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切换线性系统的聚合优化

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摘要: 分析了切换线性系统的稳定性和优化问题, 提出了一个Armijo步长优化的共轭梯度算法来寻找适当代价函数下的优化切换时间点集. 为了确保优化切换路径是可压缩的, 提出了代价函数需要满足的受限表达式. 设计了几种优化分段状态反馈切换律来搜索聚合系统的最优切换路径, 同时这些切换路径就是对应原始切换线性系统的次优化切换路径. 最后, 一个实例演示了不同切换律下的切换策略和优化代价.

关键词: 稳定性; 切换线性系统; 聚合系统; 分段状态反馈切换律; 可压缩性

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Optimal aggregation of switched linear systems

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Abstract: This paper investigates the stability and optimization problems for switched linear systems. An optimal conjugate gradient algorithm with Armijo steps is presented to search the optimal time instants under proper cost functions. To ensure that the optimal switching paths are contractive, a constrained expression of those cost functions is established. Some optimal pathwise state-feedback switching laws are designed to search the optimal switching paths of aggregated systems. The switching paths are sub-optimal switching paths of the original switched linear systems. Finally, an example is provided to demonstrate the switching strategies and optimal costs under different switching laws.

Key words: stability; switched linear systems; aggregated systems; pathwise state-feedback switching; contractivity

1 Introduction

A switched linear system has some subsystems and a rule that orchestrates the switching between them. Designing a switching law to make the switched linear system asymptotically stable is an important problem[1]. Some necessary and sufficient conditions for asymptotic stability of switched linear systems were described in [2–4]. A computation method of Lyapunov function can be found in [5]. A converse Lyapunov theorem was presented in [6].

Optimal switching is another direction. An optimal switching law not only stabilizes the switched linear system but also minimizes the cost functions. Some discrete-time switched linear systems have been studied, such as the discrete-time linear quadratic regulation problem for switched linear systems based on dynamic programming approach[7–9]. For the continuous-time switched linear system, an optimal method based on the differentiation of the cost function has been surveyed in [10–11]. However, the method encounters computational difficulties when the number of switches grows.

The maximum principle and Hamiltonian condition were often used for optimization of hybrid systems and switched systems[12]. A feedback switching law was designed to optimize the rate of convergence of switched systems[13]. We have designed a pathwise state-feedback switching law to stabilize the switched linear system without optimization[14]. When the switching sequence is preassigned or has a finite length, corresponding methods of minimizing a performance index over an infinite time horizon were mentioned in [15–16]. However, the number of switches is still large, so the computation is a heavy burden.

In this work, optimal switching time instants of contractive switching path and cost gradient expression are analyzed. An optimal conjugate gradient searching algorithm with Armijo steps is presented over finite time intervals. Some optimal pathwise state-feedback switching laws based on the switched Lyapunov function derived from Riccati mapping approach are designed to minimize the cost of the aggregated system, and they are sub-optiaml paths of the original switched linear system. Finally, an example demonstrates a non-optimal and some sub-optimal switching processes.

2 Pathwise state-feedback switching and aggregation

An expression of the continuous-time switched linear system is given by:

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ x(t_0) = x_0, \tag{1}$$

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where $x(t)$ is the continuous state, x_0 is the initial state, $\sigma(t) \in M = \{1, \cdots, m\}$ is the switching law and $A_1, \cdots, A_m$ are real constant matrices.

When switched linear system (1) is not consistently stabilizable, there is not a single switching path that can make the total state space $\mathbb{R}^n$ contractive. However, it is still possible that a switching path makes a subset of state space contractive.

Definition 1 A switching path $\theta : [0, s) \rightarrow M$ is contractive on a subset of state space Ω if it is well-defined and $||\phi(s; 0, x, \theta)|| < ||x||$, $\forall x \in \Omega$, where $\phi(t; 0, x, \theta)$ is the state of linear switched system (1) under switching law θ with initial state x, and $|| \cdot ||$ is a given norm.

There exist a natural number k and a real number $\mu \in (0, 1)$, such that

$$||\phi(s_i; 0, x, \theta_i)|| \leqslant \mu ||x||,$$
$$\sum_{i=1}^{k} \Omega_i = \mathbb{R}^n, \ \forall x \in \Omega_i, \text{ for } i = 1, \cdots, k,$$

then the following switching law asymptotically stabilizes switched linear system (1).

$$\begin{cases} i_j = \arg\{x_j \in \Omega_i\}, & i \in \{1, 2, \cdots, k\}, \\ t_{j+1} = t_j + s_{i_j}, & \\ \sigma_x(t) = \theta_{i_j}(t - t_j), & \forall t \in [t_j, t_{j+1}), \\ x_{j+1} = \phi(s_{i_j}; 0, x_j, \theta_{i_j}), & j = 0, 1, 2, \cdots, \end{cases} \tag{2}$$

where $\theta_i : [0, s_i) \mapsto M$ is well-defined and contractive on Ω_i. Switching law $\sigma_x(t)$ in (2) is called a pathwise state-feedback switching law, denoted by $\bigwedge_{i=1}^{k} \theta_i^{\Omega_i}$, which is the concatenation of switching paths $\{\theta_i\}_{i=1}^k$ through state-space partitions $\{\Omega_i\}_{i=1}^k$. Note also that each switching path θ_i corresponds to a state transition matrix G_i with the property that

$$\phi(s_i; 0, x, \theta_i) = G_i x, \text{for } \forall x \in \Omega_i.$$

The switching mechanism in (2) is mixed time-driven and state-feedback, and the above pathwise state-feedback switching law is universal and well-defined.

Lemma 1 [14] Switching law $\bigwedge_{i=1}^{k} \theta_i^{\Omega_i}$ asymptotically stabilizes switched linear system (1) if and only if the discrete-time linear system

$$z(t+1) = G_i z(t), \ z(t) \in \Omega_i, \ i = 1, 2, \cdots, k \tag{3}$$

is asymptotically stable.

For clarity, we term discrete-time switched linear system (3) as the aggregated system of switched linear system (1) w.r.t. $\{\theta_i, \Omega_i\}_{i=1}^k$.

Lemma 2 [14] Suppose that V is a continuous and positive definite function defined on $\mathbb{R}^n$, and θ_i are switching paths defined over $[0, s_i)$ for $i = 1, \cdots, k$. Then, switched linear system (1) is asymptotically stabilizable if

$$\min_{i=1}^{k} V(\phi(s_i; 0, x, \theta_i)) < V(x), \ \forall x \in \mathbb{R}^n, x \neq 0. \tag{4}$$

Condition (4) implies that aggregated system (3) is asymptotically stable, so switched linear system (1) is also asymptotically stabilizable. In this case, Let

$$\begin{cases} \hat{\Omega}_1 = \{x : V(\phi(s_1; \star)) = \min_{i=1}^{k} V(\phi(s_i; \star))\}, \\ \hat{\Omega}_j = \{x : V(\phi(s_j; \star)) = \min_{i=1}^{k} V(\phi(s_i; \star))\} - \\ \quad \bigcup_{l=1}^{j-1} \hat{\Omega}_l, \ j = 2, \cdots, k, \end{cases} \tag{5}$$

where $V(\phi(s_i; \star) = V(\phi(s_i; 0, x, \theta_i))$. Using Lemma 2, we design switching paths θ_i firstly, and then obtain state-space partitions $\hat{\Omega}_i$ by (5), instead of designing both of them at the same time.

3 Optimal contractive paths over finite time intervals

For simplicity, let the switching time instants series of θ_i be

$$\pi_1(\theta_i) = \{t_0 = 0, t_1, \cdots, t_l, t_{l+1} = s_i\}$$

and fixed switching index series be

$$\pi_2(\theta_i) = \{q_0, q_1, \cdots, q_l\}, \text{for } q_i \in M.$$

Define a cost function over $[0, s_i)$ with $x \in \Omega_i$ by

$$J(x, \theta_i) = g(x(s_i)) + \int_0^{s_i} x^{\mathrm{T}}(t) Q_{\theta_i} x(t) \mathrm{d}t, \tag{6}$$

where $g(x(s_i)) = x^{\mathrm{T}}(s_i) K x(s_i)$, Q_i and K are positive definite matrices.

If there exists a path θ_{ix}° or θ_i° in short and

$$J(x, \theta_i^{\circ}) = \inf_{\theta \in S_{[0, s_i)}} J(x, \theta), \ x \in \Omega_i, \tag{7}$$

where $\pi_2(\theta_i^{\circ}) = \pi_2(\theta) = \pi_2(\theta_i)$ and S is an admissible switching path set, then θ_i° is an optimal path over $[0, s_i)$ on Ω_i and depends on initial state x.

To minimize J, the gradient method is used. For this, we define some auxiliary functions by $p_i(t) : [t_i, t_{i+1})$ as

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = -2Q_{q_i} x(t) - A_{q_i}^{\mathrm{T}} p_i(t), \ p_i(t_{i+1}) = 0, \tag{8}$$

$$p(t) = p_i(t) + \Phi_i^{\mathrm{T}}(t_{i+1}, t_i) p(t_{i+1}), \ p(s_i) = 0, \tag{9}$$

where Φ_i is the transition matrix of subsystem $\dot{x} = A_{q_i} x$ over $[t_i, t_{i+1})$.

Theorem 1 If θ_i is a contractive switching path through Ω_i with $\pi_1(\theta_i) = \{0, t_1, \cdots, t_l, t_{l+1} = s_i\}$ and $\pi_2(\theta_i) = \{q_0, q_1, \cdots, q_l\}$, then the gradient of $J(x_0, \theta_i)$ is

$$\frac{\mathrm{d}J(\bar{t})}{\mathrm{d}\bar{t}} = [\frac{\mathrm{d}J(\bar{t})}{\mathrm{d}t_1} \ \cdots \ \frac{\mathrm{d}J(\bar{t})}{\mathrm{d}t_i} \ \cdots \ \frac{\mathrm{d}J(\bar{t})}{\mathrm{d}t_l}]^{\mathrm{T}},$$
$$i = 1, \cdots, l,$$

where $\bar{t} = [t_1 \ \cdots \ t_l]^{\mathrm{T}}$, initial state $x_0 \in \Omega_i$ and

$$\begin{aligned} \frac{\mathrm{d}J(\bar{t})}{\mathrm{d}t_i} = & x^{\mathrm{T}}(t_i)(Q_{q_{i-1}} - Q_{q_i})x(t_i) + (p^{\mathrm{T}}(t_i) + \\ & 2x^{\mathrm{T}}(s_i) K \Phi_l(s_i, t_l) \cdots \Phi_i(t_{i+1}, t_i)) \cdot \\ & (A_{q_{i-1}} - A_{q_i}) x(t_i). \end{aligned} \tag{10}$$

Proof We decompose the expression of J in (6) as

$$J(\bar{t}) = g(x(s_i)) + \int_0^{t_1} L_0(x) \mathrm{d}t + \cdots + \int_{t_l}^{s_i} L_l(x) \mathrm{d}t.$$

Let $x(t) + \triangle x(t)$ be the trajectory at switching time $t_i + \triangle t_i$ and $\hat{t} = [t_1, \cdots, t_{i-1}, t_i + \triangle t_i, t_{i+1}, t_l]$. When

$t \leqslant t_i$, $x(t)$ is independent of t_i and $\triangle x(t) = 0$. The gradient of J at t_i is

$$\frac{\mathrm{d}J(\bar{t})}{\mathrm{d}t_i} = 2x^{\mathrm{T}}(s_i)K\frac{dx(s_i)}{\mathrm{d}t_i} + x^{\mathrm{T}}(t_i)(Q_{q_{i-1}} - Q_{q_i}) \times x(t_i) + \int_{t_i+\triangle t_i}^{t_{i+1}} \frac{\partial L_i(x)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t_i}\mathrm{d}t + \cdots + \int_{t_l}^{s_i} \frac{\partial L_l(x)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t_i}\mathrm{d}t. \tag{11}$$

When $t \in [t_i, t_{i+1})$ and $i \in \{1, \cdots, l\}$, we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t_i} = \Phi_i(t, t_i)(A_{q_{i-1}} - A_{q_i})x(t_i). \tag{12}$$

Let

$$\frac{\mathrm{d}J_i}{\mathrm{d}t_i} = \int_{t_i}^{t_{i+1}} \frac{\partial L_i(x(t))}{\partial x}\Phi_i(t, t_i)(A_{q_{i-1}} - A_{q_i})x(t_i)\mathrm{d}t. \tag{13}$$

We define some auxiliary functions by

$$\hat{p}_i^{\mathrm{T}}(t_i) = \int_{t_i}^{t_{i+1}} \frac{\partial L_i(x(t))}{\partial x}\Phi_i(t, t_i)\mathrm{d}t,\ \hat{p}_i(t_{i+1}) = 0.$$

Its derivative at t_i is

$$\dot{\hat{p}}_i^{\mathrm{T}}(t_i) = -\frac{\partial L_i(x(t_i))}{\partial x} - \int_{t_i}^{t_{i+1}} \frac{\partial L_i(x(t))}{\partial x}\Phi_i(t, t_i)\mathrm{d}t = -2x^{\mathrm{T}}(t_i)Q_i - \hat{p}_i^{\mathrm{T}}(t_i)A_{q_i}. \tag{14}$$

From the expression of equations (8) and (14), we know $\hat{p}_i(t_i) = p_i(t_i)$. Rearranging equation (13), we have

$$\frac{\mathrm{d}J_i}{\mathrm{d}t_i} = p_i^{\mathrm{T}}(t_i)(A_{q_{i-1}} - A_{q_i})x(t_i). \tag{15}$$

Using the differential chain rule and (12), we get

$$\frac{\mathrm{d}x(s_i)}{\mathrm{d}t_i} = \Phi_l(s_i, t_l)\cdots\Phi_i(t_{i+1}, t_i)(A_{q_{i-1}} - A_{q_i})x(t_i), \tag{16}$$

$$\frac{\mathrm{d}J_j}{\mathrm{d}t_i} = \int_{t_j}^{t_{j+1}} \frac{\partial L_j(x(t))}{\partial x}\frac{\partial x(t)}{\partial x(t_j)}\cdots\frac{\partial x(t_{i+1})}{\partial x(t_i)}\mathrm{d}t = p_j^{\mathrm{T}}(t_j)\Phi_{j-1}(t_j, t_{j-1})\cdots\Phi_i(t_{i+1}, t_i) \times (A_{q_{i-1}} - A_{q_i})x(t_i), \tag{17}$$

where $j = i+1, \cdots, l$. Putting (13), (16) and (17) into (11) and using (9), we obtain (10). This proof is completed.

To search the optimal switching time instants of θ_i^{o}, we present an optimal conjugate gradient algorithm based on Armijo steps as follows:

Step 1 Set

$$j = 1,\ \tau(1) = [t_1^1\ \ \cdots\ \ t_l^1]^{\mathrm{T}} = [t_1\ \ \cdots\ \ t_l]^{\mathrm{T}}$$

and error $\epsilon > 0$. The initial state is x_0 and $p(s_i) = 0$.

Step 2 Compute $x_{q_i}^j(\cdot)$ and $p^j(\cdot)$ for $i = 0, \cdots, l$. Using Theorem 1, we can get gradient $\nabla J(\tau(j))$. If $||\nabla J(\tau(j))|| < \epsilon$, then $\tau(j)$ is an optimal time instants vector and stop, otherwise go to the next step.

Step 3 Set $\tau(j+1) = \tau(j) + \alpha_j d_j$, where

$$d_j = \begin{cases} -\nabla J(\tau(j)), & j = 1, \\ -\nabla J(\tau(j)) + \dfrac{||\nabla J(\tau(j))||^2}{||\nabla J(\tau(j-1))||^2}d_{j-1}, & j \geqslant 2, \end{cases}$$

and $\hat{i}, \hat{j}$ are positive integers, then $\alpha_j = v^{\hat{i}}$. Let $\tau(j+1) = [t_1^{j+1}\ \ \cdots\ \ t_l^{j+1}]^{\mathrm{T}}$ and $j = j+1$, then go back to Step 2. When $j = L$ and $||\nabla J(\tau(L))|| < \epsilon$, the optimal switching time instants vector is $\bar{t}^{\mathrm{o}} = \tau(L)$, then stop.

A question is that above optimal path θ_i^{o} might not be contractive. To ensure that θ_i^{o} is contractive, we need to find out the relationship between K and Q_i.

Theorem 2 If θ_i through Ω_i is a contractive path, then optimal path θ_i^{o} derived from (7) is also contractive on Ω_i when

$$\mu^2\left(\lambda^+(K) + \frac{\lambda^+(Q)(\mathrm{e}^{2||A||^+ s_i} - 1)}{2||A||^+}\right) < \lambda^-(K) + \frac{\lambda^-(Q)(1 - \mathrm{e}^{-2||A||^+ s_i})}{2||A||^+}, \tag{18}$$

where $\lambda^+(K)$ and $\lambda^-(K)$ are maximal and minimal eigenvalues of K, $\lambda^+(Q)$ and $\lambda^-(Q)$ are maximal and minimal eigenvalues of $\{Q_1, \cdots, Q_k\}$, and

$$||A||^+ = \max\{||A_0||, \cdots, ||A_m||\}.$$

Proof For $\forall t \in [0, s_i)$, the state $x(t)$ under arbitrary switching path θ_i^* with $\pi_2(\theta_i^*) = \pi_2(\theta_i)$ satisfies

$$\mathrm{e}^{-||A||^+(s_i - t)}||x^*(s_i)|| \leqslant ||x(t)|| \leqslant \mathrm{e}^{||A||^+(s_i - t)}||x^*(s_i)||, \tag{19}$$

where $x^*(s_i)$ is the terminal state under θ_i^*. We get the following equations from (19) as

$$\int_0^{s_i} x^{\mathrm{T}}(t)Q_{\theta_i^*}x(t)\mathrm{d}t \leqslant \lambda^+(Q)\int_0^{s_i} ||x(t)||^2\mathrm{d}t \leqslant \lambda^+(Q)||x^*(s_i)||^2\int_0^{s_i} \mathrm{e}^{2||A||^+(s_i - t)}\mathrm{d}t = \frac{\lambda^+(Q)(\mathrm{e}^{2||A||^+ s_i} - 1)}{2||A||^+}||x^*(s_i)||^2, \tag{20}$$

$$\int_0^{s_i} x^{\mathrm{T}}(t)Q_{\theta_i^*}x(t)\mathrm{d}t \geqslant \lambda^-(Q)\int_0^{s_i} ||x(t)||^2\mathrm{d}t \geqslant \lambda^-(Q)||x^*(s_i)||^2\int_0^{s_i} \mathrm{e}^{-2||A||^+(s_i - t)}\mathrm{d}t = \frac{\lambda^-(Q)(1 - \mathrm{e}^{-2||A||^+ s_i})}{2||A||^+}||x^*(s_i)||^2. \tag{21}$$

From initial state x, the optimal path θ_i^{o} has terminal state $x^{\mathrm{o}}(s_i)$ and the relationship expression is

$$x^{\mathrm{o}}(s_i)^{\mathrm{T}}Kx^{\mathrm{o}}(s_i) + \int_0^{s_i} x^{\mathrm{T}}(t)Q_{\theta_i^{\mathrm{o}}}x(t)\mathrm{d}t \leqslant x^{\mathrm{T}}(s_i)Kx(s_i) + \int_0^{s_i} x^{\mathrm{T}}(t)Q_{\theta_i}x(t)\mathrm{d}t. \tag{22}$$

Using (20) and (21), we have

$$x^{\mathrm{o}}(s_i)^{\mathrm{T}}Kx^{\mathrm{o}}(s_i) + \int_0^{s_i} x^{\mathrm{T}}(t)Q_{\theta_i^{\mathrm{o}}}x(t)\mathrm{d}t \geqslant \left(\lambda^-(K) + \frac{\lambda^-(Q)(1 - \mathrm{e}^{-2||A||^+ s_i})}{2||A||^+}\right)||x^{\mathrm{o}}(s_i)||^2, \tag{23}$$

$$x^{\mathrm{T}}(s_i)Kx(s_i) + \int_0^{s_i} x^{\mathrm{T}}(t)Q_{\theta_i}x(t)\mathrm{d}t \leqslant \left(\lambda^+(K) + \frac{\lambda^+(Q)(\mathrm{e}^{2||A||^+ s_i} - 1)}{2||A||^+}\right)||x(s_i)||^2 \leqslant \mu^2\left(\lambda^+(K) + \frac{\lambda^+(Q)(\mathrm{e}^{2||A||^+ s_i} - 1)}{2||A||^+}\right)||x||^2. \tag{24}$$

Putting (23) and (24) into (22), we obtain

$$(\lambda^-(K)+\frac{\lambda^-(Q)(1-\mathrm{e}^{-2||A||^+s_i})}{2||A||^+})||x^{\mathrm{o}}(s_i)||^2 \leqslant$$
$$\mu^2(\lambda^+(K)+\frac{\lambda^+(Q)(\mathrm{e}^{2||A||^+s_i}-1)}{2||A||^+})||x||^2.$$

When condition (18) holds, $||x^{\mathrm{o}}(s_i)|| < ||x||$ and θ_i^{o} is a contractive path. This proof is completed.

Corollary 1 Suppose that $K = aI$. Then all corresponding optimal paths θ_i^{o} of θ_i for $i = 1,\cdots,k$ are contractive if

$$a>\frac{\mu^2\lambda^+(Q)(\mathrm{e}^{2||A||^+s}-1)-\lambda^-(Q)(1-\mathrm{e}^{-2||A||^+s})}{2(1-\mu^2)||A||^+}, \tag{25}$$

where $s = \max\{s_1,\cdots,s_k\}$.

4 Optimization of the aggregated system

If $x, y \in \Omega_i$ and $x \neq y$, then $\theta_{ix}^{\mathrm{o}} = \theta_{iy}^{\mathrm{o}}$ might not be true. But for any $\epsilon > 0$, there exist $\delta > 0$ and a neighborhood of x denoted by $N(x,\delta)$ such that

$$||J(x,\theta_{ix}^{\mathrm{o}}) - J(y,\theta_{ix}^{\mathrm{o}})|| < \epsilon,\ \forall y \in N(x,\delta).$$

Under an admissible error ϵ of J, using θ_{ix}^{o} as an optimal path is reasonable for any $y \in N(x,\delta)$. It follows from the Finite Covering Theorem that there exist a natural number k_i and state $x_1,\cdots,x_{k_i}$ on Ω_i with unit norm such that $\bigcup\limits_{j=1}^{k_i} N(x_j,\delta) = H_1 \cap \Omega_i$, where H_1 is the unit sphere.

We define $\Omega_{i_j}^{\mathrm{o}}$ by $\lambda N(x_j,\delta)$ with $\lambda \neq 0$. There exists optimal paths $\theta_{i_j}^{\mathrm{o}}$ on $\Omega_{i_j}^{\mathrm{o}} \subseteq \Omega_i$ for $j = 1,\cdots,k_i$. J is radially invariant in the sense $\theta_{ix}^{\mathrm{o}} = \theta_{i\lambda x}^{\mathrm{o}}$ with property $J(\lambda x,\theta_i^{\mathrm{o}}) = \lambda^2 J(x,\theta_i^{\mathrm{o}})$, $\lambda \neq 0$. Then we have $k^{\mathrm{o}} = \sum\limits_{i=1}^{k} k_i \geqslant k$ numbers of θ_j^{o} on $\mathbb{R}^n$ for $j = 1,\cdots,k^{\mathrm{o}}$.

Using all those optimal paths θ_j^{o}, we have an aggregated system given by

$$x(t+1) = G_j^{\mathrm{o}}x(t),\ \forall x(t) \in \Omega_j^{\mathrm{o}}, \tag{26}$$

where $j \in \hat{M} = \{1,\cdots,k^{\mathrm{o}}\}$, $\exists i \in \{1,\cdots,k\}$, $\Omega_j^{\mathrm{o}} \subseteq \Omega_i$ and G_j^{o} is the state transition matrix of θ_j^{o}.

Since all optimal switching paths θ_j^{o} are contractive, aggregated system (26) is asymptotically stable under a pathwise state-feedback switching law $\bigwedge\limits_{j=1}^{k^{\mathrm{o}}}(\theta_j^{\mathrm{o}})^{\Omega_j^{\mathrm{o}}}$ defined by (2).

For aggregated system (26), we define an infinite horizon cost function by

$$J(x,\sigma,\tau_1,\cdots,\tau_n) = \int_0^{\infty} x^{\mathrm{T}}(t)Q_\sigma x(t)\mathrm{d}t + \sum_{j=1}^{n} T_j, \tag{27}$$

where

$$T_j = ax^{\mathrm{T}}(\tau_j)x(\tau_j),$$
$$0 = \tau_0 < \tau_1 < \cdots < \tau_n = \tau_{n+1} = +\infty,$$

and x is the initial state.

To design a switching law to achieve the minimal cost of (27), we set the ith running cost $L(x,i) = x^{\mathrm{T}}Q_i^{\mathrm{o}}x$ and

$$Q_i^{\mathrm{o}} = a(G_i^{\mathrm{o}})^{\mathrm{T}}G_i^{\mathrm{o}} + \int_0^{h_0^{\mathrm{o}}} \mathrm{e}^{A_{q_0}^{\mathrm{T}}t}Q_{q_0}\mathrm{e}^{A_{q_0}t}\mathrm{d}t + \cdots +$$
$$(\mathrm{e}^{A_{q_{l-1}}h_{l-1}^{\mathrm{o}}}\cdots\mathrm{e}^{A_{q_0}h_0^{\mathrm{o}}})^{\mathrm{T}}\int_0^{h_l^{\mathrm{o}}}\mathrm{e}^{A_{q_l}^{\mathrm{T}}t}Q_{q_l}\mathrm{e}^{A_{q_l}t}\mathrm{d}t \times$$
$$(\mathrm{e}^{A_{q_{l-1}}h_{l-1}^{\mathrm{o}}}\cdots\mathrm{e}^{A_{q_0}h_0^{\mathrm{o}}}), \tag{28}$$

where $h_j^{\mathrm{o}} = t_{j+1}^{\mathrm{o}} - t_j^{\mathrm{o}}$ for $j = 0,\cdots,l$. Then

$$L(x,i) = \inf_{\theta\in S_{[0,s_i)}}\{ax^{\mathrm{T}}(s_i)x(s_i) + \int_0^{s_i} x^{\mathrm{T}}(t)Q_{\theta_i}x(t)\mathrm{d}t\}.$$

The optimal cost function of (27) is defined by

$$J(x,\sigma^{\mathrm{o}},\tau_1^{\mathrm{o}},\cdots,\tau_n^{\mathrm{o}}) = \inf_{\sigma\in S_{[0,\infty)}}\sum_{t=0}^{\infty} L(\phi(t;0,x,\sigma),\sigma). \tag{29}$$

To approach the optimal cost in (29), we define a $\hat{k}$-step cost function by

$$V_{\hat{k}}(x)=\inf_{\sigma\in S_{[0,\hat{k}-1]}}\sum_{t=0}^{\hat{k}-1}L(\phi(t;0,x,\sigma),\sigma(t)),$$

where $\hat{k}$ is a natural number. We define a mapping

$$Z_i(P) = Q_i^{\mathrm{o}} + (G_i^{\mathrm{o}})^{\mathrm{T}}PG_i^{\mathrm{o}},$$

where Q_i^{o} is given by (28) and P is a positive definite matrix. The switched Riccati mapping is defined by

$$Z(Y) = \{Z_i(P_j),\ i = 1,\cdots,k^{\mathrm{o}},\ j = 1,\cdots,r\},$$

where $Y = \{P_1,\cdots,P_r\}$. Let a sequence of matrices be

$$Z_0 = \{0_{n\times n}\},\ Z_1 = \{Q_i^{\mathrm{o}}\},\ Z_j = Z(Z_{j-1}),$$

where $j = 2,\cdots$ and $i \in \hat{M}$.

Then we have

$$V_{\hat{k}}(x)=\min\{x^{\mathrm{T}}Px : P \in Z_{\hat{k}}\},\ \hat{k}=1,2,\cdots. \tag{30}$$

There exists a natural number $\hat{K}$ such that $V_{\hat{k}}(x)$ shown in (30) is a switched Lyapunov function of aggregated system (26) when $\hat{k} \geqslant \hat{K}$[8]. In this case, let $V_{\hat{k}+1}(x)$ be a switched Lyapunov function, we let the state-space partitions defined by (5) be

$$\hat{\Omega}_i^{\mathrm{o}} = \{x : \min_{P\in Z_{\hat{k}}} x^{\mathrm{T}}(Q_i^{\mathrm{o}} + (G_i^{\mathrm{o}})^{\mathrm{T}}PG_i^{\mathrm{o}})x =$$
$$\min_{j\in\hat{M},P\in Z_{\hat{k}}} x^{\mathrm{T}}(Q_j^{\mathrm{o}} + (G_j^{\mathrm{o}})^{\mathrm{T}}PG_j^{\mathrm{o}})x\} - \sum_{l=0}^{i-1}\hat{\Omega}_l^{\mathrm{o}},$$
$$i = 1,\cdots,k^{\mathrm{o}}, \tag{31}$$

where $\hat{\Omega}_0^{\mathrm{o}} = \varnothing$.

We design a pathwise state-feedback switching law called by $\bigwedge\limits_{j=1}^{k^{\mathrm{o}}}(\theta_j^{\mathrm{o}})^{\hat{\Omega}_j^{\mathrm{o}}}$ as

$$\begin{cases} i_j = \arg\{x_j \in \hat{\Omega}_i^{\mathrm{o}}\},\ j = 0,1,\cdots, \\ \tau_{j+1} = \tau_j + s_{i_j}, \\ \sigma_x^*(t) = \theta_{i_j}^{\mathrm{o}}(t-\tau_j),\ \forall t \in [\tau_j,\tau_{j+1}), \\ x_{j+1} = \phi(s_{i_j};0,x_j,\theta_{i_j}^{\mathrm{o}}) = G_{i_j}^{\mathrm{o}}x_j. \end{cases} \tag{32}$$

It is clear that switching law $\sigma_x^*(t)$ is an optimal switching path of aggregated system (26) under cost function (27) with initial state x.

Theorem 3 State-feedback switching law $\sigma_x^*(t)$ given by (32) can asymptotically stabilize switched linear system (1) with initial state x, and it is a sub-optimal switching path of switched linear system (1).

Proof For $\forall t \in [\tau_j, \tau_{j+1})$, we have

$$||\phi(t; 0, x, \sigma_x^*(t)|| \leqslant \eta^s ||x(j)||,$$

where $s = \max\limits_{i \in M}\{s_i\}$ and $\eta = \max\limits_{i \in M}\{||A_i||\}$.

$\exists \hat{K}$, $V_{\hat{k}}(x)$ shown in (30) is a switched Lyapunov function of aggregated system (26) when $\hat{k} \geqslant \hat{K}$. State-feedback switching law $\sigma_x^*(t)$ can asymptotically stabilize aggregated system (26) and then $\lim\limits_{j \to \infty} ||x(j)|| = 0$. We have

$$\lim_{t \to \infty} ||\phi(t; 0, x, \sigma_x^*(t))|| = 0,$$

so $\sigma_x^*(t)$ can asymptotically stabilize switched linear system (1). As $\sigma_x^*(t)$ is an optimal switching path of aggregated system (26), it is a sub-optimal switching path of switched linear system (1). This proof is completed.

Note that $k^{\rm o}$ is bigger and the running time is longer if ϵ is smaller. To quickly compute a sub-optimal switching path of the switched system (1), we design a switching law based on switching law (2) by

$$\begin{cases} i_j = \arg\{x_j \in \hat{\Omega}_i\},\ i \in 1, \cdots, k, \\ \tau_{j+1} = \tau_j + s_{i_j}, \qquad j = 0, 1, \cdots, \\ \bar{\sigma}_x^*(t) = \bar{\theta}_{i_j}^{\rm o}(t - \tau_j),\ \forall t \in [\tau_j, \tau_{j+1}), \\ x_{j+1} = \phi(s_{i_j}; 0, x_j, \bar{\theta}_{i_j}^{\rm o}), \end{cases} \tag{33}$$

where $\hat{\Omega}_i$ shown in (30) is an optimal partition of aggregated system (3), $\bar{\theta}_{i_j}^{\rm o}$ is an optimal path of θ_{i_j} under cost function (7) and computed by above optimal conjugate gradient algorithm directly at each step.

As $\bar{\theta}_{i_j}^{\rm o}$ is contractive when a is denoted by (25), switching law (33) asymptotically stabilizes switched linear system (1) and the switching cost is smaller than that cost under non-optimal contractive path.

5 Example

Consider a continuous-time switched linear system $\dot{x}(t) = A_i x(t),\ i = 1, 2$ with two subsystems. Its coefficient matrices are respectively:

$$A_1 = \begin{bmatrix} -0.8077 & 0.5385 & 2.2692 \\ -3.0000 & -2.1923 & -0.0769 \\ 0.2308 & 2.1538 & 0.6154 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3846 & -0.1923 & -1.0769 \\ 1.8462 & -1.9231 & 0.4231 \\ -0.3846 & -2.5385 & -0.8077 \end{bmatrix}.$$

Suppose that the sampling period is $\tau = 0.2\,{\rm s}$, contractive ratio is $\mu = 0.94$, initial state is $x = [-4\ \ 2\ \ -3]^{\rm T}$, and switching cost matrix is $Q_i = I_3$.

Computing $||A||^+ = 4.0543$ and $s = 0.6$, we set $a = 120$ which makes (25) hold. We use above switching laws to stabilize the switched system of this example respectively under cost function

$$120x^{\rm T}(s_i)x(s_i) + \int_0^{s_i} x^{\rm T}(t)Q_{\sigma(t)}x(t){\rm d}t.$$

Firstly, we get Ω_i and θ_i for $i = 1, \cdots, 5$ and corresponding aggregated system (3), compute $\hat{\Omega}_i$ by switched Laypunov function V_2 and use switching law (2) (Ω_i is substituted by $\hat{\Omega}_i$) to stabilize the switched system. The switching trajectory without optimal paths is shown in Fig.1 and the cost is 1.4696e4 over $[0, 12]$ s.

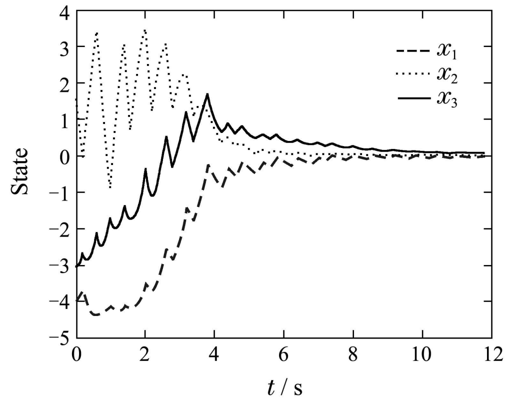

Fig. 1 Switching trajectory under $\bigwedge\limits_{i=1}^{5} (\theta_i)^{\hat{\Omega}_i}$ and V_2

Secondly, we set $k^{\rm o} = 30$ and get an aggregated system as (26). Let V_2 be a switched Laypunov function. We use an optimal switching law as (32) to asymptotically stabilize the switched linear system. The switching trajectory is shown in Fig.2. The total cost is 1.0879e4 over $[0, 12]$ s.

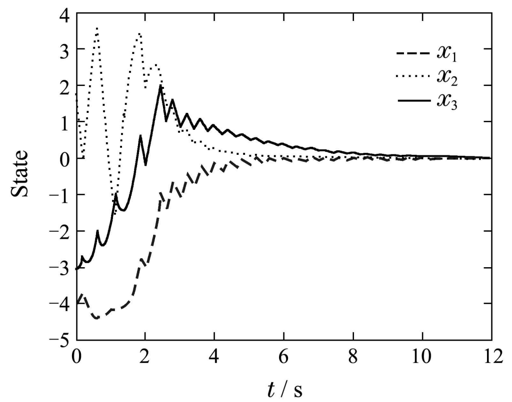

Fig. 2 Sub-optimal switching trajectory under $\sigma_x^*(t)$ and V_2

Finally, to get a smaller cost, we use switching law (33). The switching trajectory is shown in Fig.3 and the total cost is 6.2163e3 over $[0, 12]$ s.

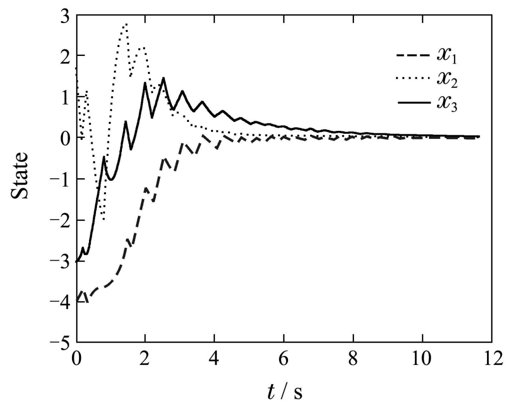

Fig. 3 Switching trajectory under $\bar{\sigma}_x^*(t)$ and V_2 with optimal path

From the above cost values under different switching laws, it can be seen that the cost over the infinite time horizon becomes smaller when we use the optimal paths in the switching laws.

6 Conclusion

The well-defined pathwise state-feedback switching law $\bigwedge_{i=1}^{k}(\theta_i)^{\Omega_i}$ provides a way to stabilize the switched linear system, but the running cost can not be optimized. On finite time intervals, a conjugate gradient algorithm with Armijo steps was presented to find the optimal path θ_i° of normal path θ_i. To make the optimal path contractive, a relationship expression between K and Q_i was found in this work. The switched Riccati mapping can find a minimum quadratic switched Lyapunov function of the aggregated system under the cost function. The corresponding switching law based on this switched Lyapunov function made the discrete-time aggregated system minimal over infinite time. As the aggregated system is a sampled system of the original switched linear system, the latter has a sub-optimal cost under the switching path.

It is a hard task to search an optimal switching path over infinite time horizon for switched linear systems. In this work, some sub-optimal switching laws were designed and have smaller running cost than those of the non-optimal switching path. It should be noted that the optimal cost is smaller if k° is bigger and error ϵ is samller, but the running time of computer programs will be higher. However, a proper sampling period might make the error between cost of optimal path and that of optimal pathwise feedback switching path very small.

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