具有通信时延和有向网络的多智能体分布式优化

杨正全[1,2†], 潘小芳[2], 张 青[2], 陈增强[3]
(1. 中国民航大学 机场综合交通研究院, 天津 300300; 2. 中国民航大学 理学院, 天津 300300; 3. 南开大学 人工智能学院, 天津 300350)

摘要: 本文对非平衡有向拓扑下一阶多智能体系统的分布式优化问题进行研究. 研究的智能体在与邻居的通信过程中都有一个变化的时延. 本文的目标是找到使得目标函数$f(x) = \sum\limits_{i=1}^{N} f_i(x_i)$最小的智能体的状态. 提出了一种基于有向图的拉普拉斯矩阵的零特征值对应的左特征向量和智能体的局部信息的控制器. 在这项研究中, 去掉了$f_i(x_i)$的梯度有界要求, 并且不要求网络平衡. 在某些假设下, 所有智能体都达到相同的状态, 同时最小化目标函数$\sum\limits_{i=1}^{N} f_i(x_i)$. 最后, 通过数值模拟验证了本文的研究结果.

关键词: 通信时延; 多智能体系统; 分布式优化; 非平衡有向网络
引用格式: 杨正全, 潘小芳, 张青, 等. 具有通信时延和有向网络的多智能体分布式优化. 控制理论与应用, 2021, 38(9): 1414 – 1420
DOI: 10.7641/CTA.2021.20243

Distributed optimization of multi-agent with communication delay and directed network

YANG Zheng-quan[1,2†], PAN Xiao-fang[2], ZHANG Qing[2], CHEN Zeng-qiang[3]
(1. Airport Comprehensive Transportation Research Institute, Civil Aviation University of China, Tianjin 300300, China;
2. College of Science, Civil Aviation University of China, Tianjin 300300, China;
3. College of Artificial Intelligence, Nankai University of China, Tianjin 300350, China)

Abstract: This paper studies the distributed optimization problem for the first-order multi-agent systems in unbalanced directed topology. Each agent has a time delay in the communication process with its neighbors. Our goal is to find a state of the agent that minimizes the objective function $f(x) = \sum\limits_{i=1}^{N} f_i(x_i)$. A controller based on the left eigenvector corresponding to the zero eigenvalue of L of the graph G and the local information of each agent is proposed. In this study, the bounded requirement of the gradient of $f_i(x_i)$ is removed, and it does not require the network to be balanced. Under some assumptions, all agents reach the same state while minimizing the objective function $\sum\limits_{i=1}^{N} f_i(x_i)$. Finally, a numerical simulation illustrates the results of this article.

Key words: communication delay; multi-agent systems; distributed optimization; unbalanced directed network
Citation: YANG Zhengquan, PAN Xiaofang, ZHANG Qing, et al. Distributed optimization of multi-agent with communication delay and directed network. *Control Theory & Applications*, 2021, 38(9): 1414 – 1420

1 Introduction

More and more scientists in the control field have studied the distributed optimization of multi-agent systems[1–9]. The purpose of the optimization is to make these agents reach a common state by using neighbor information, and the common state is the optimal solution of a function $f(x)$, where the function $f(x)$ is a sum of the function $f_i(x_i)$. In other words, it focuses on guiding all agents to cooperatively solve the global optimization problem. Distributed optimization is widely used in industry, technology, economic, and social fields, such as machine learning, smart grid, resource allocation, etc.[10–13].

In the past, consensus is a fundamental and meaningful research topic in the field of control, which focuses on designing appropriate controllers to make multiple agents reach a common state[14–17]. Furthermore, a large number of distributed optimization control strategies based on consensus have been obtained and analyzed. For example, a method based on consensus

Received 25 November 2020, accepted 15 June 2021.
†Corresponding author. E-mail: zquanyang@163.com.
Recommended by Associate Editor: YI Peng.
Supported by the National Natural Science Foundation of China (51778615), the Fund of the Airport Comprehensive Transportation Research Institute (Think Tank) and the Civil Aviation University of China (20JCZHJT-4).

is proposed to compute the intersection of convex sets in [2]. Ning et al.[4] studied the optimization strategy of multiple agents based on the fixed-time consensus method in a distributed manner. In some practical systems, the algorithm for optimizing the systems has been studied in different ways such as sub-gradient, gradient and zero-gradient sum[1–9]. For continuous time dynamic systems, Nedic et al.[5] and Gharesifard et al.[6] developed a new control framework to handle the optimization problem of the multi-agent systems. In some weight-balanced directed topological networks, Kia et al.[7] designed a continuous-time consensus control law to handle the optimization problem with discrete communication in a distributed way.

In reality, in the process of information transmission, the communication delay between agents is inevitable, especially when the number of agents is large. In order to solve this problem, some distributed optimization schemes with communication delay are proposed in [18–25]. For example, in [18–20], some consensus-based methods are developed to deal with the optimization in presence of communication delay. In some practical applications, Chen et al.[21] and Yang et al.[22] proposed the economic dispatch schemes with delay effect, and these methods solve the economic dispatch problem on time-varying digraphs. Moreover, focus on a system composed of multiple agents with constraint sets and delays, a distributed method was proposed to do with optimization problems on weight-balance directed graphs in [23]. Guo et al.[24] further proposed the algorithms based on zero-gradient-sum approach to successfully deal with distributed optimization problems under fixed and time-varying delays respectively, and gave the upper bounds of their delays. Delays can be arbitrary, time-varying but bounded.

The actual network environment is changeable and complex, and the communication between nodes is directed and asymmetric. Accordingly, unbalanced directed network is more challenging than the weight-balanced case and undirected case. It is meaningful to handle distributed optimization problem of multi-agent systems with communication delays in unbalanced directed networks. To achieve this, three challenges will be faced. Firstly, the stability analysis is usually complex when the network is directed. Secondly, the analysis is more difficult to overcome the unbalance of the network. Thirdly, once the communication delay is introduced, the analysis becomes much more complex. Most of the existing work has solved the optimization of a system composed of multiple agents with communication delays in a distributed manner, but they are only aimed at undirected and weight-balanced networks.

This paper proposes a new distributed optimization controller which not only has communication delay but also has unbalanced directed topology. Compared with the existing relevant results, the contributions of this note are summarized as follows. Compared with the distributed controller without the communication delay[1–9], this paper considers the multi-agent with communication delay, and removes the bounded requirement of the gradient of $f_i(x_i)$. Compared with [24], the optimization problem under directed topology is studied. Compared with [18–24], the communication delay is time varying, the bounded requirement of the gradient of $f_i(x_i)$ is removed, and it does not require the network to be balanced. In conclusion, compared with previous research results, the main innovations of this study are as follows. Firstly, the controller has varying time communication delay. Secondly, the network topology is directed and unbalance. Thirdly, the controller relaxes the assumption of the local function $f_i(x_i)$.

The basic structure of this article is as follows. In Sec. 2, we introduce the basic principles and the statement of graph theory. In Sec. 3, the theoretical results of this study are given. In Sec. 4, we present a simulation result of this research. In Sec. 5, we give the conclusion of this article.

2 Notations and Preliminaries

In this section, we give some concepts used in this article. Let $\mathbb{R}$ be the sets of real numbers, $\mathbb{R}^m$ represents the column vector of m-dimensional, $1_N \in \mathbb{R}^N$ represents the column vector of N-dimensional whose element is 1, $\mathbb{R}^{n\times m}$ is the matrix of size $n \times m$, $P_X(s)$ denotes the projection of the vector s onto the closed convex set X, i.e., $P_X(s) = \arg\min\limits_{x\in X} \|x - s\|$. If a function $f(s) : \mathbb{R}^m \to \mathbb{R}$ satisfies: $f(bs_1 + (1 - b)s_2) \leqslant bf(s_1) + (1 - b)f(s_2), \forall s_1, s_2 \in \mathbb{R}^m, b \in (0, 1)$, $f(s)$ is convex function. $\nabla f(x)$ is the partial derivative of $f(x)$. $\tau(t)$ is a function of communication delay and arbitrary.

In this paper, we regard the information communication between multi-agents as a directed graph G. The directed graph G is considered with the node set $\mathcal{V} = \{1, 2, \ldots, N\}$, the edge set $\mathcal{E}$ and the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. If there is communication from agent j to agent i, $(j, i) \in \mathcal{E}$. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ of the graph G is defined as 1) $a_{ii} = 0$; 2) $a_{ji} = 1$ if $(i, j) \in \mathcal{E}$; 3) $a_{ji} = 0$ if $(i, j) \notin \mathcal{E}$. Furthermore, $L = [l_{ij}]_{N\times N}$ is called the Laplacian matrix of the graph G which is denoted as $l_{ij} = -a_{ij}$, if $i \neq j$, otherwise $l_{ii} = \sum\limits_{j\neq i}^{N} a_{ij}$.

The eigenvalues of L is need to satisfy $0 = \lambda_1(L) \leqslant \lambda_2(L) \leqslant \cdots \leqslant \lambda_N(L)$, and the eigenvalues of $L^{\mathrm{T}}L$ is need to satisfy $0 = \lambda_1(L^{\mathrm{T}}L) \leqslant \lambda_2(L^{\mathrm{T}}L) \leqslant \cdots \leqslant \lambda_N(L^{\mathrm{T}}L)$. $P = [p_1\ p_2\ \cdots\ p_N]^{\mathrm{T}}$ is the left eigenvector corresponding to the zero eigenvalue of L, which satisfies $P^{\mathrm{T}}L = 0$; p_i represents the ith element of the P

vector, $\mathrm{diag}\{p_1, p_2, \cdots, p_N\}$ is a diagonal matrix with $p_1, p_2, \cdots, p_N$ as the elements on the diagonal. $d_i^{\mathrm{out}} = l_{ii}$ represents the out-degree of agent i and $d_i^{\mathrm{in}} = \sum\limits_{j\neq i}^{N} a_{ji}$ represents the in-degree of agent i. If $d_i^{\mathrm{out}} = d_i^{\mathrm{in}}$, G is a balanced graph, otherwise G is an unbalanced graph. The graph G containing at least one spanning tree is connected. If there is always a path between any two nodes in graph G, then the graph G is strongly connected.

On those topics, we study a system of multiple agents composed of $\mathbb{N}$ agents (indexed by $1, 2, \cdots, N$). The dynamic description of each agent in the system is as follows

$$\dot{x}_i(t) = u_i(t),\ i \in \mathbb{N}, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^n$ represent the state and the controller of agent i respectively.

Our goal is to design a controller for the system (1) so that each agent can track a common state and optimize the global objective function. In other words, this problem can be expressed as

$$\text{minimize } f(x) = \sum_{i=1}^{N} f_i(x_i),\ \text{s.t. } x_i = x_j, \tag{2}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector of the agent i, $f_i(x_i) : \mathbb{R}^n \to \mathbb{R}$ represents the local objective function of the agent i.

In order to solve the above-mentioned distributed optimization problem, the following lemmas and assumptions are needed.

Lemma 1[26] Given a positive matrix $\phi \in \mathbb{R}^{N\times N}, \forall a \in \mathbb{R}^N, b \in \mathbb{R}^N$, it has that

$$2ab \leqslant a^{\mathrm{T}}\phi^{-1}a + b^{\mathrm{T}}\phi b. \tag{3}$$

In particular, when $\phi \in \mathbb{R}^{N\times N}$ is an identity matrix, $2ab \leqslant a^{\mathrm{T}}a + b^{\mathrm{T}}b$.

Lemma 2[27] Let D be a symmetric matrix, it is represented as

$$D = \begin{bmatrix} D_1 & D_2 \\ D_2^{\mathrm{T}} & D_3 \end{bmatrix}, \tag{4}$$

where D_1 and D_3 are square matrix. $D < 0$ iff $D_1 < 0$, $D_3 - D_2^{\mathrm{T}}D_1^{-1}D_2 < 0$ or $D_3 < 0$, $D_1 - D_2D_3^{-1}D_2^{\mathrm{T}} < 0$.

Lemma 3[28] Let $p = [p_1\ p_2\ \cdots\ p_N]^{\mathrm{T}}$ be the left eigenvector corresponding to the zero eigenvalue of L and $\Lambda = \mathrm{diag}\{p_1, p_2, \cdots, p_N\}$, where $p_i \in \mathbb{R}, p_i > 0$ $(i = 1, 2, \cdots, N)$. Then there is any positive vector $\gamma \in \mathbb{R}^{N\times 1}$ such that $\min\limits_{\gamma^{\mathrm{T}}x=0, x\neq \mathbf{0}} \dfrac{x^{\mathrm{T}}\bar{L}x}{x^{\mathrm{T}}x} > \dfrac{\lambda_2(\bar{L})}{N}$, where $\lambda_2(\bar{L})$ is the second smallest eigenvalue of matrix $\bar{L}$ and $\bar{L} \triangleq \Lambda L + L^{\mathrm{T}}\Lambda$.

Lemma 4[29] The continuously convex and differentiable function $f(s)$ is minimized iff $\nabla f(s) = 0$.

Lemma 5 Let X be a convex and closed set and $f_i(s)$ is a convex function, from the convexity of $f_i(s)$, the following inequality hold

$$\nabla f_i(x)^{\mathrm{T}}(x - y) \geqslant f_i(x) - f_i(y),$$

where $x \in \mathbb{R}^n$ and $y \in X$.

Assumption 1 Let $\tau(t)$ be a continuously differentiable function, and it satisfies $0 \leqslant \tau(t) \leqslant d$ and $\dot{\tau}(t) \leqslant h \leqslant 1$.

Assumption 2 The graph G is a connected and unbalanced directed graph.

Assumption 3 Let $X_i = \{s_1 \in \mathbb{R}^n | \nabla f_i(s_1) = 0\}$ be a nonempty and bounded set, then, $f_i(s)$ is a convex function.

Assumption 4 Let $X = \{s_2 \in \mathbb{R}^n | \sum\limits_{i=1}^{N} \nabla f_i(s_2) = 0\}$ be a closed convex and nonempty set.

Assumption 5 In this paper, $\alpha(t)$ is need to satisfy the following conditions:

$$\lim_{t\to+\infty} \|\alpha(t)\| = 0, \tag{5}$$

$$\int_0^{+\infty} \|\alpha(t)\|\mathrm{d}t = +\infty. \tag{6}$$

Lemma 6 Suppose that Assumption 1 and 2 hold. If $d < \dfrac{\lambda_2(\bar{L})}{(\frac{\gamma_{\max}}{4} + \frac{2}{1-h})N\lambda_N(L^{\mathrm{T}}L)}$, the following inequality holds:

$$\begin{bmatrix} -B & 0 & -\Lambda L \\ 0 & -dL^{\mathrm{T}}L & L^{\mathrm{T}}L \\ -\Lambda L & L^{\mathrm{T}}L & L^{\mathrm{T}}L \end{bmatrix} < 0, \tag{7}$$

where $B = 2\Lambda L - d(\Lambda L)^{\mathrm{T}}\Lambda L - \dfrac{2d}{1-h}L^{\mathrm{T}}L$.

Proof According to Lemma 3, there exists a vector with positive entries 1_N such that $\xi^{\mathrm{T}}(t)1_N = 0$. Then, we have $2\xi(t)\Lambda L\xi(t) > \dfrac{\lambda_2(\bar{L})}{N}\xi(t)\xi(t)$, where $\bar{L} = \Lambda L + L^{\mathrm{T}}\Lambda$ and $2\xi(t)\Lambda L\xi(t) = \xi(t)\bar{L}\xi(t)$. Thus, if $d < \dfrac{\lambda_2(\bar{L})}{(\frac{\gamma_{\max}}{4} + \frac{2}{1-h})N\lambda_N(L^{\mathrm{T}}L)}$, we have

$$-\xi(t)B\xi(t) <$$
$$-(\frac{\lambda_2(\bar{L})}{N} - \frac{d}{4}\bar{L}^{\mathrm{T}}\bar{L} - \frac{2d}{1-h}L^{\mathrm{T}}L)\xi(t)\xi(t) < 0,$$

where $\gamma_{\max} = \max(p_1, p_2, \cdots, p_N)$ and $B = 2\Lambda L - d(\Lambda L)^{\mathrm{T}}\Lambda L - \dfrac{2d}{1-h}L^{\mathrm{T}}L$. Furthermore, $-B < 0$.

If let $D_1 = -B, D_2 = [0, -\Lambda L]$ and $D_3 = \begin{bmatrix} -dL^{\mathrm{T}}L & L^{\mathrm{T}}L \\ L^{\mathrm{T}}L & L^{\mathrm{T}}L \end{bmatrix}$.

It follows from Lemma 2 that (7) is equivalent to the following inequality:

$$\begin{bmatrix} -dL^{\mathrm{T}}L & L^{\mathrm{T}}L \\ L^{\mathrm{T}}L & L^{\mathrm{T}}L + L^{\mathrm{T}}\varLambda B^{-1}\varLambda L \end{bmatrix} < 0.$$

From Lemma 2, it is clear that the above formula holds.

Remark 1 In this study, the communication delay is considered. Compared with [18–24], the bounded requirement of the gradient of $f_i(x_i(t))$ is removed, and it does not require the network to be balanced.

3 Main results

In this section, we design a novel optimization controller to make all agents realize consensus and minimize the global objective functions. The controller has two parts. The first part is the summation function, which is used to achieve consistency. The second part is the sub-gradient to achieve global optimization. The distributed controller is given by:

$$u_i(t) = -\sum_{i=1}^{N} a_{ij}(x_i(t-\tau(t)) - x_j(t-\tau(t))) - \alpha(t)\frac{\nabla f_i(x_i)}{p_i}, \tag{8}$$

where $\tau(t)$ is the time-varying communication delay; $x_i(t-\tau(t)) \in \mathbb{R}^n$ represents the state vector of the agent i at time $t-\tau(t)$; $\alpha(t)$ is a function that is required to meet Assumption 6; $\nabla f_i(x_i)$ is the gradient of $f_i(x_i)$; $p = [p_1\ p_2\ \cdots\ p_N]^{\mathrm{T}}$ satisfies $p^{\mathrm{T}}L = 0$ and $p^{\mathrm{T}}1_N = 1$; p_i is the ith element of the p. The p_i in the controller is to eliminate imbalance of graph G.

Let $\xi_i(t) = x_i(t) - \bar{x}$, where $\bar{x} = \sum\limits_{j=1}^{N} p_j x_j(t)$. Then, according to $p^{\mathrm{T}}L = 0$ and $p^{\mathrm{T}}L = 1$, from (1) and (8), we have

$$\dot{\xi}_i(t) = -\sum_{i=1}^{N} a_{ij}(\xi_i(t-\tau(t)) - \xi_j(t-\tau(t))) - \alpha(t)[\frac{\nabla f_i(x_i)}{p_i} - \sum_{k=1}^{N}\nabla f_k(x_k)]. \tag{9}$$

Let

$$\xi(t) = [\xi_1(t)\ \xi_2(t)\ \cdots\ \xi_N(t)]^{\mathrm{T}},$$

$$\nabla f = \alpha(t)[\frac{\nabla f_1(x_1)}{p_1} - \sum_{k=1}^{N}\nabla f_k(x_k)\quad \frac{\nabla f_1(x_2)}{p_2} - \sum_{k=1}^{N}\nabla f_k(x_k)\ \cdots\ \frac{\nabla f_N(x_N)}{p_N} - \sum_{k=1}^{N}\nabla f_k(x_k)]^{\mathrm{T}},$$

the equation of (9) can be written as

$$\dot{\xi}(t) = -L\xi(t-\tau(t)) - \nabla f. \tag{10}$$

Below, we will give the main theorem and prove it.

Theorem 1 If Assumption 1–5 are satisfied, the control law (8) solve the optimization problem (2) in a distributed way.

Proof Firstly, in order to prove that each agent asymptotically reaches the same state, a Lyapunov-Krasovskii functional is selected.

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{11}$$

where the positive definite functions $V_1(t)$, $V_2(t)$, $V_3(t)$ are defined as

$$V_1(t) = \xi^{\mathrm{T}}\varLambda\xi, \tag{12}$$

$$V_2(t) = \frac{2d}{1-h}\int_{t-\tau(t)}^{t}\xi(s)^{\mathrm{T}}L^{\mathrm{T}}L\xi(s)\mathrm{d}s, \tag{13}$$

$$V_3(t) = \int_{-d}^{0}\int_{t+\theta}^{t}\dot{\xi}(s)^{\mathrm{T}}\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta, \tag{14}$$

where $\tau(t)$ is a function of communication delay and arbitrary and satisfies $0 \leqslant \tau(t) \leqslant d$; d is the upper bound of $\tau(t)$; $\dot{\tau}(t)$ is the derivative of $\tau(t)$ and satisfies $\dot{\tau}(t) \leqslant h \leqslant 1$; $\varLambda$ is a diagonal matrix, and $\varLambda = \mathrm{diag}\{p_1, p_2, \cdots, p_N\}$.

The derivatives of $V_1(t)$, $V_2(t)$ and $V_3(t)$ along the system (1) are given by

$$\dot{V}_1(t) = 2\xi^{\mathrm{T}}\varLambda\dot{\xi} = -2\xi^{\mathrm{T}}\varLambda L\xi(t-\tau(t)) - 2\xi^{\mathrm{T}}\varLambda L\nabla f, \tag{15}$$

$$\dot{V}_2(t) = \frac{2d}{1-h}\xi^{\mathrm{T}}L^{\mathrm{T}}L\xi - \frac{2d(1-\dot{\tau}(t))}{1-h}\times \xi^{\mathrm{T}}(t-\tau(t))L^{\mathrm{T}}L\xi^{\mathrm{T}}(t-\tau(t)) \leqslant \frac{2d}{1-h}\xi^{\mathrm{T}}L^{\mathrm{T}}L\xi - 2d\xi^{\mathrm{T}}(t-\tau(t))L^{\mathrm{T}}L\xi^{\mathrm{T}}(t-\tau(t)), \tag{16}$$

$$\dot{V}_3(t) = d\dot{\xi}^{\mathrm{T}}(t)\dot{\xi}(t) - \int_{t-d}^{t}\dot{\xi}^{\mathrm{T}}(s)\dot{\xi}(s)\mathrm{d}s = d\xi^{\mathrm{T}}(t-\tau(t))L^{\mathrm{T}}L\xi(t-\tau(t)) + 2d\xi^{\mathrm{T}}(t-\tau(t))L^{\mathrm{T}}L\nabla f + \nabla f^{\mathrm{T}}L^{\mathrm{T}}L\nabla f - \int_{t-d}^{t}\dot{\xi}^{\mathrm{T}}(s)\dot{\xi}(s)\mathrm{d}s. \tag{17}$$

According to $\xi(t-\tau(t)) = \xi(t) - \int_{t-\tau(t)}^{t}\dot{\xi}(s)\mathrm{d}s$, Lemma 1, and Assumption 1, the equation (15) can be rewritten by

$$\begin{aligned}\dot{V}_1(t) = {} & -2\xi^{\mathrm{T}}\varLambda L[\xi(t) - \int_{t-\tau(t)}^{t}\dot{\xi}(s)\mathrm{d}s] - 2\xi^{\mathrm{T}}\varLambda L\nabla f \leqslant \\ & -2\xi^{\mathrm{T}}\varLambda L[\xi(t) + \tau(t)\xi^{\mathrm{T}}(\varLambda L)^{\mathrm{T}}\varLambda L\xi(t) + \int_{t-\tau(t)}^{t}\dot{\xi}^{\mathrm{T}}(s)\dot{\xi}(s)\mathrm{d}s] - 2\xi^{\mathrm{T}}\varLambda L\nabla f \leqslant \\ & -2\xi^{\mathrm{T}}\varLambda L[\xi(t) + d\xi^{\mathrm{T}}(\varLambda L)^{\mathrm{T}}\varLambda L\xi(t) + \int_{t-d}^{t}\dot{\xi}^{\mathrm{T}}(s)\dot{\xi}(s)\mathrm{d}s] - 2\xi^{\mathrm{T}}\varLambda L\nabla f.\end{aligned} \tag{18}$$

Under Assumption 5, combined the result of (16), (17) and (18), we have

$$\dot{V}(t) =$$

$$\dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t)\leqslant$$
$$-\xi(t)[2\Lambda L-d(\Lambda L)^{\mathrm{T}}\Lambda L-\frac{2d}{1-h}L^{\mathrm{T}}L]\xi(t)-$$
$$d\xi^{\mathrm{T}}(t-\tau(t))L^{\mathrm{T}}L\xi(t-\tau(t))-2\xi^{\mathrm{T}}\Lambda L\nabla f+$$
$$2d\xi^{\mathrm{T}}(t-\tau(t))L^{\mathrm{T}}L\nabla f+\nabla f^{\mathrm{T}}L^{\mathrm{T}}L\nabla f=$$
$$\begin{bmatrix}\xi(t) & \xi(t-\tau(t)) & \nabla f\end{bmatrix}\cdot$$
$$\begin{bmatrix}-B & 0 & -\Lambda L\\ 0 & -dL^{\mathrm{T}}L & L^{\mathrm{T}}L\\ -\Lambda L & L^{\mathrm{T}}L & L^{\mathrm{T}}L\end{bmatrix}\begin{bmatrix}\xi(t)\\ \xi(t-\tau(t))\\ \nabla f\end{bmatrix}\leqslant$$
$$0, \tag{19}$$

where $B=2\Lambda L-d(\Lambda L)^{\mathrm{T}}\Lambda L-\dfrac{2d}{1-h}L^{\mathrm{T}}L$. It should be pointed out that the last formula above holds because of Lemma 6.

Thus, according to Lasalle invariance principle, we get $\xi(t)=0$, $\xi(t-\tau(t))=0$ and $\nabla f=0$ as $t\to 0$. That is $x_i(t)=\sum\limits_{i=j}^{N}p_jx_j(t)$, $x_i(t-\tau(t))=\sum\limits_{j=1}^{N}p_jx_j(t-\tau(t))$. In a word, the state of all agents tends to be the same.

Next, we will prove that this state is the optimal state such that minimizes $f(x)$.

Based on the above proof result and Assumption 4, we will prove that $x_i(t)=\sum\limits_{i=j}^{N}p_jx_j(t)$ is the optimal state.

We know $\bar{x}(t)=\sum\limits_{i=1}^{N}p_ix_i(t)$, applying the above proof results and equation (8), we have

$$\begin{aligned}\dot{\bar{x}}(t)=&\sum_{i=1}^{N}p_i\dot{x}_i(t)=\\ &-\alpha(t)\sum_{i=1}^{N}\nabla f_i(x_i(t))=\\ &-\alpha(t)\sum_{i=1}^{N}\nabla f_i(\bar{x}(t)).\end{aligned}\tag{20}$$

Because X is closed convex set and $P_X(\bar{x}(t))\in X$, there exists a positive constant M_1 such that $\|P_X(\bar{x}(t))\|\leqslant M_1$. And $\|x_i(t)-\sum\limits_{i=j}^{N}p_jx_j(t)\|\to 0$ as $t\to+\infty$, then there exists a constant T_1 such that for all $t>T_1$, $\|x_i(t)\|$ is bounded. Furthermore, there exists $M_2>0$ and $t>T_1$ such that $\|\bar{x}(t)-P_X(\bar{x}(t))\|\leqslant M_2$. Then, there exists $M>0$ and $t>T_2$ such that $\|\bar{x}(t)-P_X(\bar{x}(t))\|^2\leqslant M$.

Construct a positive condition function as follows

$$W(t)=(\bar{x}(t)-P_X(\bar{x}(t)))^{\mathrm{T}}(\bar{x}(t)-P_X(\bar{x}(t))).\tag{21}$$

The time derivative of $W(t)$ is

$$\begin{aligned}\dot{W}(t)=&2(\bar{x}(t)-P_X(\bar{x}(t)))^{\mathrm{T}}\dot{\bar{x}}(t)=\\ &-2\alpha(t)\sum_{i=1}^{N}(\bar{x}(t)-P_X(\bar{x}(t)))^{\mathrm{T}}\nabla f_i(\bar{x}(t)).\end{aligned}\tag{22}$$

Let $\rho=\min\limits_{s\in\cup X_i}\sum\limits_{i=1}^{N}(f_i(s)-f_i(P_X(s)))$. From the definition of X_i, we know that $\rho\geqslant 0$. Then from Lemma 5, we get

$$\begin{aligned}\dot{W}(t)\leqslant&-2\alpha(t)\sum_{i=1}^{N}(f_i(\bar{x}(t))-f_i(P_X(\bar{x}(t))))\leqslant\\ &-2\alpha(t)\rho.\end{aligned}\tag{23}$$

Since $\int_0^{+\infty}\|\alpha(t)\|\mathrm{d}t=+\infty$, then for all $\varepsilon>0$ there exists $T_3>T_2>0$ such that

$$\int_{T_2}^{T_3}\|\alpha(t)\|\mathrm{d}t\geqslant\frac{M-\varepsilon}{2\rho}.$$

Thus, for all $t>T_3$,

$$\int_{T_2}^{t}\dot{W}(t)\mathrm{d}t\leqslant-2\rho\int_{T_2}^{t}\alpha(t)\mathrm{d}t\leqslant-(M-\varepsilon).$$

Because $\forall t\geqslant T_2$, $0\leqslant W(t)\leqslant M$, then we have

$$W(t)\leqslant W(T_2)-(M-\varepsilon)\leqslant\varepsilon.$$

Hence, $\lim\limits_{t\to+\infty}W(t)=0$. That is, $\bar{x}(t)\in X$. According to Assumption 4, $\bar{x}(t)=\sum\limits_{i=1}^{N}p_ix_i(t)$ is the optimal state of all agents.

4 Numerical simulations and application

A simulation to verify the effectiveness of the theoretical results of this paper is given in this section. The motion of eight agents in two-dimensional space is simulated. In the algorithm (8), the function of communication delay is chosen as $\tau(t)=\dfrac{38+\sin t}{160}$ satisfying Assumption 1. The function of $\alpha(t)$ is chosen as $\alpha(t)=\dfrac{1}{t+1}$ satisfying Assumption 6. We assume the adjacency matrix $A=[a_{ij}]$ as follows:

$$A=\begin{bmatrix}0&0&1&1&0&0&1&0\\1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&0&1&0&1&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&0&1&0\end{bmatrix}.\tag{24}$$

The state of agent i is defined as $x_i(t)=[x_{i1}(t)\ x_{i2}(t)]^{\mathrm{T}}$ and the initial state is chosen as follows:

$$x_{ij}(0)=\begin{cases}-0.5\times i-0.1\times j, & i=1,2,3,4,\\ 0.5\times(i-4)+0.1\times j, & i=5,6,7,8,\end{cases}$$

where $j=1,2$. The local objective function of agent i $(i=1,2,3,4,5,6,7,8)$ is assumed as follows.

$$f_1(x_1)=\frac{1}{2}x_{11}^2+\frac{1}{2}x_{12}^2,$$
$$f_2(x_2)=\frac{1}{2}(x_{21}+1)^2+\frac{1}{2}x_{22}^2,$$

$$f_3(x_3) = \frac{1}{2}x_{31}^2 + \frac{1}{2}(x_{32}+1)^2,$$

$$f_4(x_4) = \frac{1}{2}(x_{41}+1)^2 + \frac{1}{2}(x_{42}+1)^2,$$

$$f_5(x_5) = \frac{1}{4}x_{51}^4 + \frac{1}{4}x_{52}^4,$$

$$f_6(x_6) = \frac{1}{4}(x_{61}+1)^4 + \frac{1}{4}x_{62}^4,$$

$$f_7(x_7) = \frac{1}{4}x_{71}^4 + \frac{1}{4}(x_{72}+1)^4,$$

$$f_8(x_8) = \frac{1}{4}(x_{81}+1)^4 + \frac{1}{4}(x_{82}+1)^4,$$

Obviously, each local objective function is continuously convex and differentiable. Therefore, the objective function $f(x) = \sum_{i=1}^{8} f_i(x_i)$ is also continuously convex and differentiable, and the optimal solution is unique. According to Lemma 4 and by using simple calculations, we can get the optimal value $x^* = [x_{i1}^* \ \ x_{i2}^*] = [-\frac{1}{2} \ \ -\frac{1}{2}]$ that minimizes the global objective function. Figure 1 shows the motion trajectories of 8 agents under the action of the controller (1). It is clear to see that 8 agents tend to the same position and the state of 8 agents converge to the optimal point.

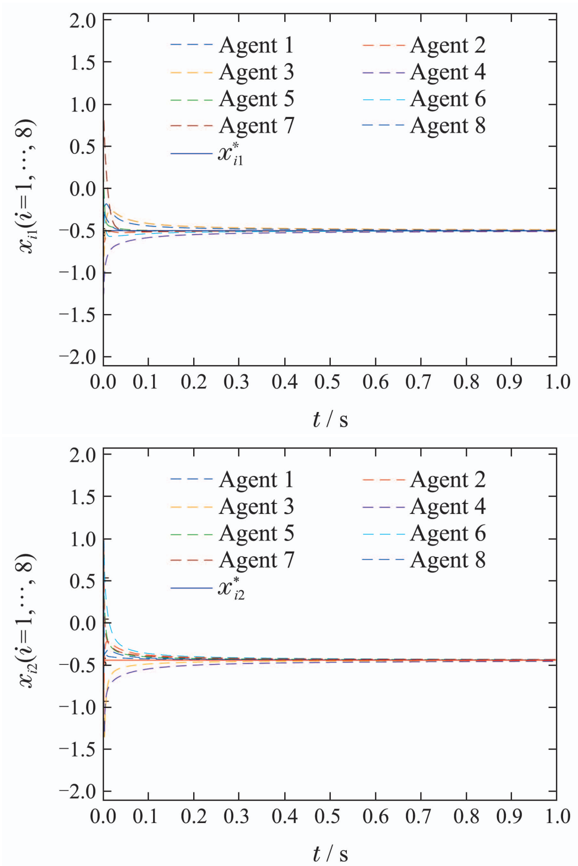

Fig. 1 The states of 8 agents

5 Conclusion

We study the optimization problem of a first-order system composed of multiple agents with communication delay and directed topology. The communication network does not require to be balanced. Each agent has states information from its neighborhood. The goal is to find the common state of all agents that minimizes the objective function $f(x)$. Based on the characteristics of P and the local communication of neighborhood, we propose a controller to do with the optimization problem. Under Assumptions 1–5 and from the convexity of $f_i(x_i)$, all agents reach consensus while minimizing the function $f(x)$. Finally, a simulation is given to illustrate the theoretical results of this article.

References:

[1] NEDIĆ A, OZDAGLAR A. Distributed sub-gradient methods for multi-agent optimization. *IEEE Transactions on Automatic Control*, 2009, 54(1): 48 – 61.

[2] SHI G, JOHANSSON K H, HONG Y. Reaching an optimal consensus: Dynamical systems that compute intersections of convex sets. *IEEE Transactions on Automatic Control*, 2013, 58(3): 610 – 622.

[3] LU J, TANG C Y. Zero-gradient-sum algorithms for distributed convex optimization: the continuous-time case. *IEEE Transactions on Automatic Control*, 2012, 57(9): 2348 – 2354.

[4] NING B, HAN L, ZUO Z. Distributed optimization for multiagent systems: an edge-based fixed-time consensus approach. *IEEE Transactions on Cybernetics*, 2019, 49(1): 122 – 132.

[5] NEDIĆ A, OZDAGLAR A. Distributed optimization over time-varying directed graphs. *IEEE Transactions on Automatic Control*, 2015, 60(3): 601 – 615.

[6] GHARESIFARD B, CORTÉS J. Distributed continuous-time convex optimization on weight-balanced digraphs. *IEEE Transactions on Automatic Control*, 2014, 59(3): 781 – 786.

[7] KIA S, CORTÉS J. Distributed convex optimization via continuous-time coordination algorithms with discrete-time communication. *Automatica*, 2015, 55(5): 254 – 264.

[8] LIN P, REN W, FARRELL A. Distributed continuous-time optimization: nonuniform gradient gains, finite-time convergence, and convex constraint set. *IEEE Transactions on Automatic Control*, 2016, 62(5): 2239 – 2253.

[9] CHEN W, REN W. Event-triggered zero-gradient-sum distributed consensus optimization over directed networks. *Automatica*, 2016, 65: 90 – 97.

[10] DENG Z, LIANG S, HONG Y. Distributed continuous-time algorithms for resource allocation problems over weight-balanced digraphs. *IEEE Transactions on Cybernetics*, 2018, 48(11): 3116 – 3125.

[11] LIN P, REN W, SONG Y. Distributed multi-agent optimization subject to nonidentical constraints and communication delays. *Automatica*, 2016, 65: 120 – 131.

[12] QIU Z, XIE L, HONG Y. Distributed optimal consensus of multiple double integrators under bounded velocity and acceleration. *Control Theory and Technology*, 2019, 17(1): 85 – 98.

[13] CHEN F, REN W. Sign projected gradient flow: a continuous-time approach to convex optimization with linear equality constraints. *Automatica*, 2020, 120: 109156.

[14] DING L, HAN Q L, X GE, et al. An overview of recent advances in event-triggered consensus of multiagent systems. *IEEE Transactions on Cybernetics*, 2018, 48(4): 1110 – 1123.

[15] HOU W, FU M, ZHANG H, et al. Consensus conditions for general second-order multi-agent systems with communication delay. *Automatica*, 2017, 75: 293 – 298.

[16] ZHU W, JIANG Z-P. Event-based leader-following consensus of multi-agent systems with input time dela. *IEEE Transactions on Automatic Control*, 2015, 60(5): 1362 – 1367.

[17] MA D, TIAN R, ZULFIQAR, et al. Bounds on delay consensus margin of second-order multi-agent systems with robust position and velocity feedback protocol. *IEEE Transactions on Automatic Control*, 2019, 64(9): 3780 – 3787.

[18] WANG H, LIAO X, HUANG T, et al. Cooperative distributed optimization in multi-agent networks with delays. *IEEE Transactions on Systems, Man and Cybernetics*, 2015, 45(2): 363 – 369.

[19] TSIANOS K I, RABBAT M G. Distributed dual averaging for convex optimization under communication delays. *American Control Conference*. Montreal: IEEE, 2012: 1067 – 1072.

[20] YANG S, LIU Q, J WANG. Distributed optimization based on a multiagent system in the presence of communication delays. *IEEE Transactions on Systems, Man and Cybernetics* , 2017, 47(5): 717 – 728.

[21] CHEN G, ZHAO Z. Delay effects on consensus-based distributed economic dispatch algorithm in microgrid. *IEEE Transactions on Power Systems*, 2018, 33(1): 602 – 612.

[22] YANG T, LU J, WU D, et al. A distributed algorithm for economic dispatch over time-varying directed networks with delays. *IEEE Transactions on Industrial Electronics*, 2017, 64(6): 5095 – 5106.

[23] WANG D, WANG Z, CHEN M, et al. Distributed optimization for multi-agent systems with constraints set and communication time-delay over a directed graph. *Information Sciences*, 2018, 438: 1 – 14.

[24] GUO Z, CHEN G. Distributed zero-gradient-sum algorithm for convex optimization with time-varying communication delays and switching networks. *International Journal of Robust and Nonlinear Control*, 2018, 28: 4900 – 4915.

[25] YANG Z, ZHANG Q, CHEN Z. Adaptive distributed convex optimization for multi-agent and its application in flocking behavior. *Journal of the Franklin Institute*, 2018, 356: 1038 – 1050.

[26] MOON Y S, PARK P, KWON W H, et al. Delay dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, 2010, 74(14): 1447 – 1455.

[27] HORN R A, JOHSON C R. *Matrix Analysis*. Cambridge, UK: Cambridge University Press, 1985.

[28] LI Z, DUAN Z. *Cooperative Control of Multi-Agent Systems: A Consensus Region Approach*. New York : CRC Press, 2014.

[29] BOYD S, VANDENBERGHE L. *Convex Optimization*. New York, NY: Cambridge University Press, 2004.

作者简介:

杨正全 副教授, 目前研究方向为多智能体系统控制与分布式优化, E-mail: zquanyang@163.com;

潘小芳 硕士研究生, 目前研究方向为多智能体系统控制与分布式优化, E-mail: 121906322@qq.com;

张 青 教授, 目前研究方向为复杂网络和多智能体系统, E-mail: qz120168@hotmail.com;

陈增强 博士生导师, 目前研究方向为多智能体系统控制、混沌系统与复杂网络, E-mail: chenzq@nankai.edu.cn.