)\cdot\frac{1}{2}=\\&\frac{1}{2}Y_{m-n}\ \text{ on } W,\end{aligned} \quad (24)$$

where R_m^C is defined in (12). Moreover, by virtue of Lemma 1 and (5) with $r_1=\nu_1$ and $r_2=\nu_2$,

$$\sum_{m=n+1}^{+\infty}\frac{1}{\sqrt{2\pi}}\int_{|x|\geqslant Y_{m-n}/2}\mathrm{e}^{-x^2/2}\,\mathrm{d}x<+\infty. \quad (25)$$

With (12) (24) and (25), it is straightforward that

$$\sum_{m=1}^{+\infty}P(V_{m+1}^C|\mathcal{F}_m^y)<+\infty\ \text{ on } W,$$

which shows $\sum\limits_{m=1}^{+\infty}I_{V_{m+1}^C}<+\infty$ almost surely on set W, in view of Borel-Cantelli-Levy theorem. As a result, as long as m is sufficiently large,

$$0<s_m<\frac{2s_{m-1}}{2+s_{m-1}}-\frac{C}{m^2}\ \text{ a.s. on } W.$$

For $m>n+1$, denote

$$\rho_m\triangleq 2-(m-n)s_m,$$
$$d_m\triangleq\frac{2s_{m-1}}{2+s_{m-1}}-s_m,$$

since $s_m<\frac{2}{2/r+m-n}<\frac{2}{m-n}$ on W, one has $\rho_m\in(0,2)$, $d_m>\frac{C}{m^2}$ and

$$(m-n)d_m=\frac{2(m-n)(2-\rho_{m-1})}{2(m-n)-\rho_{m-1}}-(2-\rho_m)=$$

$$\frac{\rho_{m-1}(2-\rho_{m-1})}{2(m-n)-\rho_{m-1}}+\rho_m-\rho_{m-1}\leqslant$$
$$\frac{1}{2(m-n-1)}+\rho_m-\rho_{m-1}.$$

Therefore,

$$m(\rho_m-\rho_{m-1})\geqslant\frac{C(m-n)}{m}-\frac{m}{2(m-n-1)},$$

which, by noting that $C>\frac{1}{2}$, infers $\liminf\limits_{m\to+\infty}m(\rho_m-\rho_{m-1})>0$, and hence $\lim\limits_{m\to+\infty}\rho_m=+\infty$. This contradicts to the fact that $\rho_m<2$. We thus conclude $P(W)=0$. That is, (20) holds and hence Theorem 1 is proved.

4 Conclusions

The stabilizability theorem in this paper, combining with Theorem 2 derived by [1], tries to elaborate on the characterization of feedback limitations in discrete-time adaptive nonlinear control. Although the stabilizability and unstabilizability conditions presented here are very close, it still calls for further efforts on the critical stabilizability condition.

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Appendix: Proof of Remarks 1 and 2

We prove Remarks 1 and 2 in this appendix.

Proof of Remark 1 Given $\beta\in(0,2]$, let $\mu=\frac{1}{12}>\frac{1}{16}$ and $g_1(x)=x^{1/16(\log(\log|x|))^\beta}$, then for any $r_2>r_1>\mathrm{e}^2$ and $x\in[r_1^{2^t},r_2^{t2^t}]$, one has

$$\begin{aligned}&x^{-\mu/t^2}g_1(x)=\\&x^{1/(16(\log(\log|x|))^\beta)-1/12t^2}\geqslant\\&x^{1/(16(\log(t2^t\log r_2))^2)-1/12t^2}=\\&x^{1/(16(t\log 2+\log t+\mathrm{O}(1))^2)-1/12t^2}\geqslant\\&r_1^{(1/(16(t\log 2+t(0.75-\log 2))^2)-1/12t^2)2^t}=\\&r_1^{2^t/36t^2},\end{aligned}$$

where t is sufficiently large. Thus,

$$\begin{aligned}&\liminf_{t\to+\infty}\inf_{x\in[r_1^{2^t},r_2^{t2^t}]}x^{-\mu/t^2}\frac{g_1(x)}{\log t}\geqslant\\&\liminf_{t\to+\infty}r_1^{2^t/36t^2}\log^{-1}t=+\infty.\end{aligned}\tag{a1}$$

Therefore, for any $f(x)$ given by Example (4) with $\beta\in(0,2]$, if we can show

$$|x|^{-1/4}f^{-1}(|x|)\geqslant g_1(x)\tag{a2}$$

holds for all sufficiently large $|x|$, then by using (a1), $f(x)$ satisfies the condition of Theorem 1 and Remark 1 follows.

To this end, denote $\varLambda_1\triangleq\log(\log|x|)$ and $\varLambda_2\triangleq\frac{1}{4}+\frac{1}{16\varLambda_1^\beta}$. Since $\log\varLambda_2\in(-\log 2,0)$ for $|x|>\mathrm{e}^e$, it yields

$$\frac{1}{4}+\varLambda_1^\beta\geqslant(\varLambda_1+\log\varLambda_2)^\beta.\tag{a3}$$

Note that for any sufficiently large $|x|$, (a3) is equivalent to

$$1\geqslant\varLambda_2\Big(4-\frac{1}{(\log(\varLambda_2\log|x|))^\beta}\Big),$$

and hence

$$|x|\geqslant|x|^{\varLambda_2(4-1/(\log(\varLambda_2\log|x|))^\beta)}.$$

So, $|x|\geqslant f(|x|^{1/4}g_1(x))$, which is exactly (a2) as desired. QED.

Proof of Remark 2 For $\beta>2$, let $g(x)=x^{1/12(\log(\log|x|))^\beta}$. Then, for any $x\in[\mathrm{e}^{2^t},+\infty)$,

$$\begin{aligned}x^{-1/16t^2}g(x)=&\,x^{1/12(\log(\log|x|))^\beta-1/16t^2}\leqslant\\&\,x^{1/(12(t\log 2)^\beta)-1/16t^2}\leqslant\\&\,\mathrm{e}^{(1/12(t\log 2)^\beta-1/16t^2)2^t}=\mathrm{o}(t^{-2}),\end{aligned}$$

as $t \to +\infty$. Consequently,

$$\sum_{t=1}^{+\infty} \sup_{x \in [\mathrm{e}^{2^t}, +\infty)} x^{-1/16t^2} g(x) <$$
$$\sum_{t=1}^{+\infty} \mathrm{e}^{(1/12(t \log 2)^{\beta} - 1/16t^2)2^t} < +\infty.$$

This implies that if for all sufficiently large $|x|$,

$$|x|^{-1/4} f^{-1}(|x|) \leqslant g(x), \tag{a4}$$

then $f(x)$ satisfies the condition of Theorem 2.

As above, denote $\Lambda_3 \triangleq \dfrac{1}{4} + \dfrac{1}{12\Lambda_1^{\beta}} \in (\dfrac{1}{4}, 1)$ for $|x| > \mathrm{e}^{e}$. Then,

$$\frac{1}{3} + \Lambda_1^{\beta} \leqslant \frac{4}{3}(\Lambda_1 + \log \Lambda_3)^{\beta},$$

and hence for any sufficiently large $|x|$,

$$|x| \leqslant |x|^{\Lambda_3(4 - 1/(\log(\Lambda_3 \log |x|))^{\beta})}.$$

Since $f(x)$ is defined by (4), similar to the argument of Remark 1, (a4) follows and we complete the proof. QED.

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