第 30 卷第 6 期
2013 年 6 月
控 制 理 论 与 应 用
Control Theory & Applications
Vol. 30 No. 6
Jun. 2013

DOI: 10.7641/CTA.2013.12020

具有结构不确定性的时滞系统的闭环鲁棒预测控制器

郑鹏远[1,2,3], 席裕庚[1,2], 李德伟[1,2†]

(1. 上海交通大学 自动化系, 上海 200240;

2. 系统控制与信息处理教育部重点实验室, 上海 200240; 3. 上海交通大学 密西根学院, 上海 200240)

摘要: 针对具有结构不确定性的时滞系统, 设计了闭环鲁棒预测控制算法. 该控制算法基于控制不变集方法, 通过采用双模控制和闭环控制策略, 增加了控制设计的自由度, 进而扩大了系统的初始可行域并能获得较优的控制性能. 仿真结果验证了该方法的有效性.

关键词: 鲁棒预测控制; 时滞; 结构不确定性; 双模控制; 控制不变集; 线性矩阵不等式

中图分类号: TP273　　　**文献标识码**: A

Closed-loop robust model predictive control for time-delay systems with structured uncertainties

ZHENG Peng-yuan[1,2,3], XI Yu-geng[1,2] , LI De-wei[1,2†]

(1. Department of Automation, Shanghai Jiao Tong University, Shanghai 200240, China;

2. Key Laboratory of System Control and Information Processing, Ministry of Education of China, Shanghai 200240, China;

3. University of Michigan-Shanghai Jiao Tong University Joint Institute, Shanghai 200240, China)

Abstract: In the controller design, we adopt both the dual-mode framework and the closed-loop strategy to augment the degrees of freedom. On the basis of the control invariant set, the closed-loop robust model predictive control approach is developed. The proposed approach enlarges the region of attraction and achieves good control performance. The effectiveness of the proposed approach is validated by a simulation.

Key words: robust model predictive control; time-delay; structured uncertainty; dual-mode; control invariant set; linear matrix inequalities

1 Introduction

Over the last decade, robust model predictive control (MPC) has attracted much attention and developed greatly. The uncertain systems with polytopic uncertainty or structured uncertainty are two familiar categories of uncertain systems. Many past works study the MPC problem of these two categories of systems, such as [1–5]. For polytopic systems, [1] proposes an efficient approach for robust MPC using control invariant sets and linear matrix inequalities (LMIs). [2] and [3] improve the technique in [1] and achieve better control performance by applying parameter dependent Lyapunov function. [4] and [5] develop robust constrained MPC algorithms which design a sequence of explicit control laws corresponding to a sequence of asymptotically stable invariant ellipsoids constructed off-line one within another in state space. For structured uncertainty, [1] and [6] also propose robust MPC design by using control invariant sets and LMIs. [7] develops a closed-loop approach for arbitrary control horizons N. [8] modifies parameter uncertainties into structured uncertainties and develops a robust MPC algorithm accordingly.

As is well known, time-delay often exists in many industrial systems and limits the control performance and robustness. For polytopic uncertainty, [1] suggests to extend the algorithm for non-delayed systems to time-delay systems. [9] presents an algorithm for polytopic uncertain systems with only one state delay. [10] and [11] put the idea in [1] into detail and develop the robust MPC approaches for time-delay systems with polytopic uncertainty. And, [12] extends the feedback MPC strategy to the time-delay systems with polytopic uncertainty. But up to now, the problem of robust MPC for time-delay systems with structured uncertainty has been rarely addressed in the literatures.

Recently, [13] proposes robust MPC based on control invariant set and design the unique state feedback control to deal with time-delay systems with structured uncertainty. Although [13] could stabilize the robust uncertain system and optimize the control performance, it leads to conservativeness due to the limited degree of

Received 14 February 2012; revised 3 April 2012.
†Corresponding author. E-mail: dwli@sjtu.edu.cn.
This work is supported by the National Natural Science Foundation of China (Nos. 60934007, 61074060, 61104078).

freedom caused by the unique state feedback approach. As a result, the region of attraction is relatively small and the control performance is somewhat poor. In order to reduce the conservativeness of unique state feedback control approach, the dual-mode control in [14] is introduced into the control design in this paper, which would introduce more freedom of design to enlarge the region of attraction and improve the control performance. But just adding deterministic control moves before the terminal invariant set would result in the loss of guranteed recursive feasibility and stability, as concluded from [16]. Observing that, the closed-loop strategy, i.e. adopting the perturbation items on a sequence of state feedback control laws as control inputs, is adopted to guarantee the recursive feasibility and closed-loop stability. Compared with [13], the proposed approach introduces more freedom of design, enlarges the region of attraction and achieves good control performance. The feasibility and stability of the closed-loop system of the proposed approach are proven.

This paper is organized as follows: Section 2 gives the formulation of time-delay systems with structured uncertainty and the conditions of control invariant set. The closed-loop robust MPC approach with dual-mode framework is developed in Section 3. An illustrative numerical example is given in Section 4 to show the merits of the proposed approach. Finally, conclusions are drawn in Section 5.

Notation: $\mathbb{R}$ represents the set of real number, $\mathbb{R}^n$ represents the n-dimensional space of real valued vectors. $\mathbb{R}^{m\times n}$ represents the set of real $m\times n$ matrices, $S^{n\times n}$ represents the set of real $n\times n$ symmetric matrices, I represents the identity matrix and O represents the block matrix of zeros with appropriate dimensions, respectively. Moreover, $I(i)$ represents the i-dimensional identity matrix and $I(i,j)$ represents the jth row of i-dimensional identity matrix. Note that the symbol $*$ in a matrix denotes the transpose of its symmetric counterpart. $x(k+i|k)$ and $u(k+i|k)$ represent the state and the future control move at time $k+i$ predicted at k, respectively. The following notations are also used:

$$\hat{x}(k+i|k)\triangleq (x^{\mathrm{T}}(k+i-d_s|k), x^{\mathrm{T}}(k+i-d_s+1|k),\cdots, x^{\mathrm{T}}(k+i|k))^{\mathrm{T}},$$

$$\bar{x}(k+i|k)\triangleq (x^{\mathrm{T}}(k+i-d_s|k), x^{\mathrm{T}}(k+i-d_{s-1}|k), \cdots, x^{\mathrm{T}}(k+i-d_1|k), x^{\mathrm{T}}(k+i|k))^{\mathrm{T}}.$$

2 Problem formulation and the conditions of control invariant set

2.1 Problem formulation

Consider the following time-delay systems with structured uncertainty:

$$x(k+1)=Ax(k)+\sum_{j=1}^{s}A_{d_j}x(k-d_j)+Bu(k)+B_pp(k), \quad (1)$$

$$q(k)=C_qx(k)+\sum_{j=1}^{s}A_{q_j}x(k-d_j)+C_{qu}u(k), \quad (2)$$

$$p(k)=\Delta(k)q(k), \quad (3)$$

where $x(k)\in\mathbb{R}^n$ is the state of the plant, $u(k)\in\mathbb{R}^m$ is the control input, $p(k)\in\mathbb{R}^p$, $q(k)\in\mathbb{R}^p$ are the additional variables accounting for the uncertainty. A, A_{d_j}, A_{q_j}, B, B_p, C_q and C_{qu} are known constant real matrices with proper dimensions. $d_j (j=1,2,\cdots,s)$ are integer state delays satisfying $d_1<d_2<\cdots<d_s$. $\Delta(k)$ describes the time-varying structured uncertainty which belongs to:

$$\boldsymbol{\Delta}\triangleq\{\mathrm{diag}\{\delta_1 I(r_1),\cdots,\delta_\xi I(r_\xi),\Delta_1,\cdots,\Delta_\eta\}: \delta_i:\|\delta_i\|\leqslant 1, i=1,2,\cdots,\xi; \Delta_j: l_2^{\theta_j}\to l_2^{\theta_j}, \|\Delta_j\|\leqslant 1,\ j=1,2,\cdots,\eta\},$$

where the operator norm on δ_i and Δ_j is the induced l_2 norm and $\sum_{i=1}^{\xi} r_i+\sum_{j=1}^{\eta}\theta_j=p$. Obviously, the structured uncertainty satisfies $\Delta^{\mathrm{T}}\Delta\leqslant I$.

In addition, systems (1)–(3) are subject to the following constraints on the control inputs:

$$|u_j(k+i|k)|\leqslant u_{j,\max},\ i\geqslant 0,\ j=1,2,\cdots,m. \quad (4)$$

At each time, the control objective of the robust MPC problem is to compute the control moves $u(k+i|k)$ by minimizing the following robust performance index:

$$\min_{u(k+i|k),i\geqslant 0}\max_{\Delta\in\boldsymbol{\Delta}} J_\infty(k)= \sum_{i=0}^{\infty}[\|x(k+i|k)\|^2_{\mathcal{L}}+\|u(k+i|k)\|^2_{\mathcal{R}}]. \quad (5)$$

Uncertainty in Eqs.(1)–(3), called structured uncertainty, is a broad class of model uncertainty descriptions, which includes affine uncertainty as a special case and is often more appropriate for accurate modeling of nonlinear systems. The uncertain systems (1)–(3), which refers to the system described in [7] with consideration of time-delays, describes the uncertainties with time-delay in the feedback loop by the second term of the right-hand side in Eq.(1). In essence, the above uncertainty structure is a modified version of that described in [1]. Obviously, the above uncertainty structure could utilize more information if it is available for a practical application. As a result, more information is helpful to improve the control design.

To deal with the uncertainty $\Delta\in\boldsymbol{\Delta}$, another scaling set is defined as follows:

$$\mathcal{D}\triangleq\{\mathrm{diag}\{D_1,\cdots,D_\xi,\rho_1 I(\theta_1),\cdots,\rho_\eta I(\theta_\eta)\}: D_i\in S^{r_i\times r_i},\ D_i>0,\ i=1,2,\cdots,\xi; \rho_j\in\mathbb{R},\ \rho_j>0,\ j=1,2,\cdots,\eta\}.$$

It is clear that for any $\Delta\in\boldsymbol{\Delta}$ and $\mathfrak{D}\in\mathcal{D}$,

$\varDelta\mathfrak{D} = \mathfrak{D}\varDelta$ holds.

Before proposing the main results, the following lemma in [15] is introduced.

Lemma 1 [15] Let W, M and E be corresponding matrices with appropriate dimensions. Moreover, assume W is also symmetric. Thus, for any F with $F^{\mathrm{T}}F < I$, the following inequality:

$$W + MFE + E^{\mathrm{T}}F^{\mathrm{T}}M^{\mathrm{T}} < 0$$

is satisfied if and only if there exists a parameter $\varepsilon > 0$ which guarantees

$$W + \varepsilon MM^{\mathrm{T}} + \varepsilon^{-1}E^{\mathrm{T}}E < 0.$$

2.2 Conditions of control invariant set and the unique state feedback control

In [13], we study the conditions of control invariant set and design the unique state feedback control for systems (1)–(4), which will be introduced briefly below.

Following the procedure in [1], the following Lyapunov-Krasovskii function is chosen on the augmented state $\hat{x}(k)$.

$$V(\hat{x}(k|k)) = x^{\mathrm{T}}(k|k)P_0x(k|k) + \sum_{j=1}^{d_1} x^{\mathrm{T}}(k-j|k)P_1x(k-j|k) + \sum_{j=d_1+1}^{d_2} x^{\mathrm{T}}(k-j|k)P_2x(k-j|k) + \cdots + \sum_{j=d_{s-1}+1}^{d_s} x^{\mathrm{T}}(k-j|k)P_sx(k-j|k).$$

Assume there exists a feedback $u(k+i|k) = Fx(k+i|k)$, $i \geqslant 0$ and impose the following robust stable conditions on the above Lyapunov-Krasovskii function:

$$V(\hat{x}(k+i+1|k)) - V(\hat{x}(k+i|k)) < -[\|x(k+i|k)\|^2_{\mathcal{L}} + \|u(k+i|k)\|^2_{\mathcal{R}}],\ i \geqslant 0, \tag{6}$$

where

$$x(k+i+1|k) = (A+BF)x(k+i|k) + \sum_{j=1}^{s} A_{d_j}x(k+i-d_j|k) + B_pp(k+i|k). \tag{7}$$

The following lemma gives us conditions for the existence of the appropriate $P_i > 0 (i = 0, 1, \cdots, s)$ satisfying Eq.(6) and the corresponding state feedback matrix F.

Lemma 2 For the uncertain systems (1)–(3) without constraints on inputs, the state feedback matrix F in the control law which guarantees Eq.(6) is given by

$$F = YQ_0^{-1}.$$

with $P_i = \gamma Q_i^{-1}$ if there exist $\varepsilon > 0$, $\gamma > 0$, $Y \in \mathbb{R}^{m\times n}$, $Q_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \cdots, s$, satisfying the following conditions:

$$\begin{bmatrix} \varGamma_1 & * & * \\ \varGamma_2 & \varGamma_3 & * \\ \varGamma_4 & O & \varGamma_5 \end{bmatrix} > 0, \tag{8}$$

where

$$\varGamma_1 = \begin{bmatrix} Q_0 & * & * & * \\ O & Q_1 & * & * \\ O & O & \ddots & * \\ O & O & O & Q_s \end{bmatrix},$$

$$\varGamma_2 = \begin{bmatrix} AQ_0 + BY & A_{d_1}Q_1 \cdots A_{d_s}Q_s \\ C_qQ_0 + C_{qu}Y & A_{q_1}Q_1 \cdots A_{q_s}Q_s \\ \mathcal{R}^{\frac{1}{2}}Y & O \quad \cdots \quad O \\ \mathcal{L}^{\frac{1}{2}}Q_0 & O \quad \cdots \quad O \end{bmatrix},$$

$$\varGamma_3 = \begin{bmatrix} Q_0 - \varepsilon B_pB_p^{\mathrm{T}} & * & * & * \\ O & \varepsilon I & * & * \\ O & O & \gamma I & * \\ O & O & O & \gamma I \end{bmatrix},$$

$$\varGamma_4 = \begin{bmatrix} Q_0 & O & O & O & O \\ O & Q_1 & O & O & O \\ O & O & \ddots & O & O \\ O & O & O & Q_{s-1} & O \end{bmatrix},\ \varGamma_5 = \begin{bmatrix} Q_1 & * & * & * \\ O & Q_2 & * & * \\ O & O & \ddots & * \\ O & O & O & Q_s \end{bmatrix}.$$

Proof See Appendix A.

Based on condition (8), the upper bound of performance cost under the unique feedback control $u(k+i|k) = Fx(k+i|k)$ can be obtained. By summing Eq.(6) from $i = 0$ to $i = \infty$, we can get

$$\sum_{i=0}^{\infty} [\|x(k+i|k)\|^2_{\mathcal{L}} + \|u(k+i|k)\|^2_{\mathcal{R}}] \leqslant V(\hat{x}(k|k)).$$

Let $V(\hat{x}(k|k)) \leqslant \gamma$ with γ a nonnegative parameter, which means γ is the upper bound of performance cost. This can be guaranteed by

$$\begin{bmatrix} 1 & * \\ \hat{x}(k|k) & Q \end{bmatrix} \geqslant 0, \tag{9}$$

where $Q = \mathrm{diag}\{\underbrace{Q_s, \cdots, Q_s}_{d_s-d_{s-1}}, \cdots, \underbrace{Q_2, \cdots, Q_2}_{d_2-d_1}, \underbrace{Q_1, \cdots, Q_1}_{d_1}, Q_0\}$, $Q_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \cdots, s$.

In the following, input constraints are incorporated into the robust MPC controller as sufficient LMI constraints.

Lemma 3 The input constraints Eq.(4) would be satisfied if there exist $\varepsilon > 0$, $\gamma > 0$, $X \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{m\times n}$, $Q_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \cdots, s$, satisfying conditions (8)–(9) as well as following conditions:

$$\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Q_0 \end{bmatrix} \geqslant 0,\ X_{jj} \leqslant u^2_{j,\max},\ j = 1, 2, \cdots, m. \tag{10}$$

Proof It can be deduced in a similar way as in [1], here it is omitted.

From the above analysis, Eqs.(8)–(10) are the conditions of control invariant set and guarantee that γ is an upper bound of performance cost. Therefore, the min-max optimization problem (5) can be converted into a linear objective minimization problem, where the structured uncertainty can be addressed and the upper bound γ can be minimized by utilizing LMI tools[13]. Based on that, the following state feedback MPC approach is proposed:

Controller C_1 Solve the following linear objective minimization problem

$$\min_{\varepsilon,\gamma,Q_0,Q_1,\cdots,Q_s,X,Y} \gamma, \text{ s.t. Eqs.(8)–(10)},$$

where $\varepsilon > 0$, $\gamma > 0$, $X \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{m\times n}$ and $Q_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \cdots, s$ are optimization variables. The corresponding unique state feedback control law is given by $F = YQ_0^{-1}$.

Remark 1 Obviously, controller C_1 is constructed based on the design of a control invariant set, where an unique state feedback control law is utilized as the future control strategy of the MPC controller. As [7] pointed out, the unique state feedback control strategy leads to the conservativeness due to the limited degree of freedom. Hence, in the following, we will adopt a sequence of control actions followed by a state feedback control law as the control strategy to the design of MPC, which complies with the dual-mode control framework and the invariant set corresponding to the state feedback control law is used as the terminal set. Since more freedom is introduced, the enlarged attractive region and improved control performance can be expected.

3 The closed-loop robust MPC approach with dual-mode framework

Based on the results in Section 2, to enlarge the region of attraction and improve the control performance, this section develops the robust MPC approach with dual-mode framework. As can be concluded from [16], just adding N deterministic control moves before the terminal invariant set would result in the loss of guaranteed recursive feasibility and stability. Here, we adopt both the closed-loop strategy and the dual-mode framework, which add N perturbation items on a sequence of state feedback laws followed by the unique feedback control law. That is

$$\begin{cases} u(k+i|k) = F(k+i|k)x(k+i|k) + v(k+i|k), \\ \qquad 0 \leqslant i \leqslant N-1, \\ u(k+i|k) = F(k+N|k)x(k+i|k), \\ \qquad i \geqslant N, \end{cases} \tag{11}$$

where $v(k+i|k)$, $0 \leqslant i \leqslant N-1$ and $F(k+i|k)$, $0 \leqslant i \leqslant N-1$ refer to the perturbation items and the future feedback control laws at time $k+i$ predicted at time k respectively, $F(k+N|k)$ refers to the linear feedback control law of the target set at time $k+N$ predicted at time k, i.e. the feedback control law in the control invariant set. The N perturbation items on the state feedback sequence would steer an augmented state sequence with respect to the time-delay systems into a control invariant set and a unique state feedback law would drive the terminal state to the equilibrium.

Aiming at designing the robust MPC with strategy (11), we need to calculate the state sequence, the control input sequence, the terminal state and the corresponding performance. In the following, we will show how to calculate these related variables and then formulate the optimization problem of robust MPC.

To simplify the presentation, the augmented variables are defined:

$$\begin{cases} \mathcal{X}(k) \triangleq [x^{\mathrm T}(k|k), \cdots, x^{\mathrm T}(k+N-1|k)]^{\mathrm T}, \\ \hat{F}(k) \triangleq \mathrm{diag}\{F(k|k), \cdots, F(k+N-1|k\}, \\ \mathcal{V}(k) \triangleq [v^{\mathrm T}(k|k), \cdots, v^{\mathrm T}(k+N-1|k)]^{\mathrm T}, \\ U(k) \triangleq [u^{\mathrm T}(k|k), \cdots, u^{\mathrm T}(k+N-1|k)]^{\mathrm T}, \\ \mathcal{P}(k) \triangleq [p^{\mathrm T}(k|k), \cdots, p^{\mathrm T}(k+N-1|k)]^{\mathrm T}, \\ \mathcal{G}(k) \triangleq [q^{\mathrm T}(k|k), \cdots, q^{\mathrm T}(k+N-1|k)]^{\mathrm T}. \end{cases} \tag{12}$$

Provided that $F(k+i|k)$, $0 \leqslant i \leqslant N-1$ have been known at time k, $F(k+N|k)$ and $v(k+i|k)$ can be obtained by solving optimization problem under the dual-mode control (11). For the control strategy, the following shifting method is used to get the known $F(k+i|k+1)$ at time $k+1$:

$$\begin{cases} F(i|0) = 0,\ 0 \leqslant i \leqslant N-1,\ k = 0, \\ F(k+i|k+1) = F(k+i|k),\ 1 \leqslant i \leqslant N,\ k \geqslant 0. \end{cases}$$

For example, the state feedback sequence is initiated as $\hat{F}(0) = (0, \cdots, 0)$ at time $k = 0$, and the unique feedback control law $F(N|0)$ and perturbation items sequence $\mathcal{V}(0)$ can be obtained by online solving an optimization problem under the dual-mode control (11). At next time $k = 1$, the shifting method is used to generate $\hat{F}(1) = (0, \cdots, 0, F(N|0))$, and $F(1+N|1)$, $\mathcal{V}(1)$ will be optimized by online solving the optimization problem. The above procedure repeats to generate $\hat{F}(k)$, $F(k+N|k)$ and $\mathcal{V}(k)$ at all the following times $k = 2, 3, \cdots$.

At first, we will calculate an augmented state sequence consisting of $\mathcal{X}(k)$ and $x(k+N|k)$, which refer to the state sequence before entering the terminal invariant set and the terminal state respectively. By using $\hat{x}(k|k)$, $\mathcal{V}(k)$ and $\mathcal{P}(k)$, the augmented state sequence can be formulated as follows:

$$\begin{bmatrix} \mathcal{X}(k) \\ x(k+N|k) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{\mathrm{out}} \\ \mathcal{A}_N \end{bmatrix} \hat{x}(k|k) + \begin{bmatrix} \mathcal{B}_{\mathrm{out}} \\ \mathcal{B}_N \end{bmatrix} \mathcal{V}(k) + \begin{bmatrix} \mathcal{B}_{p\mathrm{out}} \\ \mathcal{B}_{p,N} \end{bmatrix} \mathcal{P}(k), \tag{13}$$

where the calculation of matrices $\mathcal{A}_{\mathrm{out}}$, $\mathcal{B}_{\mathrm{out}}$, $\mathcal{B}_{p\mathrm{out}}$, $\mathcal{A}_N$, $\mathcal{B}_N$ and $\mathcal{B}_{p,N}$ will be introduced below in detail.

Denote matrix $\Im(i) = I(d_s+1, i) \otimes I(n)$ where

$\otimes$ denotes the Kronecker product. Then, the following augmented matrices are given:

$$\begin{bmatrix}\mathcal{A}_{-d_s}\\ \mathcal{A}_{1-d_s}\\ \vdots\\ \mathcal{A}_{-1}\end{bmatrix} \triangleq \begin{bmatrix}\Im(1)\\ \Im(2)\\ \vdots\\ \Im(d_s)\end{bmatrix},\quad \begin{bmatrix}\mathcal{B}_{-d_s}\\ \mathcal{B}_{1-d_s}\\ \vdots\\ \mathcal{B}_{-1}\end{bmatrix} \triangleq \begin{bmatrix}O\\ O\\ \vdots\\ O\end{bmatrix},\quad \begin{bmatrix}\mathcal{B}_{p,-d_s}\\ \mathcal{B}_{p,1-d_s}\\ \vdots\\ \mathcal{B}_{p,-1}\end{bmatrix} \triangleq \begin{bmatrix}O\\ O\\ \vdots\\ O\end{bmatrix}.$$

According to the system dynamics, matrices $\mathcal{A}_{\rm out}$, $\mathcal{B}_{\rm out}$, $\mathcal{B}_{p{\rm out}}$, $\mathcal{A}_N$, $\mathcal{B}_N$ and $\mathcal{B}_{p,N}$ can be described by the following expressions:

$$\begin{bmatrix}\mathcal{A}_{\rm out}\\ \mathcal{A}_N\end{bmatrix} = \begin{bmatrix}\mathcal{A}_0\\ \mathcal{A}_1\\ \vdots\\ \mathcal{A}_{N-1}\\ \mathcal{A}_N\end{bmatrix} = \begin{bmatrix}\Im(d_s+1)\\ A_0\mathcal{A}_0 + A_{d_1}\mathcal{A}_{-d_1} + \cdots + A_{d_s}\mathcal{A}_{-d_s}\\ \vdots\\ A_{N-2}\mathcal{A}_{N-2} + A_{d_1}\mathcal{A}_{N-2-d_1} + \cdots + A_{d_s}\mathcal{A}_{N-2-d_s}\\ A_{N-1}\mathcal{A}_{N-1} + A_{d_1}\mathcal{A}_{N-1-d_1} + \cdots + A_{d_s}\mathcal{A}_{N-1-d_s}\end{bmatrix},$$

$$\begin{bmatrix}\mathcal{B}_{\rm out}\\ \mathcal{B}_N\end{bmatrix} = \begin{bmatrix}\mathcal{B}_0\\ \mathcal{B}_1\\ \vdots\\ \mathcal{B}_{N-1}\\ \mathcal{B}_N\end{bmatrix} = \begin{bmatrix}O\\ A_0\mathcal{B}_0 + A_{d_1}\mathcal{B}_{-d_1} + \cdots + A_{d_s}\mathcal{B}_{-d_s} + BI(N,1)\\ \vdots\\ A_{N-2}\mathcal{B}_{N-2} + A_{d_1}\mathcal{B}_{N-2-d_1} + \cdots + A_{d_s}\mathcal{B}_{N-2-d_s} + BI(N,N-1)\\ A_{N-1}\mathcal{B}_{N-1} + A_{d_1}\mathcal{B}_{N-1-d_1} + \cdots + A_{d_s}\mathcal{B}_{N-1-d_s} + BI(N,N)\end{bmatrix},$$

$$\begin{bmatrix}\mathcal{B}_{p{\rm out}}\\ \mathcal{B}_{p,N}\end{bmatrix} = \begin{bmatrix}\mathcal{B}_{p,0}\\ \mathcal{B}_{p,1}\\ \vdots\\ \mathcal{B}_{p,N-1}\\ \mathcal{B}_{p,N}\end{bmatrix} = \begin{bmatrix}\mathrm{O}\\ A_0\mathcal{B}_{p,0} + A_{d_1}\mathcal{B}_{p,-d_1} + \cdots + A_{d_s}\mathcal{B}_{p,-d_s} + B_pI(N,1)\\ \vdots\\ A_{N-2}\mathcal{B}_{p,N-2} + A_{d_1}\mathcal{B}_{p,N-2-d_1} + \cdots + A_{d_s}\mathcal{B}_{p,N-2-d_s} + B_pI(N,N-1)\\ A_{N-1}\mathcal{B}_{p,N-1} + A_{d_1}\mathcal{B}_{p,N-1-d_1} + \cdots + A_{d_s}\mathcal{B}_{p,N-1-d_s} + B_pI(N,N)\end{bmatrix},$$

where $A_i = A+BF(k+i|k)$ denotes the closed-loop system state matrices at time $k+i$.

Next, we will calculate the augmented terminal states $\hat{x}(k+N|k) = [x(k+N-d_s|k)^{\rm T}\ x(k+N-d_s+1|k)^{\rm T}\ \cdots\ x(k+N|k)^{\rm T}]^{\rm T}$, which is required to belong to the control invariant set. It can be formulated as follows:

$$\hat{x}(k+N|k) = \mathcal{A}_{{\rm termi}}\hat{x}(k) + \mathcal{B}_{{\rm termi}}\mathcal{V}(k) + \mathcal{B}_{p_{\rm termi}}\mathcal{P}(k), \tag{14}$$

where $\mathcal{A}_{{\rm termi}}$, $\mathcal{B}_{{\rm termi}}$ and $\mathcal{B}_{p_{\rm termi}}$ need to be determined according to the comparison of the value of N and that of d_s.

Here, we discuss how to obtain matrices $\mathcal{A}_{{\rm termi}}$, $\mathcal{B}_{{\rm termi}}$ and $\mathcal{B}_{p_{\rm termi}}$ in detail.

If $N \geqslant d_s$, then

$$\mathcal{A}_{{\rm termi}} = [\mathcal{A}_{N-d_s}^{\rm T}\ \mathcal{A}_{N-d_s+1}^{\rm T}\ \cdots\ \mathcal{A}_N^{\rm T}]^{\rm T},$$
$$\mathcal{B}_{{\rm termi}} = [\mathcal{B}_{N-d_s}^{\rm T}\ \mathcal{B}_{N-d_s+1}^{\rm T}\ \cdots\ \mathcal{B}_N^{\rm T}]^{\rm T},$$
$$\mathcal{B}_{p_{\rm termi}} = [\mathcal{B}_{p,N-d_s}^{\rm T}\ \mathcal{B}_{p,N-d_s+1}^{\rm T}\ \cdots\ \mathcal{B}_{p,N}^{\rm T}]^{\rm T}.$$

If $N < d_s$, then

$$\mathcal{A}_{{\rm termi}} = [\Im(N+1)^{\rm T}\ \cdots\ \Im(d_s)^{\rm T}\ \mathcal{A}_{\rm out}^{\rm T}\ \mathcal{A}_N^{\rm T}]^{\rm T},$$
$$\mathcal{B}_{{\rm termi}} = [O\ \mathcal{B}_{\rm out}^{\rm T}\ \mathcal{B}_N^{\rm T}]^{\rm T},$$
$$\mathcal{B}_{p_{\rm termi}} = [O\ \mathcal{B}_{p{\rm out}}^{\rm T}\ \mathcal{B}_{p,N}^{\rm T}]^{\rm T}.$$

Subsequently, the input sequence before entering the terminal invariant set will be calculated. It can be formulated as follows:

$$U(k) = \hat{F}(k)\mathcal{X}(k) + \mathcal{V}(k) = \hat{F}(k)\mathcal{A}_{\rm out}\hat{x}(k|k) + (I + \hat{F}(k)\mathcal{B}_{\rm out})\mathcal{V}(k) + \hat{F}(k)\mathcal{B}_{p_{\rm out}}\mathcal{P}(k). \tag{15}$$

Finally, synthesizing Eq.(2) and definitions of $\mathcal{G}(k)$, $\mathcal{X}(k)$ and $U(k)$, we will get the additional variable sequence $\mathcal{G}(k)$ accounting for the uncertainty.

$$\mathcal{G}(k) = \hat{C}_q\mathcal{X}(k) + \hat{C}_{qu}U(k) + \hat{\mathcal{A}}_q \times [x^{\rm T}(k-d_s|k)\ \cdots\ x^{\rm T}(k-1|k)\ \mathcal{X}^{\rm T}(k)]^{\rm T}, \tag{16}$$

where

$$\hat{C}_q = {\rm diag}(\underbrace{C_q, \cdots, C_q}_{N}),\ \hat{C}_{qu} = {\rm diag}(\underbrace{C_{qu}, \cdots, C_{qu}}_{N}),$$
$$\mathcal{A}_q = [A_{q_s},\ O, A_{q_{s-1}}, O, \cdots, A_{q_1}, O],$$
$$\hat{\mathcal{A}}_q = {\rm diag}\{\underbrace{\mathcal{A}_q, \cdots, \mathcal{A}_q}_{N}\}.$$

Let

$$\bar{\mathcal{A}} = [\Im^{\rm T}(1)\ \Im^{\rm T}(2)\ \cdots\ \Im^{\rm T}(d_s)\ \mathcal{A}_{\rm out}^{\rm T}]^{\rm T},$$
$$\bar{\mathcal{B}} = [O\ \mathcal{B}_{\rm out}^{\rm T}]^{\rm T},\ \bar{\mathcal{B}}_p = [O\ \mathcal{B}_{p{\rm out}}^{\rm T}]^{\rm T}.$$

Then, Eq.(16) can be rewritten as

$$\mathcal{G}(k) = \hat{C}_q\mathcal{X}(k) + \hat{C}_{qu}U(k) + \hat{\mathcal{A}}_q(\bar{\mathcal{A}}\hat{x}(k|k) + \bar{\mathcal{B}}\mathcal{V}(k) + \bar{\mathcal{B}}_p\mathcal{P}(k)). \tag{17}$$

Combining Eqs.(13) (15)(17), it can be obtained

$$\mathcal{G}(k) = \mathcal{S} + (\hat{C}_q\mathcal{B}_{p_{\text{out}}} + \hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\text{out}}} + \hat{A}_q\bar{\mathcal{B}}_p)\mathcal{P}(k), \tag{18}$$

where

$$\mathcal{S} \triangleq (\hat{C}_q\mathcal{A}_{\text{out}} + \hat{C}_{qu}\hat{F}(k)\mathcal{A}_{\text{out}} + \hat{A}_q\bar{\mathcal{A}})\hat{x}(k) + (\hat{C}_q\mathcal{B}_{\text{out}} + \hat{C}_{qu}\hat{F}(k)\mathcal{B}_{\text{out}} + \hat{C}_{qu} + \hat{A}_q\bar{\mathcal{B}})\mathcal{V}(k). \tag{19}$$

Synthesizing Eq.(3) with definitions of $\mathcal{P}(k)$ and $\mathcal{G}(k)$, it is obvious

$$\mathcal{P}(k) = \text{diag}\,\{\Delta(k), \Delta(k+1|k), \cdots, \Delta(k+N-1|k)\}\mathcal{G}(k). \tag{20}$$

According to definitions of Δ and Eq.(20), it can be concluded that for any $\mathfrak{D}_\ell \in \mathcal{D}$, $\ell = 0, 1, \cdots, N-1$,

$$\mathcal{P}^{\text{T}}(k)\text{diag}\{\mathfrak{D}_0, \mathfrak{D}_1, \cdots, \mathfrak{D}_{N-1}\}^{-1}\mathcal{P}(k) \leqslant \mathcal{G}^{\text{T}}(k)\text{diag}\{\mathfrak{D}_0, \mathfrak{D}_1, \cdots, \mathfrak{D}_{N-1}\}^{-1}\mathcal{G}(k). \tag{21}$$

From the above derivation, the terminal state, the state sequence and the control input sequence before entering the terminal invariant set have been obtained. Therefore, the whole control performance can be optimized and the dual-mode controller can be designed. Before the controller is proposed, two lemmas and an algorithm would be established for the time-delay uncertain system.

Lemma 4 For the uncertain systems (1)–(3) with initial state $\hat{x}(k|k)$, if there exist $\gamma > 0$, $\mathcal{V}(k)$, Λ_1, $Q_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \cdots, s$, satisfying the following inequality:

$$\begin{bmatrix} 1 & * & * & * & * & * \\ O & \Lambda_1 & * & * & * & * \\ \hat{\mathcal{L}}^{\frac{1}{2}}(\mathcal{A}_{\text{out}}\hat{x}(k|k) + \mathcal{B}_{\text{out}}\mathcal{V}(k)) & \hat{\mathcal{L}}^{\frac{1}{2}}\mathcal{B}_{p_{\text{out}}}\Lambda_1 & \gamma I & * & * & * \\ \hat{\mathcal{R}}^{\frac{1}{2}}[\hat{F}(k)\mathcal{A}_{\text{out}}\hat{x}(k|k) + (I+\hat{F}(k)\mathcal{B}_{\text{out}})\mathcal{V}(k)] & \hat{\mathcal{R}}^{\frac{1}{2}}\hat{F}(k)\mathcal{B}_{p_{\text{out}}}\Lambda_1 & O & \gamma I & * & * \\ \mathcal{A}_{\text{termi}}\hat{x}(k|k) + \mathcal{B}_{\text{termi}}\mathcal{V}(k) & \mathcal{B}_{p_{\text{termi}}}\Lambda_1 & O & O & Q & * \\ \mathcal{S} & (\hat{C}_q\mathcal{B}_{p_{\text{out}}} + \hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\text{out}}} + \hat{A}_q\bar{\mathcal{B}}_p)\Lambda_1 & O & O & O & \Lambda_1 \end{bmatrix} \geqslant 0, \tag{22}$$

the following conditions can be guaranteed:

$$\|\mathcal{X}(k)\|^2_{\hat{\mathcal{L}}} + \|U(k)\|^2_{\hat{\mathcal{R}}} + \hat{x}^{\text{T}}(k+N|k)P\hat{x}(k+N|k) \leqslant \gamma,$$

$$\begin{bmatrix} 1 & * \\ \hat{x}(k+N|k) & Q \end{bmatrix} \geqslant 0,$$

where

$$\hat{\mathcal{L}} = \text{diag}\{\underbrace{\mathcal{L}, \cdots, \mathcal{L}}_{N}\},\ \hat{\mathcal{R}} = \text{diag}\{\underbrace{\mathcal{R}, \cdots, \mathcal{R}}_{N}\},$$

$$\Lambda_1 = \text{diag}\{\mathfrak{D}_0, \mathfrak{D}_1, \cdots, \mathfrak{D}_{N-1}\},$$

$\mathfrak{D}_\ell \in \mathcal{D}$, $\ell = 0, 1, \cdots, N-1$, $P = \gamma Q^{-1}$ with Q as proposed in Eq.(9) and $\mathcal{S}$ proposed in Eq.(19).

Proof See Appendix B.

In the following, we shall show that constraints on the inputs can also be incorporated into our robust MPC approach as sufficient LMI constraints.

Lemma 5 For the uncertain systems (1)–(4) with initial state $\hat{x}(k|k)$, the input constraints before the switching horizon N can be guaranteed if there exist $\mathcal{V}(k)$ and Λ_2 satisfying the following inequality

$$\begin{bmatrix} 1 & * & * & * \\ O & \Lambda_2 & * & * \\ \mathcal{S} & (\hat{C}_q\mathcal{B}_{p_{\text{out}}} + \hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\text{out}}} + \hat{A}_q\bar{\mathcal{B}}_p)\Lambda_2 & \Lambda_2 & * \\ I(Nm, im+j)[\hat{F}(k)\mathcal{A}_{\text{out}}\hat{x}(k|k) + (I+\hat{F}(k)\mathcal{B}_{\text{out}})\mathcal{V}(k)] & I(Nm, im+j) \times (\hat{F}(k)\mathcal{B}_{p_{\text{out}}}\Lambda_2) & O & u^2_{j,\max} \end{bmatrix} \geqslant 0, \tag{23}$$

where $j = 1, 2, \cdots, m$, $i = 0, 1, \cdots, N-1$, $\Lambda_2 = \text{diag}(\widetilde{\mathfrak{D}}_0, \widetilde{\mathfrak{D}}_1, \cdots, \widetilde{\mathfrak{D}}_{N-1})$, $\widetilde{\mathfrak{D}}_\ell \in \mathcal{D}$, $\ell = 0, 1, \cdots, N-1$.

Proof See Appendix C.

Thus, we derive the sufficient LMI constraints which guarantee the satisfaction of input constraints.

From lemma 4, it follows that the index γ is the upper bound of the performance cost for the uncertain systems (1)–(3), which means minimizing the corresponding index approximately optimizes the control performance. Thus, the following algorithm can be proposed:

Algorithm 1 Let $\hat{x}(k|k)$ be the initial state of the uncertain systems (1)–(3), and constraints on the input are described as in Eq.(4). Then the upper bound on the robust performance objective function can be obtained from the solution of the following linear objective minimization problem

$$\min_{\varepsilon,\gamma,Q_0,\cdots,Q_s,X,Y,\mathcal{V}(k),\Lambda_1,\Lambda_2} \gamma, \quad \text{s.t. Eqs.(8)(10)(22)–(23)}, \tag{24}$$

where $\varepsilon > 0$, $\gamma > 0$, $X \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{m\times n}$, $Q_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \cdots, s$, $\Lambda_1 = \text{diag}\{\mathfrak{D}_0, \mathfrak{D}_1, \cdots, \mathfrak{D}_{N-1}\}$, $\Lambda_2 = \text{diag}\{\widetilde{\mathfrak{D}}_0, \widetilde{\mathfrak{D}}_1, \cdots, \widetilde{\mathfrak{D}}_{N-1}\}$, $\mathfrak{D}_\ell \in \mathcal{D}$, $\widetilde{\mathfrak{D}}_\ell \in \mathcal{D}$, $\ell = 0, 1, \cdots, N-1$.

Base on all above developments, we propose the following dual-mode control approach with N perturbation items on a state feedback sequence followed by a unique state feedback law.

Controller C_2 Initialization: Given N and $k = 0$, $F(i|0) = 0$, $0 \leqslant i \leqslant N-1$.

Generic step:

Step 1 At each time $k \geqslant 0$, solve Eq.(24) to obtain the optimal solution

$$[\varepsilon_{\rm opt}, \gamma_{\rm opt}, Q_{0,{\rm opt}}, Q_{1,{\rm opt}} \cdots, Q_{s,{\rm opt}}, X_{\rm opt}, Y_{\rm opt}, \mathcal{V}_{\rm opt}(k), \varLambda_{1,{\rm opt}}, \varLambda_{2,{\rm opt}}].$$

Step 2 Act with $u(k|k) = F(k|k)x(k|k) + v_{\rm opt}(k|k)$.

Step 3 $F(k+i|k+1) = F(k+i|k)$, $1 \leqslant i \leqslant N-1$; $F(k+N|k+1) = Y_{\rm opt}Q_{0,{\rm opt}}^{-1}$.

Step 4 $k \leftarrow k+1$ and go to Step 1.

Compared with the unique feedback controller C_1, the dual-mode controller C_2 can introduce extra degrees of freedom through the use of perturbations, which would enlarge the region of attraction and improve the control performance.

Remark 2 In general, for the design based on dual-mode control framework, the closed-loop system would achieve better control performance if $F(k+i|k+1)$ is designed as optimization variables and obtained by online solving the optimization problem in next step $k+1$. However, if $F(k+i|k+1)$, $1 \leqslant i \leqslant N$ is solved in next step $k+1$, the optimization problem would not be a convex optimization problem and lead to very heavy on-line computational burden. In order to avoid this problem, the shifting method is used to generate the sequence of state feedback laws as $F(k+i|k+1) = F(k+i|k)$, $1 \leqslant i \leqslant N$. And the unique feedback control law $F(k+1+N|k+1)$ remains to be determined by solving the optimization problem at sample time $k+1$. In this way, the optimization problem can be formulated as a convex optimization and solved online by semi-definite programming.

For the closed-loop system under controller C_2, the feasibility and stability property of the closed-loop system can be asserted as follows:

Theorem 1 If there is a feasible solution for the dual-mode controller C_2 with initial state $\hat{x}(k|k)$ at time k, there will also exist a feasible solution for all times $t > k$, and the closed-loop system is asymptotically stable.

Proof See Appendix D.

Remark 3 Note that controller C_1 requires the current state to be strictly in the control invariant set, which results in somewhat conservativeness. For the proposed C_2, the state can be allowed to move from outside of the control invariant set and finally into this invariant set. The conditions in controller C_1 can be recovered by imposing $N = 0$ in the proposed approach, and included as its special case. Therefore the proposed C_2 has more freedom and less conservativeness of design compared with controller C_1. Consequently, it would enlarge the region of attraction and improve the control performance.

Remark 4 The proposed method is available for the systems with fixed time-delays. If the time-delay is time-varying, the original system is a switching system. Although the idea of the proposed method can be used for this case after some necessary modifications, the controller should be modified greatly and is not discussed by this paper.

4 Case studies

Considering the following time-delay system with structured uncertainty

$$x(k+1) = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} -0.05 & 0.05 \\ 0.3 & -0.05 \end{bmatrix} x(k-1) + \begin{bmatrix} -0.05 & 0.05 \\ 0.6 & -0.05 \end{bmatrix} x(k-3) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} p(k),$$

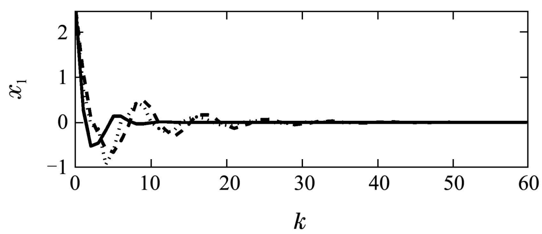
$$q(k) = [0.1 \;\; 0.1]x(k) + [0.01 \;\; 0]x(k-1) + [0.06 \;\; 0]x(k-3) + 0.1u(k),$$

$$p(k) = \varDelta(k)q(k),$$

where structured uncertainty $\varDelta(k)$ is a random variable uniformly distributed in the interval [0 1] and the input constraint $|u| \leqslant 1$. The initial conditions are given as $x(0) = [1 \;\; 1]^{\rm T}$, $x(-1) = x(-2) = x(-3) = [2.5 \;\; -5]^{\rm T}$. The weighting matrices are given as $\mathcal{L} = {\rm diag}\{1, 1\}$, $\mathcal{R} = 1$.

In the following, the region of attraction and control performance will be compared between the proposed approach and that in [13]. For the proposed controller C_2, it adopts $N = 3$ and $N = 1$ respectively.

The control results under the three controllers are shown in Fig.1 and Fig.2. Fig.1 shows C_2 achieves better control performance. It also can be seen that with controller C_2, the increase of N could improve the control performance. Fig.2 demonstrates that over the entire horizon the input constraints are satisfied. The regions of attraction are plotted in Fig.3. From Fig.3, we can conclude that controller C_2 achieves a much larger region of attraction, and the increase of N could enlarge the region of attraction.



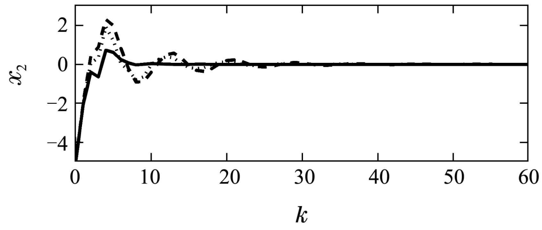


Fig. 1 The closed-loop state responses: solid line for controller C_2 $(N = 3)$, dotted line for controller C_2 $(N = 1)$, dashed line for [13]

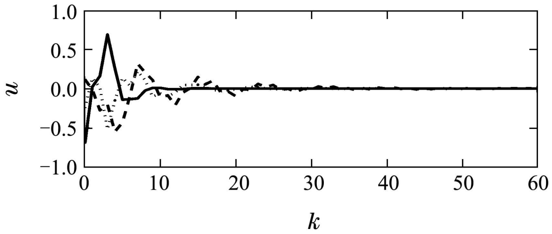


Fig. 2 The control signals: solid line for controller C_2 $(N = 3)$, dotted line for controller C_2 $(N = 1)$, dashed line for [13]

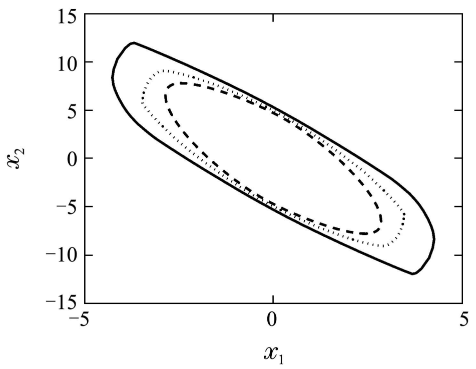


Fig. 3 The regions of attraction: solid line for controller C_2 $(N = 3)$, dotted line for controller C_2 $(N = 1)$, dashed line for [13]

5 Conclusions

This paper presents a closed-loop robust MPC approach for time-delay systems with structured uncertainty. The dual-mode framework and the closed-loop strategy are adopted to enlarge the region of attraction and improve the control performance. With the proposed control approach, the model uncertainty can be addressed with guaranteed robust closed-loop stability and good control performance.

References:

[1] KOTHARE M, BALAKRISHNAN V, MORARI M. Robust constrained model predictive control using linear matrix inequalities [J]. *Automatica*, 1996, 32(10): 1361 – 1379.

[2] CUZZOLA F, GEROMEL J C, MORARI M. An improved approach for constrained robust model predictive control [J]. *Automatica*, 2002, 38(7): 1183 – 1189.

[3] MAO W. Robust stabilization of uncertain time-varying discrete systems and comments on "an improved approach for constrained robust model predictive control" [J]. *Automatica*, 2003, 39(6): 1109 – 1112.

[4] WAN Z, KOTHARE M. An efficient off-line formulation of robust model predictive control using linear matrix inequalities [J]. *Automatica*, 2003, 39(5): 837 – 846.

[5] 丁宝苍, 邹涛, 李少远. 时变不确定系统的变时域离线鲁棒预测控制 [J]. 控制理论与应用, 2006, 23(2): 240 – 244.
(DING Baocang, ZOU Tao, LI Shaoyuan. Varying-horizon off-line robust predictive control for time-varying uncertain systems [J]. *Control Theory & Applications*, 2006, 23(2): 240 – 244.)

[6] WU F. LMI-based robust model predictive control and its application to an industrial CSTR problem [J]. *Journal of Process Control*, 2001, 11(6): 649 – 659.

[7] CASAVOLA A, FAMULARO D, FRANZE G. Robust constrained predictive control of uncertain norm-bounded linear systems [J]. *Automatica*, 2004, 40(11): 1865 – 1876.

[8] PARK P, JEONG S. Constrained RHC for LPV systems with bounded rates of parameter variations [J]. *Automatica*, 2004, 40(5): 865 – 872.

[9] JEONG S, PARK P. Constrained MPC algorithm for uncertain time-varying systems with state-delay [J]. *IEEE Transactions on Automatic Control*, 2005, 50(2): 257 – 263.

[10] DING B, HUANG B. Constrained robust model predictive control for time-delay systems with polytopic description [J]. *International Journal of Control*, 2007, 80(4): 509 – 522.

[11] DING B, XIE L, CAI W. Robust MPC for polytopic uncertain systems with time-varying delays [J]. *International Journal of Control*, 2008, 81(8): 1239 – 1252.

[12] LI D, XI Y. Constrained feedback robust model predictive control for polytopic uncertain systems with time-delays [J]. *International Journal of Systems Science*, 2011, 42(10): 1651 – 1660.

[13] ZHENG P, XI Y, LI D. Robust model predictive control for time-delay systems with structured uncertainty [C] //*The 8th World Congress on Intelligent Control and Automation*. Jinan, China: IEEE, 2010: 1174 – 1178.

[14] MICHALSKA H, MAYNE D. Robust receding horizon control of constrained nonlinear systems [J]. *IEEE Transactions on Automatic Control*, 1993, 38(11): 1623 – 1633.

[15] XIE L. Output feedback H_∞ control of systems with parameter uncertainty [J]. *International Journal of Control*, 1996, 63(4): 741 – 750.

[16] PLUYMERS B, SUYKENS J, MOOR B. Min-max feedback MPC using a time-varying terminal constraint set and comments on 'efficient robust constrained model predictive control with a time-varying terminal constraint set' [J]. *Systems & Control Letters*, 2005, 54(12): 1143 – 1148.

Appendix A

From the definition of quadratic function $V(\hat{x}(k+i|k))$, it follows

$$V(\hat{x}(k+i+1|k)) - V(\hat{x}(k+i|k)) = \begin{bmatrix} \bar{x}(k+i|k) \\ x(k+i+1|k) \end{bmatrix}^{\mathrm{T}} \mathrm{diag}\{-P_s, P_s - P_{s-1}, \cdots, P_1 - P_0, P_0\} \cdot \begin{bmatrix} \bar{x}(k+i|k) \\ x(k+i+1|k) \end{bmatrix}. \tag{A1}$$

Adopting $u(k+i|k) = Fx(k+i|k)$ into system described by Eqs.(1)–(3), it can be concluded

$$x(k+i+1|k) = [A + BF + B_p\varDelta(C_q + C_{qu}F)]x(k+i|k) + \sum_{j=1}^{s}(A_{d_j} + B_p\varDelta A_{q_j})x(k+i-d_j|k). \tag{A2}$$

Synthesizing Eqs.(A1)−(A2), it leads to

$$\bar{x}^{\mathrm{T}}(k+i|k)\varPsi\bar{x}(k+i|k) = V(\hat{x}(k+i|k)) - V(\hat{x}(k+i+1|k)) - \|x(k+i|k)\|^2_{\mathcal{L}} - \|u(k+i|k)\|^2_{\mathcal{R}}. \tag{A3}$$

where $\varPsi$ is given as follows:

$$\Psi=\begin{bmatrix} \Xi & * & * & * & * & * \\ -\bar{A}^{\mathrm{T}}_{d_{s-1}}P_0\bar{A}_{d_s} & \Xi_1 & * & * & * & * \\ -\bar{A}^{\mathrm{T}}_{d_{s-2}}P_0\bar{A}_{d_s} & -\bar{A}^{\mathrm{T}}_{d_{s-2}}P_0\bar{A}_{d_{s-1}} & \Xi_2 & * & * & * \\ \vdots & \vdots & \vdots & \ddots & * & * \\ -\bar{A}^{\mathrm{T}}_{d_1}P_0\bar{A}_{d_s} & -\bar{A}^{\mathrm{T}}_{d_1}P_0\bar{A}_{d_{s-1}} & -\bar{A}^{\mathrm{T}}_{d_1}P_0\bar{A}_{d_{s-2}} & \cdots & \Xi_{s-1} & * \\ -\bar{A}^{\mathrm{T}}P_0\bar{A}_{d_s} & -\bar{A}^{\mathrm{T}}P_0\bar{A}_{d_{s-1}} & -\bar{A}^{\mathrm{T}}P_0\bar{A}_{d_{s-2}} & \cdots & -\bar{A}^{\mathrm{T}}P_0\bar{A}_{d_1} & \Xi_s \end{bmatrix},$$

with

$\bar{A}=A+BF+B_p\Delta(C_q+C_{qu}F)$; $\bar{A}_{d_j}=A_{d_j}+B_p\Delta A_{q_j}$;

$\Xi=P_s-\bar{A}^{\mathrm{T}}_{d_s}P_0\bar{A}_{d_s}$; $\Xi_1=P_{s-1}-P_s-\bar{A}^{\mathrm{T}}_{d_{s-1}}P_0\bar{A}_{d_{s-1}}$;

$\Xi_2=P_{s-2}-P_{s-1}-\bar{A}^{\mathrm{T}}_{d_{s-2}}P_0\bar{A}_{d_{s-2}}$;

$\Xi_{s-1}=P_1-P_2-\bar{A}^{\mathrm{T}}_{d_1}P_0\bar{A}_{d_1}$;

$\Xi_s=P_0-P_1-\bar{A}^{\mathrm{T}}P_0\bar{A}-F^{\mathrm{T}}\mathcal{R}F-\mathcal{L}$.

Thus, it can be concluded that Eq.(6) can be guaranteed by the following inequality:

$$\Psi>0.$$

Substituting $P_i=\gamma Q_i^{-1}$, using Schur complement and after some straightforward transformations, it can be concluded that the inequality above would hold true if there exist Q_i, $i=0,1,\cdots,s$ and $Y=FQ_0$, such that

$$\begin{bmatrix} \Gamma_1 & * & * \\ T & \Upsilon & * \\ \Gamma_4 & O & \Gamma_5 \end{bmatrix}>0, \tag{A4}$$

where

$$T=\begin{bmatrix} \bar{A}Q_0 & \bar{A}_{d_1}Q_1 & \cdots & \bar{A}_{d_s}Q_s \\ \mathcal{R}^{\frac{1}{2}}Y & O & \cdots & O \\ \mathcal{L}^{\frac{1}{2}}Q_0 & O & \cdots & O \end{bmatrix},\ \Upsilon=\begin{bmatrix} Q_0 & * & * \\ O & \gamma I & * \\ O & O & \gamma I \end{bmatrix}.$$

It is obvious that Eq.(A4) is equivalent to

$$\begin{bmatrix} \Gamma_1 & * & * \\ T & \Upsilon & * \\ \Gamma_4 & O & \Gamma_5 \end{bmatrix}+\begin{bmatrix} O \\ B_p \\ O \end{bmatrix}\Delta\cdot$$
$$[C_qQ_0+C_{qu}Y\ \ A_{q_1}Q_1\ \cdots\ A_{q_s}Q_s\ O]+$$
$$\left[C_qQ_0+C_{qu}Y\ A_{q_1}Q_1\cdots\ A_{q_s}Q_s\ O\right]^{\mathrm{T}}\Delta^{\mathrm{T}}\begin{bmatrix} O \\ B_p \\ O \end{bmatrix}^{\mathrm{T}}>0,$$

where

$$T=\begin{bmatrix} AQ_0+BY & A_{d_1}Q_1 & \cdots & A_{d_s}Q_s \\ \mathcal{R}^{1/2}Y & O & \cdots & O \\ \mathcal{L}^{1/2}Q_0 & O & \cdots & O \end{bmatrix}.$$

Moreover, the inequality above can be further rewritten as

$$\begin{bmatrix} -\Gamma_1 & * & * \\ -T & -\Upsilon & * \\ -\Gamma_4 & O & -\Gamma_5 \end{bmatrix}+\begin{bmatrix} O \\ -B_p \\ O \end{bmatrix}\Delta\cdot$$
$$\left[C_qQ_0+C_{qu}Y\ A_{q_1}Q_1\cdots\ A_{q_s}Q_s\ O\right]+$$
$$\left[C_qQ_0+C_{qu}Y\ \ A_{q_1}Q_1\cdots\ A_{q_s}Q_s\ \ O\right]^{\mathrm{T}}\Delta^{\mathrm{T}}\begin{bmatrix} O \\ -B_p \\ O \end{bmatrix}^{\mathrm{T}}<0. \tag{A5}$$

Choosing

$$W=\begin{bmatrix} -\Gamma_1 & * & * \\ -T & -\Upsilon & * \\ -\Gamma_4 & O & -\Gamma_5 \end{bmatrix},\ M=\begin{bmatrix} O \\ -B_p \\ O \end{bmatrix},$$
$$E=\left[C_qQ_0+C_{qu}Y\ \ \ A_{q_1}Q_1\ \cdots\ A_{q_s}Q_s\ \ \ O\right],$$

and using Lemma 1, it can be concluded from Eq.(A5) that there exists a parameter $\varepsilon>0$ which satisfies

$$\begin{bmatrix} -\Gamma_1 & * & * \\ -T & -\Upsilon & * \\ -\Gamma_4 & O & -\Gamma_5 \end{bmatrix}+\varepsilon\begin{bmatrix} O \\ -B_p \\ O \end{bmatrix}\begin{bmatrix} O \\ -B_p \\ O \end{bmatrix}^{\mathrm{T}}+$$
$$\varepsilon^{-1}\left[C_qQ_0+C_{qu}Y\ \ \ A_{q_1}Q_1\ \ \cdots\ \ A_{q_s}Q_s\ \ O\right]^{\mathrm{T}}\times$$
$$\left[C_qQ_0+C_{qu}Y\ \ \ A_{q_1}Q_1\ \ \cdots\ \ A_{q_s}Q_s\ \ O\right]<0.$$

Immediately, we can get the following condition

$$\begin{bmatrix} \Gamma_1 & * & * \\ T & \Upsilon & * \\ \Gamma_4 & O & \Gamma_5 \end{bmatrix}-\varepsilon\begin{bmatrix} O \\ B_p \\ O \end{bmatrix}\begin{bmatrix} O \\ B_p \\ O \end{bmatrix}^{\mathrm{T}}-$$
$$\varepsilon^{-1}\left[C_qQ_0+C_{qu}Y\ \ \ A_{q_1}Q_1\ \ \cdots\ \ A_{q_s}Q_s\ \ O\right]^{\mathrm{T}}\times$$
$$\left[C_qQ_0+C_{qu}Y\ \ \ A_{q_1}Q_1\cdots\ A_{q_s}Q_s\ \ O\right]>0.$$

By Schur complement, satisfying the above inequality leads to Eq.(8), which completes the proof.

Appendix B

Multiplying inequality (22) by $\mathrm{diag}\{1,\Lambda_1^{-1},I,I,I,\Lambda_1^{-1}\}$ to both the right side and the left side, we obtain that

$$\begin{bmatrix} 1 & * & * & * & * & * \\ O & \Lambda_1^{-1} & * & * & * & * \\ \hat{\mathcal{L}}^{\frac{1}{2}}(\mathcal{A}_{\mathrm{out}}\hat{x}(k|k)+\mathcal{B}_{\mathrm{out}}\mathcal{V}(k)) & \hat{\mathcal{L}}^{\frac{1}{2}}\mathcal{B}_{p_{\mathrm{out}}} & \gamma I & * & * & * \\ \hat{\mathcal{R}}^{\frac{1}{2}}[\hat{F}(k)\mathcal{A}_{\mathrm{out}}\hat{x}(k|k)+(I+\hat{F}(k)\mathcal{B}_{\mathrm{out}})\mathcal{V}(k)] & \hat{\mathcal{R}}^{\frac{1}{2}}\hat{F}(k)\mathcal{B}_{p_{\mathrm{out}}} & O & \gamma I & * & * \\ \mathcal{A}_{\mathrm{term}i}\hat{x}(k|k)+\mathcal{B}_{\mathrm{term}i}\mathcal{V}(k) & \mathcal{B}_{p_{\mathrm{term}i}} & O & O & Q & * \\ \Lambda_1^{-1}\mathcal{S} & \Lambda_1^{-1}(\hat{C}_q\mathcal{B}_{p_{\mathrm{term}i}}+\hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\mathrm{term}i}}+\hat{\mathcal{A}}_q\bar{\mathcal{B}}_p) & O & O & O & \Lambda_1^{-1} \end{bmatrix}\geqslant 0.$$

Using Schur complement and matrix transformation, the inequality above can be rewritten as

$$\begin{bmatrix} \gamma & * \\ O & \gamma\Lambda_1^{-1} \end{bmatrix}-\begin{bmatrix} \mathcal{A}_{\mathrm{term}i}\hat{x}(k|k)+\mathcal{B}_{\mathrm{term}i}\mathcal{V}(k) & \mathcal{B}_{p_{\mathrm{term}i}} \\ \mathcal{S} & (\hat{C}_q\mathcal{B}_{p_{\mathrm{out}}}+\hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\mathrm{out}}}+\hat{\mathcal{A}}_q\bar{\mathcal{B}}_p) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \gamma^{-1} & * \\ O & \gamma\Lambda_1^{-1} \end{bmatrix}\times$$
$$\begin{bmatrix} \mathcal{A}_{\mathrm{term}i}\hat{x}(k|k)+\mathcal{B}_{\mathrm{term}i}\mathcal{V}(k) & \mathcal{B}_{p_{\mathrm{term}i}} \\ \mathcal{S} & (\hat{C}_q\mathcal{B}_{p_{\mathrm{out}}}+\hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\mathrm{out}}}+\hat{\mathcal{A}}_q\bar{\mathcal{B}}_p) \end{bmatrix}-$$
$$\begin{bmatrix} \hat{\mathcal{L}}^{\frac{1}{2}}(\mathcal{A}_{\mathrm{out}}\hat{x}(k|k)+\mathcal{B}_{\mathrm{out}}\mathcal{V}(k)) & \hat{\mathcal{L}}^{\frac{1}{2}}\mathcal{B}_{p_{\mathrm{out}}} \\ \hat{\mathcal{R}}^{\frac{1}{2}}[\hat{F}(k)\mathcal{A}_{\mathrm{out}}\hat{x}(k|k)+(I+\hat{F}(k)\mathcal{B}_{\mathrm{out}})\mathcal{V}(k)] & \hat{\mathcal{R}}^{\frac{1}{2}}\hat{F}(k)\mathcal{B}_{p_{\mathrm{out}}} \end{bmatrix}^{\mathrm{T}}\times$$

$$\begin{bmatrix} \hat{\mathcal{L}}^{\frac{1}{2}}(\mathcal{A}_{\rm out}\hat{x}(k|k)+\mathcal{B}_{\rm out}\mathcal{V}(k)) & \hat{\mathcal{L}}^{\frac{1}{2}}\mathcal{B}_{p_{\rm out}} \\ \hat{\mathcal{R}}^{\frac{1}{2}}[\hat{F}(k)\mathcal{A}_{\rm out}\hat{x}(k|k)+(I+\hat{F}(k)\mathcal{B}_{\rm out})\mathcal{V}(k)] & \hat{\mathcal{R}}^{\frac{1}{2}}\hat{F}(k)\mathcal{B}_{p_{\rm out}} \end{bmatrix} \geqslant 0.$$

Multiplying the inequality above by $[1\ \ \mathcal{P}^{\rm T}(k)]$ to the left side and its transpose to the right side, we can obtain (using Eqs.(13)–(15)(18))

$$\|\mathcal{X}(k)\|^2_{\hat{\mathcal{L}}}+\|U(k)\|^2_{\hat{\mathcal{R}}}+\hat{x}^{\rm T}(k+N|k)P\hat{x}(k+N|k)\leqslant$$
$$\gamma+\mathcal{P}^{\rm T}(k)\gamma\Lambda_1^{-1}\mathcal{P}(k)-\mathcal{G}^{\rm T}(k)\gamma\Lambda_1^{-1}\mathcal{G}(k).$$

Combining the inequality above with Eq.(21), it immediately gets

$$\|\mathcal{X}(k)\|^2_{\hat{\mathcal{L}}}+\|U(k)\|^2_{\hat{\mathcal{R}}}+\hat{x}^{\rm T}(k+N|k)P\hat{x}(k+N|k)\leqslant\gamma.$$

Moreover, we can conclude that $\hat{x}(k+N|k)^{\rm T}Q^{-1}\hat{x}(k+N|k)\leqslant 1$, which completes the proof.

Appendix C

Eq.(15) leads to

$$\begin{cases} u_j(k+i|k)=I(Nm,im+j)\times U(k)= \\ I(Nm,im+j)\times[\hat{F}(k)\mathcal{A}_{\rm out}\hat{x}(k|k)+ \\ (I+\hat{F}(k)\mathcal{B}_{\rm out})\mathcal{V}(k)+\hat{F}(k)\mathcal{B}_{p_{\rm out}}\mathcal{P}(k)], \\ j=1,2,\cdots,m,\ i=0,1,\cdots,N-1. \end{cases} \quad \text{(A6)}$$

Multiplying inequality (23) by $\mathrm{diag}\{1,\Lambda_2^{-1},\Lambda_2^{-1},1\}$ to both the right side and the left side, we obtain that

$$\begin{bmatrix} 1 & * & * & * \\ O & \Lambda_2^{-1} & * & * \\ \Lambda_2^{-1}\mathcal{S} & \Lambda_2^{-1}(\hat{C}_q\mathcal{B}_{p_{\rm out}}+\hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\rm out}}+\hat{\mathcal{A}}_q\bar{\mathcal{B}}_p) & \Lambda_2^{-1} & * \\ I(Nm,im+j)[\hat{F}(k)\mathcal{A}_{\rm out}\hat{x}(k|k)+(I+\hat{F}(k)\mathcal{B}_{\rm out})\mathcal{V}(k)] & I(Nm,im+j)\times(\hat{F}(k)\mathcal{B}_{p_{\rm out}}) & O & u^2_{j,\max} \end{bmatrix}\geqslant 0.$$

By Schur complement, the inequality above can be rewritten as

$$\begin{bmatrix} 1 & * \\ O & \Lambda_2^{-1}\end{bmatrix} - \begin{bmatrix} \mathcal{S} & \hat{C}_q\mathcal{B}_{p_{\rm out}}+\hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\rm out}}+\hat{\mathcal{A}}_q\bar{\mathcal{B}}_p \\ I(Nm,im+j)\times[\hat{F}(k)\mathcal{A}_{\rm out}\hat{x}(k|k)+(I+\hat{F}(k)\mathcal{B}_{\rm out})\mathcal{V}(k)] & I(Nm,im+j)\times(\hat{F}(k)\mathcal{B}_{p_{\rm out}}) \end{bmatrix}^{\rm T}\times$$
$$\begin{bmatrix} \Lambda_2^{-1} & * \\ O & u^{-2}_{j,\max}\end{bmatrix}\begin{bmatrix} \mathcal{S} & \hat{C}_q\mathcal{B}_{p_{\rm out}}+\hat{C}_{qu}\hat{F}(k)\mathcal{B}_{p_{\rm out}}+\hat{\mathcal{A}}_q\bar{\mathcal{B}}_p \\ I(Nm,im+j)\times[\hat{F}(k)\mathcal{A}_{\rm out}\hat{x}(k|k)+(I+\hat{F}(k)\mathcal{B}_{\rm out})\mathcal{V}(k)] & I(Nm,im+j)\times(\hat{F}(k)\mathcal{B}_{p_{\rm out}}) \end{bmatrix}\geqslant 0.$$

Multiplying the inequality above by $[1\ \ \mathcal{P}^{\rm T}(k)]$ to the left side and its transpose to the right side, then we obtain that (using Eqs. (15) (18) (A6))

$$u_j(k+i|k)^{\rm T}u^{-2}_{j,\max}u_j(k+i|k)\leqslant$$
$$1+\mathcal{P}^{\rm T}(k)\Lambda_2^{-1}\mathcal{P}(k)-\mathcal{G}^{\rm T}(k)\Lambda_2^{-1}\mathcal{G}(k),$$
$$j=1,2,\cdots,m,\ i=0,1,\cdots,N-1.$$

Combining the inequality above with condition (21), it immediately results in

$$u_j^{\rm T}(k+i|k)u^{-2}_{j,\max}u_j(k+i|k)\leqslant 1,$$
$$j=1,2,\cdots,m,\ i=0,1,\cdots,N-1,$$

which in turn guarantees constraint (4).

Appendix D

Suppose there is a feasible solution for C_2 with initial state $\hat{x}(k|k)$ at time k, and $\gamma^*(k)$ is the corresponding upper bound, $v^*(k|k), v^*(k+1|k),\cdots,v^*(k+N-1|k)$ the corresponding perturbation items on the feedback control sequence. The LMIs in the problem depending explicitly on the current state $\hat{x}(k)$ are inequalities (22)–(23). Thus, to prove the feasibility, we only need proving that (22) and (23) are feasible for state $\hat{x}(k+1)$ at time $k+1$.

To prove the feasibility at time $k+1$, we construct a feasible solution $\gamma(k+1)=\gamma^*(k)$, $v(k+i|k+1)=v^*(k+i|k)$, $1\leqslant i\leqslant N-1$, $v(k+N|k+1)=0$. The feasibility of (23) at time k implies $u_j^*(k+i|k)\leqslant u_{j,\max}$, $0\leqslant i\leqslant N-1$. Since $v(k+i|k+1)=v^*(k+i|k)$, $1\leqslant i\leqslant N-1$, $v(k+N|k+1)=0$, it is obvious that $u_j(k+i|k+1)\leqslant u_{j,\max}$, $1\leqslant i\leqslant N$, which means that the constructed solution satisfies Eq.(23). We choose the terminal weighting function as $V(\hat{x}(k+i|k))=\hat{x}^{\rm T}(k+i|k)P\hat{x}(k+i|k)$ where P is defined in Lemma 4. The feasibility of Eq.(22) at time k implies

$$\gamma^*(k)\geqslant\|\mathcal{X}^*(k)\|^2_{\hat{\mathcal{L}}}+\|U^*(k)\|^2_{\hat{\mathcal{R}}}+V(\hat{x}^*(k+N|k)). \quad \text{(A7)}$$

Inequality (8) leads to inequality (6), which in turn guarantees that

$$V(\hat{x}(k+N+1|k))-V(\hat{x}(k+N|k))<$$
$$-[\|x(k+N|k)\|^2_{\hat{\mathcal{L}}}+\|u(k+N|k)\|^2_{\hat{\mathcal{R}}}]. \quad \text{(A8)}$$

Synthesizing Eqs.(A7)–(A8), it leads to

$$\gamma^*(k)\geqslant\|\mathcal{X}^*(k)\|^2_{\hat{\mathcal{L}}}+\|U^*(k)\|^2_{\hat{\mathcal{R}}}+V(\hat{x}(k+N+1|k))+$$
$$\|x^*(k+N|k)\|^2_{\hat{\mathcal{L}}}+\|u^*(k+N|k)\|^2_{\hat{\mathcal{R}}}\geqslant$$
$$\|\mathcal{X}(k+1)\|^2_{\hat{\mathcal{L}}}+\|U(k+1)\|^2_{\hat{\mathcal{R}}}+V(\hat{x}(k+N+1|k)).$$

Since $\gamma(k+1)=\gamma^*(k)$, it can be concluded

$$\gamma(k+1)\geqslant$$
$$\|\mathcal{X}(k+1)\|^2_{\hat{\mathcal{L}}}+\|U(k+1)\|^2_{\hat{\mathcal{R}}}+V(\hat{x}(k+N+1|k)),$$

which means that the constructed solution satisfies Eq.(22). Hence, the constructed solution is feasible for C_2 with state $\hat{x}(k+1|k)$ at time $k+1$.

At time k and time $k+1$, the corresponding performance cost can be described respectively as follows:

$$J^*(k)=\left\|\mathcal{X}^*(k)\right\|^2_{\hat{\mathcal{L}}}+\left\|U^*(k)\right\|^2_{\hat{\mathcal{R}}}+V(\hat{x}^*(k+N|k)), \quad \text{(A9)}$$
$$\bar{J}(k+1)=\|\mathcal{X}(k+1)\|^2_{\hat{\mathcal{L}}}+\|U(k+1)\|^2_{\hat{\mathcal{R}}}+$$
$$V(\hat{x}(k+N+1|k)). \quad \text{(A10)}$$

Synthesizing Eqs.(A8)–(A10), we obtain that $\bar{J}(k+1)-J^*(k)\leqslant-\|x^*(k|k)\|^2_{\hat{\mathcal{L}}}-\|u^*(k|k)\|^2_{\hat{\mathcal{R}}}$. Thus, it can be concluded that $J^*(k+1)\leqslant\bar{J}(k+1)<J^*(k)$, which proves the asymptotically stability for the closed-loop system.

作者简介:

郑鹏远 (1975–), 男, 博士, 做博士后研究, 目前研究方向为模型预测控制和实时优化理论, E-mail: pyzheng@sjtu.edu.cn;

席裕庚 (1946–), 男, 教授, 目前研究方向为模型预测控制和大系统理论, E-mail: ygxi@sjtu.edu.cn;

李德伟 (1971–), 男, 副教授, 目前研究方向为模型预测控制和鲁棒控制, E-mail: dwli@sjtu.edu.cn.