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一类混杂系统的最优控制

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摘要: 研究了一类脉冲依赖于状态的混杂系统的最优控制问题. 与传统的变分方法不同, 通过将跳跃瞬间转化为一个新的待优化参数, 得到了该混杂系统的必要最优性条件, 从而将最优控制问题转化为一边界值问题, 该边界值问题可由数值方法或解析方法解决. 此外, 利用广义微分的理论, 将该必要最优性条件推广到Frechet微分形式. 结论表明, 在混杂动态系统运行的连续部分, 最优解所满足的必要性条件和传统的连续系统相同. 在混杂动态系统的脉冲点处, 哈密尔顿函数满足连续性条件, 协态变量则满足一定的跳跃条件. 最后, 通过两个实例分析, 表明该方法是有效的.

关键词: 最优控制; 混杂系统; 必要最优性条件; Frechet微分

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Optimal control of a class of hybrid systems

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Abstract: An optimal control problem is investigated for a class of hybrid systems, where the impulsive instants are state-dependent. Instead of relying on the usual technique of variational approach, necessary optimality conditions of this hybrid system are obtained by parameterizing the impulsive instants. Then, the optimal control problem is transformed to a boundary value problem, which can be solved by numerical method or analytic method. Moreover, taking advantage of the theory of generalized differential, necessary optimality conditions are extended to Frechet differential form. It is shown that, at the continuous part of this hybrid dynamic system, the necessary optimality conditions have the same form as traditional continuous system. At the impulsive points of this system, the Hamiltonian function is continuous and the adjoint variable satisfies certain condition. Finally, two examples are presented to illustrate validity of the methods.

Key words: optimal control; hybrid system; necessary optimality condition; Frechet differential

1 Introduction

Over the last few decades, there has been much research on a special class of complex systems, which are called hybrid systems. Hybrid systems can be characterized by a combination of continuous-valued and discrete-valued variables. In many cases, it is not only desirable but also natural to use hybrid models to describe the dynamical behavior of the systems. For example, modeling a car with four gears. Therefore, hybrid models have been used extensively in many application fields, such as the field of behavior-based robotics, multi-agent network control systems, chemical processes, manufacturing systems and electrical circuit systems, etc. In [1], Michael Branicky et al. provided a unified framework to model and control hybrid systems. They observe four phenomena that occur in a real-world system, including autonomous and controlled switching of the state variables. More examples, theory and applications of hybrid systems are introduced in [2]. See [3] and references therein for a recent survey on hybrid systems.

In recent years, the issues of optimal control in hybrid systems have attracted increasing interest in both theoretical research and practical applications. Various methods emerged to find optimal solutions for hybrid systems, such as the maximum principle[4], the viscosity solution technique[5], some numerical optimal algorithms[6–8], the embedding approach[9] and the method of smoothed approximation[10].

In particular, special attention has been focused on necessary optimality conditions for hybrid systems because of its theoretical and practical value. In general, it is easy to get the necessary conditions for continuous dynamics subsystems of hybrid problem. The difficult and challenging part is the necessary conditions at the switching times. Many researchers have done a lot of work in this aspect. Following the direct techniques of variations, nec-

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essary conditions of optimality are established for a class of impulsive hybrid systems[11] and the switching control system with nonsmooth cost functionals[12–13]. By employing such direct techniques or indirect methods, different kinds of necessary conditions are given to different hybrid problems in [14–21]. Most of these papers concern optimal control of hybrid systems with time-driven dynamics.

However, in many real situations, some transition of hybrid systems depends on event instead of time. For example, in the thermostats, the system will change its state when the temperature attains some values. Similarly, in the design of robot, the robot may be required to change its state when it reaches some special state. In both cases, the transitions are event-driven.

In this paper, we consider necessary optimality conditions for a class of impulsive hybrid systems, in which the transitions depend on the state. By introducing a new time variable as the method of [16–17, 21], we get necessary optimality conditions for this problem. Moreover, we extend such necessary optimality conditions to the case with nonsmooth cost functional. Usually, the advantage of hybrid system lies in its nonsmooth trajectory. Therefore, the use of nonsmooth objective functional can reflect this advantage better.

The rest of the paper is organized as follows. In Section 2, some knowledge of nonsmooth analysis are given. The problem is formulated in Section 3. Section 4 is devoted to the statement of the necessary optimality conditions. In Section 5, nonsmooth form of necessary optimality conditions is presented. The conclusion is drawn in Section 6.

2 Basic knowledge of nonsmooth analysis

In this section, some knowledge of Frechet differential is introduced. For more information, the reader can see [12–13] and [22–23].

If φ is lower semicontinuous at x, the Frechet subdifferential of φ at x is defined by

$$\hat{\partial}\varphi(x) := \{x^* \in \mathbb{R}^n | \liminf_{y\to x} \frac{\varphi(y)-\varphi(x)-< x^*, y-x >}{\|y-x\|} \geqslant 0\}.$$

Similarly, If φ is supper semicontinuous at x, the Frechet supperdifferential of φ at x is defined by

$$\hat{\partial}^+\varphi(x) := \{x^* \in \mathbb{R}^n | \limsup_{y\to x} \frac{\varphi(y)-\varphi(x)-< x^*, y-x >}{\|y-x\|} \leqslant 0\}.$$

From the above definitions, we have

$$\hat{\partial}^+\varphi(x) = -\hat{\partial}(-\varphi)(x).$$

If φ is continuous, the Frechet differential of φ at x is defined by

$$\hat{\partial}^+\varphi(x) := \{x^* \in \mathbb{R}^n | \lim_{y\to x} \frac{\varphi(y)-\varphi(x)-< x^*, y-x >}{\|y-x\|} = 0\}.$$

From the above definitions, we can get the following relationships among Frechet subdifferential, Frechet supperdifferential and Frechet differential.

Proposition 1 Let $\varphi : X \to \mathbb{R}$ with $|\varphi(x)| < \infty$. Then $\hat{\partial}\varphi(x) \neq \varnothing$ and $\hat{\partial}^+\varphi(x) \neq \varnothing$ if and only if φ is Frechet differentiable at x, where

$$\hat{\partial}\varphi(x) = \hat{\partial}^+\varphi(x) = \nabla\varphi(x).$$

When $\hat{\partial}\varphi(x)$ is a singleton, φ may not be Frechet differentiable at x. For example, let $\varphi(x) = \max\{0, x\sin(1/x)\}$ if $x \neq 0$ with $\varphi(0) = 0$, then $\hat{\partial}\varphi(0) = 0$ and $\hat{\partial}^+\varphi(0) = \varnothing$.

3 Problem statement

Let us consider the following control system:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), & h(x(t)) \neq 0, \\ x(t) \mapsto g(x(t), u(t)), & h(x(t)) = 0, \\ r(x(a), x(b)) = 0, \end{cases} \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ are twice continuously differentiable with their variables.

Suppose that $h(x(t)) = 0$ have a finite number of isolated roots $t_j, j = 1, \cdots, N-1$ and

$$a < t_1 < \cdots < t_{N-1} < b.$$

Now we give the following definitions as [24].

Definition 1 (Hybrid time set) A hybrid time set $\tau = \{I_i\}_{i=1}^N$ of (1) is a finite sequence of intervals of the real line, such that $I_i = [t_{i-1}, t_i]$, $i = 1, \cdots, N$.

Definition 2 (Run) A run of (1) is a collection (τ, x, u) with

$$\tau = \{I_i\}_{i=1}^N,\ x = \{y_i(\cdot)\}_{i=1}^N,\ u = \{v_i(\cdot)\}_{i=1}^N,$$

that satisfies:

Continuous evolution: for $h(x(t)) \neq 0$, $y_i(\cdot)$ is the solution to differential $\dot{y}(t) = f(y_i(t), v_i(t))$.

Discrete evolution: for

$$h(x(t_i)) = 0,\ y_{i+1}(t_i) = g(y_i(t_i), v_i(t_i)).$$

An admissible tuple (τ, x, u) for (1) is a process satisfying the constraint of (1).

Now we give the following optimal control problem (P): Minimize the functional

$$J = \sum_{i=1}^N \varphi_i(x(t_i)) + \int_a^b L(x(t), u(t))\mathrm{d}t \tag{2}$$

over the set of all admissible pairs (τ, x, u), where φ, L are twice continuously differentiable with their variables.

An admissible pair (τ^0, x^0, u^0) is called a weak local minimum point for problem (P) if for some $\epsilon > 0$, (τ^0, x^0, u^0) minimizes (2) over all admissible processes (τ, x, u) satisfying

$$|t_i - t_i^0| \leqslant \epsilon,\ i = 1, \cdots, N-1,\ \|y_i - y_i^0\|_\infty \leqslant \epsilon,$$
$$\|v_i - v_i^0\|_\infty \leqslant \epsilon,\ i = 1, \cdots, N.$$

4 Necessary optimality conditions

At the beginning of this section, we give one convention which will be used in this section. Let $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ be a C^1 function, then

$$D_{(x(a),x(b))}r(x(a), x(b)) =$$

$$\begin{pmatrix} \dfrac{\partial r_1}{\partial x_1(a)} & \cdots & \dfrac{\partial r_1}{\partial x_n(a)} & \dfrac{\partial r_1}{\partial x_1(b)} & \cdots & \dfrac{\partial r_1}{\partial x_n(b)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial r_l}{\partial x_1(a)} & \cdots & \dfrac{\partial r_l}{\partial x_n(a)} & \dfrac{\partial r_l}{\partial x_1(b)} & \cdots & \dfrac{\partial r_l}{\partial x_n(b)} \end{pmatrix}$$

denotes the Jacobian of function r.

Suppose the state of (1) is left continuous at the switching points t_i, $i=1,\cdots,N-1$, that is

$$x(t_i)=\lim_{t\to t_i^-}x(t).$$

Let

$$x(t_{i+1})=\lim_{t\to t_i^+}x(t),\ t_0=a,\ t_N=b,$$

according to the discussion of Section 3, problem (P) can be written as the following problem (P1): Minimize

$$\sum_{i=1}^{N}\varphi_i(y_i(t_i))+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}L(y_i(t),v_i(t))\mathrm{d}t,$$

subject to

$$\begin{cases}\dot{y}_i(t)=f(y_i(t),v_i(t)),\ t\in[t_{i-1},t_i],\\ \qquad i=1,\cdots,N,\\ y_{i+1}(t_i)=g(y_i(t_i),v_i(t_i)),\ i=1,\cdots,N-1,\\ h(y_i(t_i))=0,\ i=1,\cdots,N-1,\\ r(x(a),x(b))=0.\end{cases}$$

For $i=1,\cdots,N$, introduce a new time variable $s\in[0,1]$ with $t=t_{i-1}+s(t_i-t_{i-1})$ and define

$$y_i(s)=y(t_{i-1}+s(t_i-t_{i-1})),$$
$$v_i(s)=v(t_{i-1}+s(t_i-t_{i-1})).$$

Then problem (P1) can be reformulated as the following equivalent problem (P2): Minimize

$$\sum_{i=1}^{N}\varphi_i(y_i(1))+\sum_{i=1}^{N}\int_0^1[(t_i(s)-t_{i-1}(s))L(y_i(s),v_i(s))]\mathrm{d}s,$$

subject to

$$\begin{cases}\dot{y}_i(s)=(t_i(s)-t_{i-1}(s))f(y_i(s),v_i(s)),\\ \qquad i=1,\cdots,N,\\ \dot{t}_i(s)=0,\ i=1,\cdots,N-1,\\ y_{i+1}(0)=g(y_i(1),v_i(1)),\ i=1,\cdots,N-1,\\ h(y_i(1))=0,\ i=1,\cdots,N-1,\\ r(y_1(0),y_N(1))=0.\end{cases}$$

Now we give a necessary condition for problem (P).

Theorem 1 Let (τ^0,x^0,u^0) be a weak local minimum point of problem (P), the rank of the Jocabian $D_{(x(a),x(b))}r(x^0(a),x^0(b))$ is l, then there exist a piecewise continuous differential variable $\lambda(t):[a,b]\to\mathbb{R}^n$, multipliers $\mu\in\mathbb{R}^l$, $\omega_i\in\mathbb{R}^n$, $\xi_i\in\mathbb{R}$, $i=1,\cdots,N-1$, and $\lambda_0\in\mathbb{R}$, such that for $H(x,u,\lambda)=L(x,u)+\lambda^{\mathrm{T}}f(x,u)$, the following equations hold

$$\lambda(a)=-D_{x(a)}[\mu^{\mathrm{T}}r(x^0(a),x^0(b))],$$
$$\lambda(b)=D_{x(b)}[\lambda_0\varphi(x^0(b))+\mu^{\mathrm{T}}r(x^0(a),x^0(b))],\tag{3}$$
$$\dot{\lambda}(t)=-\frac{\partial H(x^0(t),u^0(t),\lambda(t))}{\partial x},\ t\in[a,b],\tag{4}$$
$$\frac{\partial H(x^0(t),u^0(t),\lambda(t))}{\partial u}=0,\ t\in[a,b].\tag{5}$$

At the switching instants, the following conditions are satisfied

$$H[t_i^{0+}]=H[t_i^{0-}],\tag{6}$$
$$\lambda(t_i^{0-})=\lambda_0\frac{\partial[\varphi_i(x^0(t_i^0))]}{\partial x(t_i)}+\xi_i\frac{\partial[h(x^0(t_i^0))]}{\partial x(t_i)}+\frac{\partial[\lambda^{\mathrm{T}}(t_i^{0+})g(x^0(t_i^0),u^0(t_i^0))]}{\partial x(t_i)},\tag{7}$$

where $i=1,\cdots,N-1$.

Proof Let (τ^0,x^0,u^0) be a weak local minimum of Problem (P), then it is a solution to Problem (P2). Applying classical necessary conditions to Problem (P2), there exist $\lambda_i(s)\in\mathbb{R}^n$, $i=1,\cdots,N-1$, multipliers $\mu\in\mathbb{R}^l$, $\xi_i\in\mathbb{R}$, $\omega_i\in\mathbb{R}^n$, $i=1,\cdots,N-1$, and $\lambda_0\in\mathbb{R}$, such that for

$$\hat{H}(x,u,\lambda)=\sum_{i=1}^{N}\{(t_i-t_{i-1})[L(y_i,v_i)+\lambda_i^{\mathrm{T}}f(y_i,v_i)]\}=\sum_{i=1}^{N}[(t_i-t_{i-1})H_i],$$

$$\phi(x,u)=\lambda_0\sum_{i=1}^{N}\varphi_i(y_i(1))+\mu^{\mathrm{T}}r(y_1(0),y_N(1))+\sum_{i=1}^{N-1}\xi_ih(y_i(1))+\sum_{i=1}^{N-1}\omega_i^{\mathrm{T}}(y_{i+1}(0)-g(y_i(1),v_i(1))),$$

we have

$$\dot{\lambda}_i^*(s)=-\frac{\partial\hat{H}(x^0,u^0,\lambda_i)}{\partial t_i},\ i=1,\cdots,N-1,\tag{8}$$
$$\lambda_i^*(0)=-\frac{\partial\phi(x^0,u^0)}{\partial t_i(0)}=0,\ i=1,\cdots,N-1,\tag{9}$$
$$\lambda_i^*(1)=\frac{\partial\phi(x^0,u^0)}{\partial t_i(1)}=0,\ i=1,\cdots,N-1.\tag{10}$$

Because the Hamiltonian is constant along the optimal trajectory, the right hand of (8) is constant on $[0,1]$. Thus by equations (8)–(10), we get the continuity condition (6).

On the other hand, we have

$$\dot{\lambda}_i(s)=-\frac{\partial\hat{H}(x^0,u^0,\lambda_i)}{\partial y_i}=-(t_i-t_{i-1})\frac{\partial H_i(y_i^0,v_i^0,\lambda_i)}{\partial y_i},\tag{11}$$
$$\lambda_1(0)=-\frac{\partial\phi(x^0,u^0)}{\partial y_1(0)}=-D_{y_1(0)}[\mu^{\mathrm{T}}r(y_1^0(0),y_N^0(1))],\tag{12}$$
$$\lambda_N(1)=\frac{\partial\phi(x^0,u^0)}{\partial y_N(1)}=D_{y_N(1)}[\lambda_0\varphi(y_N^0(1))+\mu^{\mathrm{T}}r(y_1^0(0),y_N^0(1))],\tag{13}$$
$$\lambda_{i+1}(0)=-\frac{\partial\phi(x^0,u^0)}{\partial y_{i+1}(0)}=-\omega_i,\tag{14}$$
$$\lambda_i(1)=\frac{\partial\phi(x^0,u^0)}{\partial y_i(1)}=$$

$$\lambda_0\frac{\partial\phi_i(y_i(1))}{\partial y_i(1)}+\xi_i\frac{\partial h(y_i(1))}{\partial y_i(1)}-\frac{\partial[\omega_i^{\mathrm{T}}g(y_i(1),v_i(1))]}{\partial y_i(1)}, \tag{15}$$

$$\frac{\partial \hat{H}_u(x^0(s),u^0(s)),\lambda(s)}{\partial u}=0, \tag{16}$$

where $i=1,\cdots,N-1$,.

Recombine the adjoint variable

$$\lambda(t)=\lambda_i(\frac{t-t_{i-1}^0}{t_i^0-t_{i-1}^0}),t\in[t_{i-1}^0,t_i^0],\ i=1,\cdots,N,$$

the state variables and control function accordingly, then equations (12) and (13) lead to the natural boundary conditions (3). Equations (14) and (15) yield in the jump condition (7). The adjoint (4) and the minimum condition (5) come from equations (11) and (16), respectively. This completes the proof of the theorem.

Now we consider the following system, which extend problem (P) from one switching function to several functions:

$$\begin{cases}\dot{x}(t)=f(x(t),u(t)),\ \ \prod\limits_{j=1}^{M}h_j(x(t))\neq 0,\\ x(t)\mapsto g_j(x(t),u(t)),h_j(x(t))=0,\\ r(x(a),x(b))=0,\end{cases} \tag{17}$$

where t_i are the roots of

$$\prod_{j=1}^{M}h_j(x(t))=0,\ j=1,\cdots,M.$$

If $h_j(x(t))=0$ have the same root for

$$j\in\{i_1,i_2,\cdots,i_k\},\ i_1<i_2<\cdots<i_k,$$

then a rule can be set to choose the minimal index and the state of $(\hat{P})$ jumps as the requirement of $h_{i_1}(x(t_i))=0$.

Now we give the following problem $(\hat{P})$: Minimize the functional

$$J=\sum_{i=1}^{N}\varphi_i(x(t_i))+\int_a^b L(x(t),u(t))\mathrm{d}t,$$

subject to (17).

Similar to the proof of Theorem 1, we can prove the following theorem for $(\hat{P})$.

Theorem 2 Let (τ^0,x^0,u^0) be a weak local minimum point of problem $(\hat{P})$, and if all the conditions in Theorem 1 are met, then the results in Theorem 1 hold except the jumping condition (7) replaced by

$$\lambda(t_i^{0-})=\lambda_0\frac{\partial[\phi_i(x^0(t_i^0))]}{\partial x(t_i)}+\xi_i\frac{\partial[h_j(x^0(t_i^0))]}{\partial x(t_i)}\times \frac{\partial[\lambda^{\mathrm{T}}(t_i^{0+})g_j(x^0(t_i^0),u^0(t_i^0))]}{\partial x(t_i)},$$

where t_i are the roots of

$$\prod_{j=1}^{M}h_j(x(t))=0,\ i=1,\cdots,N-1.$$

Remark 1 The method used in proving Theorem 2 is similar to the methods in [16–17,21]. However, our problem is different from theirs. First, the problems in [17,21] are time-driven, but our problem is event-driven. Second, the trajectory of the state in this paper is discontinuous, while the track of state is continuous in [16]. Third, as stated in Theorem 2, the state space can be divided into a number of different parts by several different switching functions. The model used in [16] can only divide the state space into two parts as the switching function becomes positive from negative.

Example 1 We consider a mobile robot navigation problem. The task of the robot is to reach a goal point from a fixed starting point. Let

$$\begin{cases}\dot{x}_1(t)=2u_1(t),\\ \dot{x}_2(t)=u_2(t),\end{cases}\ x_1(t)+2x_2(t)\neq 3, \tag{18}$$

$$\begin{cases}x_1(t)\mapsto 2x_1(t),\\ x_2(t)\mapsto 2x_2(t),\end{cases}\ x_1(t)+2x_2(t)=3, \tag{19}$$

with boundary condition

$$x_1(0)=0,\ x_2(0)=0,\ x_1(2)=4,\ x_2(2)=3, \tag{20}$$

where $(x_1(t),x_2(t))$ is the position of the robot in $\mathbb{R}^2$, $u_1(t)$, $u_2(t)$ represent the speed of robot along the x-axis and y-axis. The meaning of (19) is that the robot will occurs a jump in state when it encounters some obstacle. Our aim is to find the optimal jump points t_i, $i=1,\cdots,N-1$, and the control variables $u_1(t)$, $u_2(t)$, such that the cost functional

$$\sum_{i=1}^{N-1}x_i^2(t_i)+\int_0^2(\frac{1}{2}u_1^2+\frac{1}{2}u_2^2)\mathrm{d}t$$

is minimized during these processes.

For simplicity of calculation, we suppose the switching function $x_1(t)+2x_2(t)=3$ has one root in $[0,2]$.

Let

$$H=\frac{1}{2}u_1^2+\frac{1}{2}u_2^2+2\lambda_1u_1+\lambda_2u_2.$$

By the continuity condition (6), we have

$$4(\lambda_1^2(t^-)-\lambda_1^2(t^+))+\lambda_2^2(t^-)-\lambda_2^2(t^+)=0. \tag{21}$$

By the jumping condition (7), we get that

$$\begin{cases}\lambda_1(t^-)=2x_1(t)+\xi+2\lambda_1(t^+),\\ \lambda_2(t^-)=2x_2(t)+2\xi+2\lambda_2(t^+),\end{cases} \tag{22}$$

combining equations (18)–(21) with equations (4)–(5), we can obtain six nonlinear equations with six unknown quantities after some calculation. By Newton iterative methods of solving nonlinear equations and MATLAB procedure, we get that the system jumps at $t=0.8864\,\mathrm{s}$ and the control variables are

$$u_1(t)=\begin{cases}0.5439,\ \ t\in[0,0.8864]\,\mathrm{s},\\ 0.9299,\ \ t\in(0.8864,2]\,\mathrm{s},\end{cases}$$

$$u_2(t)=\begin{cases}1.1484,\ \ t\in[0,0.8864]\,\mathrm{s},\\ 0.8658,\ \ t\in(0.8864,2]\,\mathrm{s}.\end{cases}$$

The behavior of control, adjoint and state variable is shown in Fig.1.

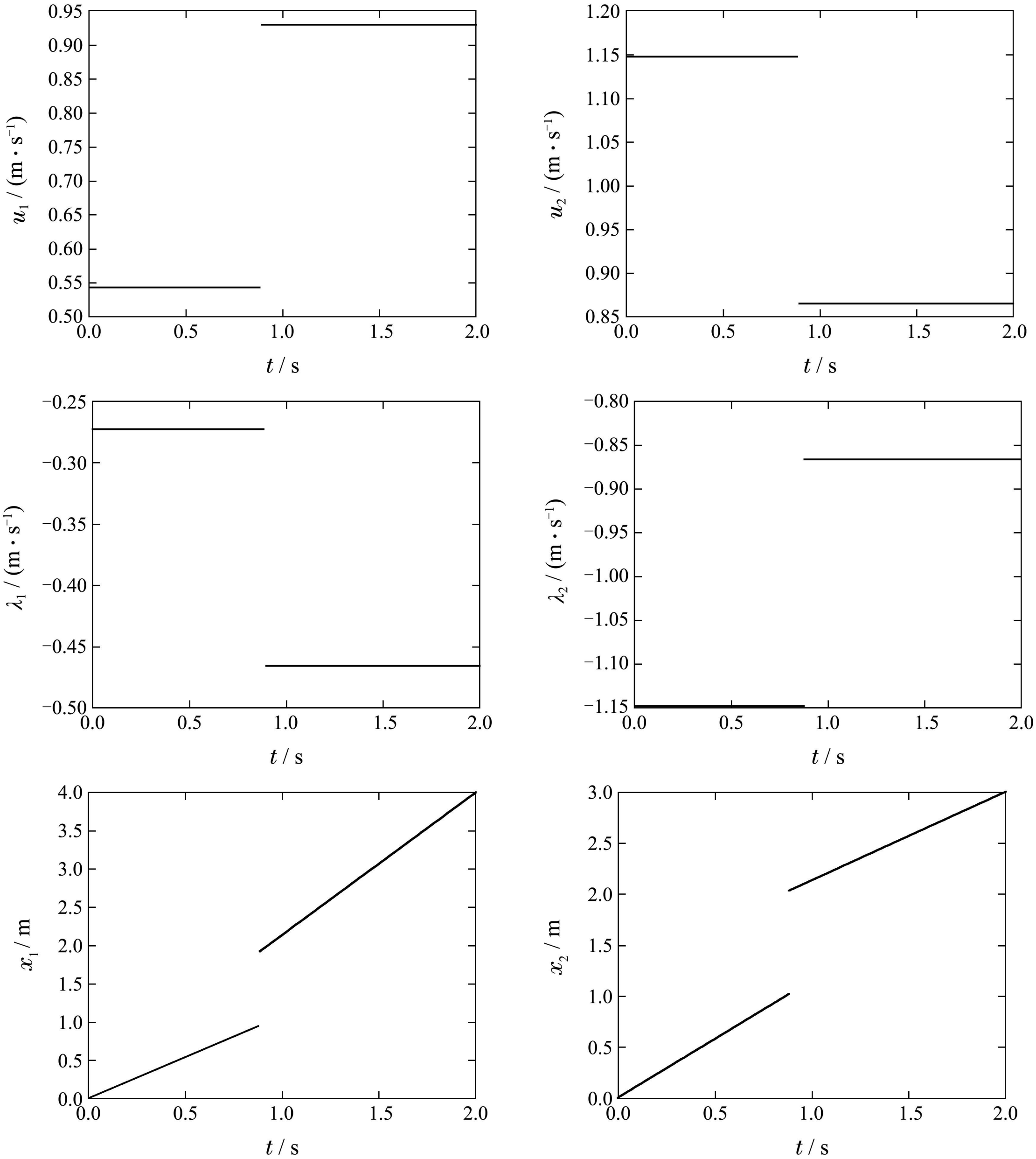

Fig. 1 Control, adjoint and state variable in Example 1

5 Generalized differential form of necessary optimality conditions

We start with a lemma about Frechet differential, which can be found in [12,23].

Lemma 1 Let $\varphi : X \to \mathbb{R}$, $|\varphi(\bar{x})| < \infty$. Then for any $x^* \in \hat{\partial}\varphi(x)$, there exists a function $s : X \to \mathbb{R}$ with $s(\bar{x}) = \varphi(\bar{x})$, $s(x) \leqslant \varphi(x)$ whenever $x \in X$, such that $s(\cdot)$ is Frechet differentiable at $\bar{x}$ with $\nabla s(\bar{x}) = x^*$.

Now we give the supperdifferential form of necessary optimality conditions by the method of [13].

Theorem 3 Let (τ^0, x^0, u^0) be a weak local minimum point of problem (P), φ_i is Frechet supperdifferentiable at y_i^0, then for every $x^*(t_i) \in \hat{\partial}^+\varphi_i(x^0(t_i))$, $i = 1, 2, \cdots, N$, the result of Theorem 1 hold except that (3) and (7) replaced by

$$\begin{cases} \lambda(a) = -D_{x(a)}[\mu^{\mathrm{T}} r(x^0(a), x^0(b))], \\ \lambda(b) = \lambda_0 x^*(b) + D_{x(b)}\mu^{\mathrm{T}} r(x^0(a), x^0(b)), \end{cases} \tag{23}$$

$$\lambda(t_i^{0-}) = \lambda_0 x^*(t_i^0) + \xi_i \frac{\partial[h(x^0(t_i^0))]}{\partial x} + \frac{\partial[\lambda^{\mathrm{T}}(t_i^{0+}) g(x^0(t_i^0), u^0(t_i^0))]}{\partial x(t_i)}, \quad i = 1, \cdots, N-1. \tag{24}$$

Proof For any $x^*(t_i^0) \in \hat{\partial}^+\varphi_i(x^0(t_i))$, using Lemma 1 to $-x^*(t_i^0)$, there exists s_i satisfying

$$s_i(x^0(t_i^0)) = \varphi_i(x^0(t_i^0)), \ s_i(x(t_i)) = \varphi_i(x(t_i))$$

in some neighborhood of $s_i(x^0(t_i^0))$. Moreover, s_i is Frechet differentiable at $x^0(t_i^0)$ with

$$\nabla s_i(x^0(t_i^0)) = x^*(t_i^0), \ i = 1, \cdots, N.$$

Therefore (τ^0, x^0, u^0) is a weak local minimum of problem (P3): Minimize the functional

$$J = \sum_{i=1}^{N} s_i(x(t_i)) + \int_a^b L(x(t), u(t))\mathrm{d}t$$

over the set of all admissible pairs (τ, x, u). Combining the results of Theorem 2 and Lemma 1, we complete the proof of the theorem.

Example 2 Minimize the cost functional

$$J = \sum_{i=1}^{N}[-|x_1(t_i)|] + \int_0^2 (\frac{1}{2}u_1^2 + \frac{1}{2}u_2^2 - x_1(t) - 3tx_2(t))\mathrm{d}t$$

subject to equations (18)–(20).

As in Example 1, we also suppose the switching function has only one root in $[0, 2]$. Using the jumping condition (24), we derive that

$$\begin{cases} \lambda_1(t^-) = d + \xi + 2\lambda_1(t^+), \\ \lambda_2(t^-) = 2\xi + 2\lambda_2(t^+), \end{cases} \tag{25}$$

where $d \in [-1, 1]$ is Frechet supperdifferential at the switching point.

Taking advantage of similar method in Example 1, we get that the feasible switching time belongs to the set $[0.2469, 0.2475]$. Comparing the values of the cost functional in this feasible set, we obtain that the optimal switching time is $t = 0.2475\,\mathrm{s}$ and the control variables are

$$u_1(t) = \begin{cases} -2t - 1.813, & t \in [0, 0.2745]\,\mathrm{s}, \\ -2t + 2.944, & t \in (0.2745, 2]\,\mathrm{s}, \end{cases}$$

$$u_2(t) = \begin{cases} -1.5t^2 + 3.9641, & t \in [0, 0.2745]\,\mathrm{s}, \\ -1.5t^2 + 2.8016, & t \in (0.2745, 2]\,\mathrm{s}, \end{cases}$$

The behavior of control, adjoint and state variable is shown in Fig.2.

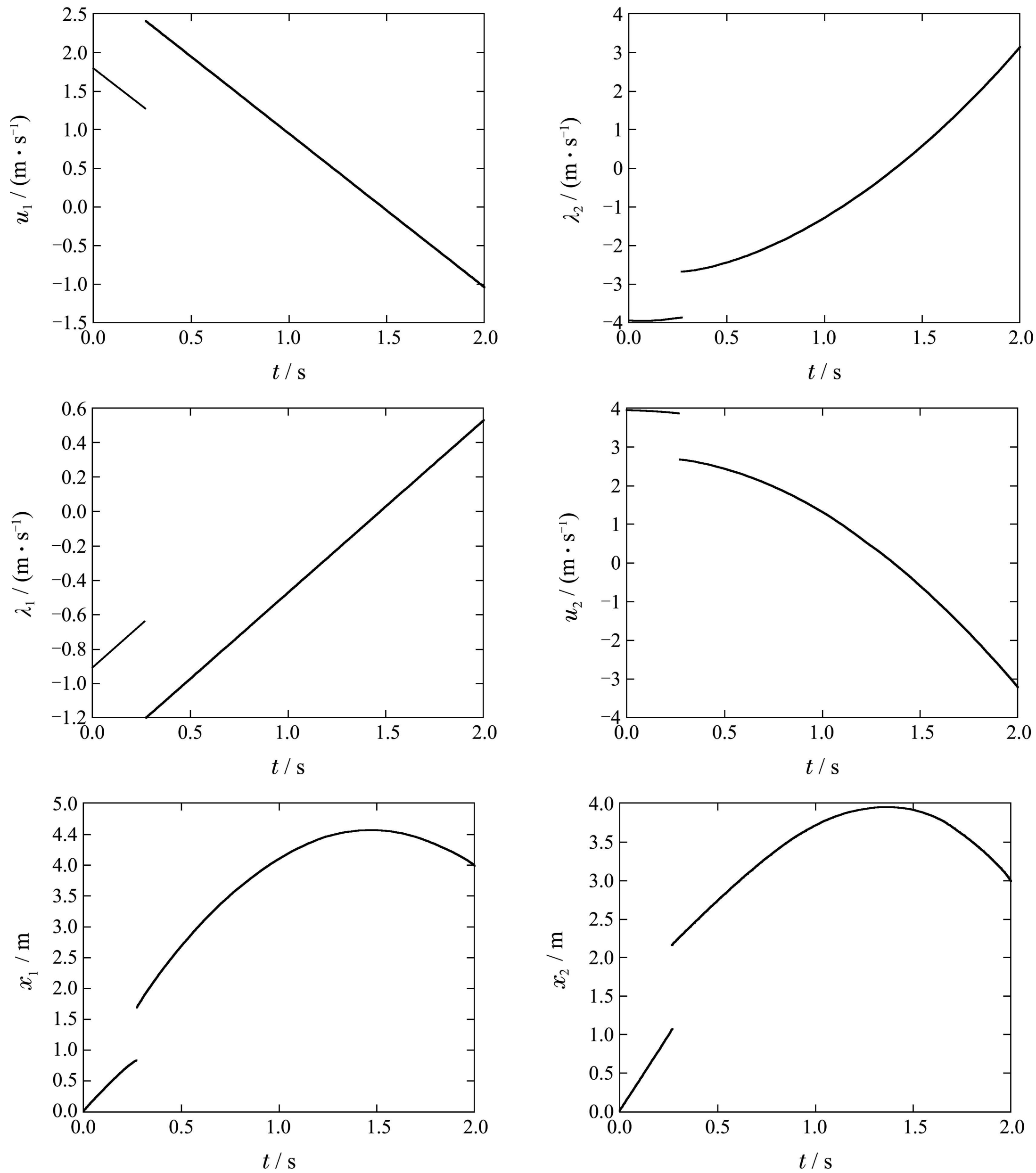

Fig. 2 Control, adjoint and state variable in Example 2

Remark 2 The cost functional in Example 2 is nonsmooth, so we cann't use Theorem 1 to get the optimal switching point. The necessary optimality condition of Frechet differential form is a good way to solve this problem.

6 Conclusions

This paper has addressed optimal control problems for a class of impulsive hybrid systems, where the transitions are state-driven. By parameterizing the switching instants and reducing all the state and control variables to a common fixed time interval $[0, 1]$, we obtain the necessary optimality condition for this hybrid system. At last, using the theory of Frechet subdifferential, we extend the necessary optimality conditions to the nonsmooth case.

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