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具有切换拓扑结构的非恒等节点复杂网络同步化判据

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摘要: 网络拓扑结构与节点动态在复杂网络的同步化过程中起着关键性的作用, 针对具有切换拓扑结构与非恒等节点的同步化问题还没有非常有效的判据. 本文研究了具有切换拓扑与非恒等节点的复杂网络同步化问题, 针对非恒等节点不存在公共平衡解的情况, 选取所有节点的平均状态作为同步化目标, 并在此基础上建立起误差动态方程. 基于所有外部耦合矩阵可以同时三角化的条件下, 构建了低维系统的公共Lyapunov函数, 提出了在误差向量范数有界意义下的复杂网络全局同步化判据, 保证系统在任意切换策略下实现复杂网络的同步化. 最后通过数值仿真验证了结果的有效性.

关键词: 复杂网络; 非恒等节点; 同步化; 切换系统

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A synchronization criterion for dynamical networks with non-identical nodes and switching topology

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Abstract: Network topology and node dynamics play a key role in forming synchronization of complex networks. Unfortunately there is no effective synchronization criterion for dynamical networks with non-identical nodes and switching topology. This paper studies the synchronization problem of a complex network with non-identical nodes and switching connection topology. Considering the general case where no common equilibrium solution is assumed to exist, we select the average state of all nodes as the target of synchronization and establish the dynamical error equations. Under the condition of simultaneous triangularization of the outer connection matrices, a common Lyapunov function is constructed by those of several lower dimensional dynamic systems, a global synchronization criterion in the sense of boundedness of the maximum state deviation between the nodes is proposed under arbitrary switching topology. Finally, numerical simulations are provided to show the effectiveness of the results.

Key words: complex networks; non-identical nodes; synchronization; switching systems

1 Introduction

In recent years, complex dynamical networks as an emerging topic have received a great deal of attention. A complex network is a large set of interconnected nodes, where the nodes and connections can be everything. Many natural and man-made systems can be described as complex dynamical networks[1–3], such as social communities, biological neural networks, the Internet, electrical power grids, and the world wide web. In the past few decades, the synchronization problem of complex networks has been a hot topic due to its potential applications[4–5].

Assuming that all network nodes are identical, the synchronization of complex networks has been intensively studied[6–14]. In [6], the direct Lyapunov method was introduced to analyze synchronization of coupled systems. Pecora and Carroll developed the master stability function for arrays of coupled oscillators in [7]. Wang and Chen investigate the synchronization problems for the small-world dynamical networks and the scale-free dynamical networks in [8]. In [9–10], the delayed complex dynamical networks were considered, and some synchronization criteria have been proposed based on Lyapunov function theory. The synchronization of the dynamical networks with uncertainties was studied in [11], from the perspective of control theory, the controllers have been designed to synchronize the dynamical networks. In [12], the smart variable structure control for asymptotical synchronization to its equilibrium is developed based on the ergodicity char-

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acteristic of chaos nodes. In [13–14], the complex dynamical networks have been synchronized by pinning part of nodes. However, it is not always practical to assume that all network nodes are identical since many real-world complex networks may consist of different types of nodes [15]. Taking a multi-robot system for example[16], the multi-robot system can be viewed as a dynamical network where the nodes consist of robots. Since individual robots usually have distinct structures or have different parameters such as masses and inertias, the robot models result in different dynamics and the multi-robot system is a dynamical network with non-identical nodes. If the complex networks have non-identical nodes, the synchronization criteria for networks with identical nodes will no longer work in general. By now, there have been few studies on this subject. Based on constructing a common Lyapunov function for all the nodes, a synchronization criterion was given under the assumption that all non-identical nodes have the same equilibrium in [17]. But usually, there is no common equilibrium solution for each isolated node, so the synchronization problem become very difficult. The traditional sense of complete synchronization may not exist, but it may still exhibit some other kind of synchronization behaviors, such as bounded synchronization. Based on constructing the average node dynamics, a global bounded synchronization criterion was proposed[18]. In [19–20], synchronization of complex networks with non-identical chaotic systems were investigated. In [21–22], the synchronization problem of dynamical networks with non-identical was investigated by designing an impulsive consensus control scheme. In [23], the static gain feedback controller has been proposed to synchronize the complex network with non-identical nodes and free coupling matrix. Further on, pinning control method has been used to solved the synchronization problem of complex networks with different kinds of nodes in [24].

On the other hand, in most studies of complex networks, it is assumed to have constant outer connection matrix. Since the connection topology plays a key role in forming synchronization behaviors of a complex dynamical networks, time-varying connection topology is more realistic and covers more situations in practice. Assuming that the connection topology changes continuously with time and that the upper bound of change rate is known, various synchronization criteria and methods have been given[25–27]. However, the connection structure of a network may jump suddenly in the real world. For the switching topology is discontinuously fast-varying topology, so the problem of synchronization cannot be handled as general time varying topology. When the connection outer matrices can be simultaneously diagonalized, several synchronization criteria have been proposed [28–29]. Obviously the simultaneous diagonalization condition is very difficult to satisfy. Therefore, searching for relaxed conditions on outer connection matrices is very meaningful and challenging. By using the simultaneous triangularization asspumption, the synchronization problem for complex dynamical networks and switching topology has studied in [30], and several synchronization criteria have been established by means of constructing a common Lyapunov function, single Lyapunov and multiple Lyapunov functions, respectively. However, identical nodes are still assumed in this result. As far as we know, no synchronization criteria have been reported for dynamical networks with non-identical nodes and switching topology under the simultaneous triangularization assumption, which is motivates our work.

We will study the synchronization problem of a complex network with non-identical nodes and switching topology, neither an equilibrium for each isolated node nor a synchronization manifold is assumed to exist. Deviation equations are established by introducing the average dynamics of all nodes. Under the condition that the connection matrices can be simultaneously triangularized, a synchronization criterion in the sense of boundedness under arbitrary switchings is given by the common Lyapunov function method.

2 Model description and preliminaries

This section gives model description and some necessary preliminaries to derive the main results of the paper. Consider a dynamical network with switching topology which consists of N non-identical nodes, the state equations of the network are given below:

$$\dot{x}_i(t)=f_i(x_i(t))+c\sum_{j=1}^{N} l_{ij}^{\sigma(t)} Hx_j(t),\ i=1,\cdots,N, \tag{1}$$

where $x_i=(x_{i1},\cdots,x_{in})^{\mathrm{T}}\in\mathbb{R}^n$ is the state variable of node i; $H\in\mathbb{R}^{n\times n}$ is the inner-coupling matrix; $f_i(\cdot)$ are continuously differentiable with Jacobian Df_i; $\sigma:[0,\infty)\to Q=\{1,2,\cdots,m\}$ is a switching signal; $c>0$ is the coupling strength; for each $k\in Q$, the matrix $L_k=(l_{ij}^k)_{N\times N}$ is the outer coupling matrix representing topology structure of the network. Suppose that the diagonal elements of L_k satisfy $l_{ii}^k=-\sum\limits_{\substack{j=1\\ j\neq i}}^{N} l_{ij}^k$.

Consider the case where there is no common solution to the isolated non-identical nodes, the average state trajectory is selected as target of synchronization. Now, we will give the definition of the average node dynamics

$$\bar{f}(x)=\frac{1}{N}\sum_{i=1}^{N} f_i(x). \tag{2}$$

The average state trajectory is

$$s(t)=\frac{1}{N}\sum_{i=1}^{N} x_i(t). \tag{3}$$

Define the error vector by $e_i = x_i(t) - s(t)$, we can get the following error dynamical systems:

$$\begin{aligned}\dot{e}_i = {} & f_i(x_i) + c\sum_{j=1}^{N} l_{ij}^{\sigma} H x_j - \\ & \frac{1}{N}\sum_{k=1}^{N}(f_k(x_k) + c\sum_{j=1}^{N} l_{kj}^{\sigma} H x_j) = \\ & f_i(s+e_i) - \frac{1}{N}\sum_{k=1}^{N} f_k(s+e_k) + c\sum_{j=1}^{N} l_{ij}^{\sigma} H e_j = \\ & f_i(s) + c\sum_{j=1}^{N} l_{ij}^{\sigma} H e_j + \int_0^1 Df_k(s+\tau e_i)e_i \mathrm{d}\tau - \\ & \frac{1}{N}\sum_{k=1}^{N}(f_k(s) + \int_0^1 Df_k(s+\tau e_k)e_k \mathrm{d}\tau) = \\ & D\bar{f}(s)e_i + c\sum_{j=1}^{N} l_{ij}^{\sigma} H e_j + \\ & \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i \mathrm{d}\tau - \\ & \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k \mathrm{d}\tau + f_i(s) - \bar{f}(s).\end{aligned} \tag{4}$$

Let $e = (e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \cdots, e_N^{\mathrm{T}})^{\mathrm{T}}$, then we can get the error system of the network

$$\begin{aligned}&\dot{e} = \\ &(I_N \otimes D\bar{f}(s) + cL^{\sigma} \otimes H)e + g(t,e) - \frac{1}{N}\cdot \\ &\begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau \end{pmatrix} e + \\ &\begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s) \end{pmatrix},\end{aligned} \tag{5}$$

where $g(t,e) = (g_1(t,e_1)^{\mathrm{T}}, \cdots, g_N(t,e_N)^{\mathrm{T}})^{\mathrm{T}}$, and

$$g_i(t,e_i) = \int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))e_i \mathrm{d}\tau. \tag{6}$$

To get the result of this paper, we need the following assumptions.

Assumption 1 There exists a constant $\theta > 0$, such that

$$\|g_i(t,e_i)\| \leqslant \theta \|e_i\|. \tag{7}$$

Assumption 2 There exists a unitary matrix $\Phi = (\phi_{ij})_{N\times N}$, with $\Phi^{-1} = \Phi^{\mathrm{T}} = (\psi_{ij})_{N\times N}$, which makes

$$\Phi^{\mathrm{T}} L_k \Phi = \begin{bmatrix} \hat{l}_{11}^k & \hat{l}_{12}^k & \cdots & \hat{l}_{1N}^k \\ 0 & \hat{l}_{22}^k & \cdots & \hat{l}_{2N}^k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{l}_{NN}^k \end{bmatrix} = \hat{L}_k, \tag{8}$$

where $\hat{l}_{ii}^k = \lambda_i^k$ the eigenvalues of L_k, Φ_i is the ith column of the Φ with $\Phi_1 = (\frac{1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}})^{\mathrm{T}}$ and $\lambda_1 = 0$.

Compared with the simultaneous diagonalization assumption, assumption Eq.(8) is much weaker. To simplify Eq.(5), let us take the coordinate transformation

$$w = (w_1^{\mathrm{T}}, w_2^{\mathrm{T}}, \cdots, w_N^{\mathrm{T}})^{\mathrm{T}} = (\Phi^{\mathrm{T}} \otimes I_n)e, \tag{9}$$

system (5) can be transformed into

$$\begin{aligned}&\dot{w} = (\Phi^{\mathrm{T}} \otimes I_N)\dot{e} = \\ &(I_N \otimes D\bar{f}(s) + c\hat{L}_{\sigma} \otimes H)w + (\Phi^{\mathrm{T}} \otimes I_N)\times \\ &g(t,e) - \frac{1}{N}(\Phi^{\mathrm{T}} \otimes I_N)\cdot \\ &\begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau \end{pmatrix}\cdot \\ &(\Phi^{\mathrm{T}} \otimes I_N)w + (\Phi^{\mathrm{T}} \otimes I_N)\begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s) \end{pmatrix}.\end{aligned} \tag{10}$$

Let

$$M = \begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau \end{pmatrix},$$

$$\mu(t) = \begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s) \end{pmatrix}. \tag{11}$$

Then Eq.(10) become

$$\begin{aligned}\dot{w} = {} & (I_N \otimes D\bar{f}(s) + c\hat{L}_{\sigma} \otimes H)w + (\Phi^{\mathrm{T}} \otimes I_N) \times \\ & g(t,e) - \frac{1}{N}(\Phi^{\mathrm{T}} \otimes I_N) \times M \times (\Phi^{\mathrm{T}} \otimes I_N)w + \\ & (\Phi^{\mathrm{T}} \otimes I_N) \times \mu(t).\end{aligned} \tag{12}$$

Note that

$$\begin{aligned}M = {} & \sqrt{N}[(\Phi_1, 0, \cdots, 0) \otimes \int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau] + \\ & \sqrt{N}[(0, \Phi_1, \cdots, 0) \otimes \int_0^1 Df_2(s+\tau e_2)\mathrm{d}\tau] + \\ & \cdots + \sqrt{N}[(0, 0, \cdots, \Phi_1) \otimes \\ & \int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau],\end{aligned} \tag{13}$$

since Φ is a unitary matrix, we can get

$$\frac{1}{N}(\Phi^{\mathrm{T}} \otimes I_N)M(\Phi \otimes I_N) =$$

$$\frac{1}{\sqrt{N}}\begin{pmatrix}\phi_{11} & \phi_{12} & \cdots & \phi_{1N}\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}\otimes\int_0^1 Df_1(s+\tau e_1)\mathrm{d}\tau+$$

$$\frac{1}{\sqrt{N}}\begin{pmatrix}\phi_{21} & \phi_{22} & \cdots & \phi_{2N}\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}\otimes\int_0^1 Df_2(s+\tau e_1)\mathrm{d}\tau+$$

$$\cdots+\frac{1}{\sqrt{N}}\begin{pmatrix}\phi_{N1} & \phi_{N2} & \cdots & \phi_{NN}\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}\otimes$$

$$\int_0^1 Df_N(s+\tau e_N)\mathrm{d}\tau. \tag{14}$$

Substituting Eq.(14) into Eq.(10), we can get

$$\begin{aligned}\dot{w} =& (I_N\otimes D\bar{f}(s)+c\hat{L}_\sigma\otimes H)w+\\ &(\varPhi^{\mathrm{T}}\otimes I_N)g(t,e)-\begin{pmatrix}* & * & \cdots & *\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}w+\\ &(\varPhi^{\mathrm{T}}\otimes I_N)\mu(t).\end{aligned} \tag{15}$$

It is easy to see that the deviations e_i satisfy $\sum\limits_{i=1}^{N} e_i = 0$. Therefore, $w_1\equiv 0$, we only need to consider $w_2,\cdots,w_N$. So the component form of Eq.(15) is

$$\begin{cases}\dot{w}_2 =\\ (D\bar{f}(s)+\lambda_2^\sigma cH)w_2+\hat{l}_{23}^\sigma cHw_3+\cdots+\\ \hat{l}_{2N}^\sigma cHw_N+\psi_{21}g_1+\psi_{22}g_2+\cdots+\\ \psi_{2N}g_N+(\varPhi_2^{\mathrm{T}}\otimes I_n)\mu(t),\\ \dot{w}_3 =\\ (D\bar{f}(s)+\lambda_3^\sigma cH)w_3+\hat{l}_{34}^\sigma cHw_4+\cdots+\\ \hat{l}_{3N}^\sigma cHw_N+\psi_{31}g_1+\psi_{32}g_2+\cdots+\\ \psi_{3N}g_N+(\varPhi_3^{\mathrm{T}}\otimes I_n)\mu(t),\\ \quad\vdots\\ \dot{w}_{N-1} =\\ (D\bar{f}(s)+\lambda_{N-1}^\sigma cH)w_{N-1}+\hat{l}_{(N-1)N}^\sigma cHw_N+\\ \psi_{(N-1)1}g_1+\psi_{(N-1)2}g_2+\cdots+\psi_{(N-1)N}g_N+\\ (\varPhi_{N-1}^{\mathrm{T}}\otimes I_n)\mu(t),\\ \dot{w}_N =\\ (D\bar{f}(s)+\lambda_N^\sigma cH)w_N+\psi_{N1}g_1+\psi_{N2}g_2+\cdots+\\ \psi_{NN}g_N+(\varPhi_N^{\mathrm{T}}\otimes I_n)\mu(t).\end{cases} \tag{16}$$

There are several ways to check simultaneous triangularizability. The most useful method is to test nilpotency of $\{A_k\}_{LA}$, the Lie algebra generated by the connection matrices $\{A_k\}$[30].

3 Synchronization criterion under arbitrary switchings

In this section, we will develop a synchronization criterion in the sense of boundness under arbitrary switchings. The following lemma is needed to derive our main result.

Lemma 1[18] Suppose there exist a strictly positive definite matrix $P(t)\in PC^1_{n\times n}$ and a constant $\delta>0$ such that the derivative of

$$V(x,t)=x^{\mathrm{T}}P(t)x,$$

along the trajectory of the system:

$$\dot{x}=f(x,t),\ x\in\mathbb{R}^n,\ t\in[0,\infty),$$

satisfies

$$\dot{V}\leqslant-\delta\|x\|^2,\ \text{if}\ \|x\|\geqslant g(t),$$

where $g(t)$ is a non-negative bounded function defined on $[0,\infty)$, for any $t>0$, let $Q_t=\{x|V(x,t)\leqslant\sup\limits_{y\in\varOmega,s\geqslant0}\{V(y,s)\}\}$ and

$$c=\overline{\lim_{t\to\infty}}(\max\{\|x\||x\in Q_t\}),$$

then $x(t)$ converges to the set $M=\{x|\ \|x\|\leqslant c\}$.

Theorem 1 Suppose assumption (7) and (8) hold, the network (1) synchronizes to the set

$$\varOmega=\{e|\|e\|\leqslant 8\sqrt{\frac{a}{b}}\frac{\beta}{(\tilde{\alpha}-2\gamma)}\overline{\lim_{t\to\infty}}\bar{\mu}(t)\},$$

under arbitrary switchings, if the following condition is satisfied:

1) There exist positive definite matrices $P_i(t)\in PC^1_{n\times n}$ and constants $a>0,\ b>0$ such that

$$\begin{cases}a\|x\|^2\leqslant x^{\mathrm{T}}P_i(t)x\leqslant b\|x\|^2,\ \forall t\in\mathbb{R}^+,\\ x\in\mathbb{R}^n,\ i=2,\cdots,N.\end{cases} \tag{17}$$

2) There exist constants $\alpha_i>0$ such that

$$\begin{cases}\dot{P}_i(t)+P_i(t)(D\bar{f}(s)+c\lambda_i^kH)+\\ (D\bar{f}(s)+c\lambda_i^kH)^{\mathrm{T}}P_i(t)+\alpha_iI_n<0,\\ i=2,\cdots,N,\ k=1,\cdots,m,\end{cases} \tag{18}$$

and let $\bar{\mu}(t)=\|\mu(t)\|$ be bounded. Define

$$\bar{\alpha}_i=\alpha_i-2\theta\|P_i\|\sum_{k=1}^{N}|\psi_{ik}\varPhi_{ki}|,$$

$$\upsilon_{ij}=2\max_{1\leqslant q\leqslant m}|\hat{l}_{ij}^qc|\delta_i\|P_iH\|+ 2\theta\delta_i\|P_i\|\sum_{k=1}^{N}|\psi_{ik}\phi_{kj}|,$$

$$\delta_i=\frac{N-2}{2\bar{\alpha}_i}\sum_{p<i}\frac{\upsilon_{pi}^2}{\bar{\alpha}_p}+1,\ \delta_2=1,\ \bar{P}_i=\delta_iP_i,$$

$$\tilde{\alpha}=\min_{1\leqslant i\leqslant N}(\bar{\alpha}_i-(N-2)\sum_{j>i}\frac{\upsilon_{ji}^2}{\bar{\alpha}_j}),$$

$$\beta=(\sum_{i=2}^{N}(\delta_i\|P_i\|^2))^{\frac{1}{2}},$$

where $i=2,\cdots,N$, if $\bar{\alpha}_i>0,\ \bar{\alpha}_i-(N-2)\sum\limits_{j>i}\dfrac{\upsilon_{ji}^2}{\bar{\alpha}_j}>$

$0,\ \gamma > 0$ and $\tilde{\alpha} - 2\gamma > 0$.

Proof Under the coordinate transformation

$$w = (w_1^{\mathrm{T}}, w_2^{\mathrm{T}}, \cdots, w_N^{\mathrm{T}}) = (\Phi^{-1} \otimes I_n)e,$$

we have expression (16). Choose $V_2(w_2) = w_2^{\mathrm{T}} \bar{P}_2 w_2$. When the q-th subnetwork is connected, we can get

$$\begin{aligned}\dot{V}_2 = {} & w_2^{\mathrm{T}}(\dot{P}_2 + (D\bar{f} + \lambda_2^q cH)^{\mathrm{T}} P_2 + P_2(D\bar{f} + \\ & \lambda_2^q cH))w_2 + 2\hat{l}_{23}^q c w_2^{\mathrm{T}} P_2 H w_3 + \cdots + \\ & 2\hat{l}_{2N}^q c w_2^{\mathrm{T}} P_2 H w_N + 2w_2^{\mathrm{T}} P_2 \sum_{p=1}^{N} \psi_{2p} g_p(t, e_p) + \\ & 2w_2^{\mathrm{T}} P_2 (\Phi_2^{\mathrm{T}} \otimes I_N)\mu(t). \end{aligned} \tag{19}$$

Note that the last term of Eq.(19) satisfies

$$2w_2^{\mathrm{T}} P_2 (\Phi_2^{-1} \otimes I_N)\mu(t) \leqslant 2\|P_2\|\|w_2\|\bar{\mu}(t). \tag{20}$$

In view of $e_j = \phi_{j1}w_1 + \cdots + \phi_{jN}w_N$, we can get

$$\|\sum_{p=1}^{N} \psi_{ip} g_p(t, e_p)\| \leqslant \theta \sum_{j=1}^{N} (\sum_{k=1}^{N} |\psi_{ik}\phi_{kj}|)\|w_j\|. \tag{21}$$

Using Eqs.(7)–(8)(20)–(21), it turns out that

$$\begin{aligned}\dot{V}_2 \leqslant {} & -\alpha_2\|w_2\|^2 + 2|\hat{l}_{23}^q c|\|P_2 H\|\|w_2\|\|w_3\| + \\ & \cdots + 2|\hat{l}_{2N}^q c|\|P_2 H\|\|w_2\|\|w_N\| + \\ & 2\theta\|P_2\| \sum_{j=1}^{N} (\sum_{k=1}^{N} |\psi_{2k}\phi_{kj}|)\|w_2\|\|w_j\| + \\ & 2\|P_2\|\|w_2\|\bar{\mu}(t) \leqslant \\ & -(\alpha_2 - 2\theta\|P_2\| \sum_{k=1}^{N} |\psi_{2k}\Phi_{k2}|)\|w_2\|^2 + \\ & \sum_{j=3}^{N} 2 \max_{1\leqslant q\leqslant m} |\hat{l}_{2j}^q c|\delta_2\|P_2 H\|\|w_2\|\|w_j\| + \\ & \sum_{j=3}^{N} 2\theta\delta_2\|P_2\| \sum_{k=1}^{N} |\psi_{2k}\phi_{kj}|\|w_2\|\|w_j\| + \\ & 2\|P_2\|\|w_2\|\bar{\mu}(t) \leqslant \\ & -\bar{\alpha}_2\|w_2\|^2 + \sum_{j=3}^{N} \upsilon_{2j}\|w_2\|\|w_j\| + \\ & 2\|P_2\|\|w_2\|\bar{\mu}(t). \end{aligned} \tag{22}$$

On the basis of Young's inequality, we have

$$\begin{aligned}\|w_2\|\|w_j\| \leqslant {} & \frac{\bar{\alpha}_2}{2\upsilon_{2j}(N-2)}\|w_2\|^2 + \\ & \frac{\upsilon_{2j}(N-2)}{2\bar{\alpha}_2}\|w_j\|^2. \end{aligned} \tag{23}$$

Thus lead to

$$\begin{aligned}\dot{V}_2 \leqslant {} & -\frac{1}{2}\bar{\alpha}_2\|w_2\|^2 + \sum_{j=3}^{N} \frac{\upsilon_{2j}^2(N-2)}{2\bar{\alpha}_2}\|w_j\|^2 + \\ & 2\|P_2\|\|w_2\|\bar{\mu}(t). \end{aligned} \tag{24}$$

By analogy, after having $V_{i-1}(w_{i-1})$, let define $V_i(w_i) = w_i^{\mathrm{T}} \bar{P}_i w_i$, then, we have

$$\begin{aligned}\dot{V}_i \leqslant {} & -(\frac{(N-2)}{2} \sum_{p<i} \frac{\upsilon_{pi}^2}{\bar{\alpha}_p} + \frac{1}{2}\bar{\alpha}_i)\|w_i\|^2 + \\ & \sum_{j=2, j\neq i}^{N} \frac{\upsilon_{ij}^2(N-2)}{2\bar{\alpha}_i}\|w_j\|^2 + 2\delta_i\|P_i\|\|w_i\|\bar{\mu}(t). \end{aligned} \tag{25}$$

Construct the common Lyapunov function $V(w) = \sum_{i=2}^{N} V_i(w_i)$, we have

$$\begin{aligned}\dot{V} \leqslant {} & -\frac{1}{2}\sum_{i=2}^{N} (\bar{\alpha}_i - (N-2)\sum_{j>i} \frac{\upsilon_{ji}^2}{\bar{\alpha}_j})\|w_i\|^2 + \\ & 2\sum_{i=2}^{N} \delta_i\|P_i\|\|w_i\|\bar{\mu}(t) \leqslant \\ & -\frac{1}{2}\tilde{\alpha}\|w\|^2 + 2\bar{\mu}(t)\|w\|(\sum_{i=2}^{N} (\delta_i\|P_i\|)^2)^{\frac{1}{2}} = \\ & -\frac{1}{2}\tilde{\alpha}\|w\|^2 + 2\beta\bar{\mu}(t)\|w\|, \end{aligned} \tag{26}$$

if $\|w\| \geqslant \dfrac{4\beta}{\tilde{\alpha} - 2\gamma}\bar{\mu}(t)$, then

$$\dot{V} \leqslant -\gamma\|w\|^2.$$

By Lemma 1, it is easy to see that network (1) synchronizes to the set, which completes the proof.

Remark 1 When $\overline{\lim\limits_{t\to\infty}}\,\bar{\mu}(t) = 0$, we have asymptotic synchronization in the classical sense.

Remark 2 In [17], it only consider the simple case that all non-identical nodes have the same equilibrium. In [18], the bounded synchronization of complex networks with non-identical nodes was studied, but it's results are based on the assumption that the connection matrix is symmetric and can be diagonalized. In addition, [17] and [18] did not considered the general case that the complex networks have switching topology, in [30], synchronization of complex networks with switching topology was studied, and several synchronization criteria have been established, but unfortunately, identical nodes are still assumed in this result. Compared with the existing results of [17–18, 30], the results we obtained have three distinct features. First of all, we consider the dynamical networks with non-identical nodes where neither an equilibrium for each isolated node nor a synchronization manifold exists. Secondly, we consider the dynamical networks with switching topology structure, and the different outer coupling configuration can be switched arbitrarily. Thirdly, the result of this paper is based on the assumption that all the connection outer matrices are simultaneous triangularized which is much weaker than the assumption of simultaneous diagonalization. Therefore, our result covers more general cases of the dynamical networks in the real world, and it is more extensive than the existing results.

4 Example

Consider the following dynamical network with 3 non-identical nodes:

$$\dot{x}_i = A_i x_i + g(x_i) + \sum_{j=1}^{N} l_{ij}^{\sigma} H x_j,\ i = 1, 2, 3,\ \sigma \in [1, 2], \tag{27}$$

where

$$L_1 = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix},\ L_2 = \begin{bmatrix} -4 & 2 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2.4 & -0.4 & 0.4 \\ 0.4 & -2.3 & -0.4 \\ -0.3 & 0.4 & -2.2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1.6 & 0.4 & -0.4 \\ -0.4 & -1.7 & 0.4 \\ 0.3 & -0.4 & -1.8 \end{bmatrix},\ H = \mathrm{diag}\{2, 2, 2\},$$

$$x_0 = (1, 0.5, -1, 1, -2, -1, -1, 1.5, 1)^{\mathrm{T}},$$

and

$$g(x_i) = (-2\sin(\frac{\pi x_{i1}}{3.2} + \pi), -2\sin(\frac{\pi x_{i2}}{3.2} + \pi), -2\sin(\frac{\pi x_{i3}}{3.2} + \pi))^{\mathrm{T}}.$$

Applying Theorem 1 we know synchronization in the sense of boundedness under arbitrary switching is achieved. Simulation results are depicted in Fig.1.

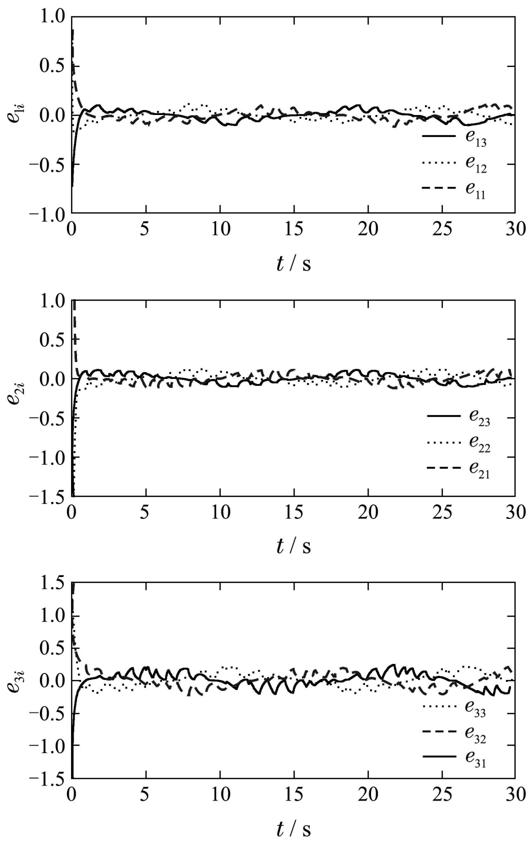

Fig. 1 The synchronization errors of the switched network

Form Fig.1, it is easy to see that the state of error system is very large at the initial time, but after a while, it converge to a bounded area quickly, and the state trajectory never leave the band region from then on, so the complex network is achieved bounded synchronization, which verify the validity of the Theorem 1.

5 Conclusion

We have studied the synchronization problem of a complex network with non-identical nodes and switching topology. Based on constructing common Lyapunov function, a criterion for global synchronization under arbitrary switchings is presented. To reduce the conservative and expand the scope of application, the assumption is relaxed to simultaneous triangularization of the connection outer matrices, which is much weaker than the assumption of simultaneous diagonalization required by some existing papers. This paper only discuss the arbitrary switchings case. How to synchronize this kind of complex network by design switching laws is a challenging and difficult problem which deserve to do future research.

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