第 30 卷第 7 期
2013 年 7 月

控 制 理 论 与 应 用
Control Theory & Applications

Vol. 30 No. 7
Jul. 2013

DOI: 10.7641/CTA.2013.12088

基于伪谱展开的抛物型系统镇定与边界观测研究

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摘要: 模型截断设计方法在无限维系统控制中得到广泛的应用, 但是其自身存在信息丢失的缺陷而限制了控制器对高频扰动抑制的性能. 本文研究具有内部和Neumann边界控制的抛物型系统, 其中系统采用边界测量. 内部控制采用比例反馈形式, 其中反馈增益核由Sturm-Liouville系统稳定性分析来待定; 类似地, 边界反馈的设计也采用待定反馈增益核的方式, 最终对描述系统稳定性的Sturm-Liouville系统采用伪谱方法进行求解. 数字仿真结果表明了该方法的有效性.

关键词: 截断再设计; 设计再截断; 伪谱展开; 抛物型偏微分方程

中图分类号: O231.4　　**文献标识码**: A

Pseudospectral expansion-based model reduction for control and boundary observation of unstable parabolic partial differential equations

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Abstract: The reduce-then-design approach is widely used for controller synthesis of infinite dimensional systems. A drawback of the reduce-then-design method is the inherent loss of information due to the truncation before control design. Moreover, the order of the model truncation is a trade-off between model accuracy and real time computation. The stabilization of an unstable linear parabolic partial differential equation (PDE) system with both Neumann boundary control and interior control is considered in this work. Point output measurement is available at one end of the physical domain. A proportional state feedback is proposed for the interior control with a symmetric kernel function, and the pseudospectral method is used to solve the stability conditions governed by the Sturm-Liouville systems. In addition, an observer is designed using the point measurement at one end of the physical domain, and used to propose an observer - based feedback controller for the PDE system. Both controller and observer gains are designed numerically to make the eigenvalues of the associated Sturm-Liouville problems stable. Simulations show the effectiveness of the proposed controller.

Key words: reduce-then-design; design-then-reduce; pseudospectral expansion; parabolic PDEs

1 Introduction

Control of partial differential equations (PDEs) and associated numerical solutions have been widely motivated in the process engineering (e.g., [1–4]). In [5], the dynamic model of the heat exchanger is given by two partial differential equations that are used without spatial discretization to design the control law. In [6], the control of a DOC (diesel oxidation catalyst) is studied which is inherently a distributed parameter system due to its elongated geometry where a gas stream is in contact with a spatially distributed catalyst. In [7], the authors considered the optimal control of convection-diffusion systems modeled by parabolic PDEs with time-dependent spatial domains for application to the crystal temperature regulation problem in the Czochralski (CZ) crystal growth process. In [8], a P-type steady-state iterative learning control (ILC) scheme is applied to the boundary control of a class of nonlinear processes described by PDEs, which

Received 29 April 2012; revised 21 January 2013.
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This work was supported by the National Natural Science Foundation of China (No. F030119), the Natural Science Foundation of Zhejiang Province (Nos. Y1110354，Y6110751), and the Natural Science Foundation of Ningbo (No. 2010A610096).

cover many important industrial processes such as heat exchangers, industrial chemical reactors, biochemical reactors, and biofilters. In [9], the approximation of completely resonant nonlinear wave systems via deterministic learning is studied where the plants are distributed parameter systems (DPS) describing homogeneous and isotropic elastic vibrating strings with fixed endpoints.

How to establish a bridge to connect the mathematical PDE system control and finite dimensional synthesis tool is a quite interesting topic for both mathematicians and engineers. The main interests in the mathematical control community focus on the fundamental problems such as controllability, observability and optimization of distributed parameter systems using quite advanced mathematical knowledge which are usually not accessible to general practical engineers and applied sciences. In the last decade, two main categories of approaches developed to handle practical control problems of PDEs arising in engineering practices became quite popular due to the usage of accessible engineering mathematical tools, i.e., the inertial manifold method and the backstepping technique. The inertial manifold method uses the attractors of the dissipative parabolic PDE to split the infinite dimensional system into fast and slow dynamics, then the controller synthesis approaches by virtue of finite dimensional control theories can be used (e.g., [10] and references therein). The backstepping has been proved as a powerful approach to PDE control with both Neumann and Dirichlet boundary actuation where explicit control laws have been proposed for both parabolic and hyperbolic PDEs (e.g., [11] and references therein). Additionally, extensions of adaptive control to infinite dimensional systems (with interior or boundary control mechanisms) is a very exciting research field (e.g. [9, 12–15] and references therein).

Instead of implementing the model reduction techniques before control design, we follow a design-then-reduce method in this work. We consider a proportional type interior control for the unstable PDE system. An integral operation for the product of the proportional feedback kernel gain and the system state is used for the PDE stabilization in this paper, e.g.,

$$v(t) = \int_0^1 k_{\mathrm{v}} f(y)\psi(y,t)\mathrm{d}y,$$

where $f(y)$ is the control actuation function and k_{v} is the to-be-designed gain. By substituting the proposed proportional control law into the unstable PDE system, we use the variable separation method to obtain a self-adjoint Sturm-Liouville problem associated with the closed-loop system (i.e., the spectral conditions of the closed-loop C_0-semigroup), which includes the to-be-designed feedback controller gain (e.g., kernel function). The closed-loop Sturm-Liouville system is an integro-differential-type two boundary value problem which does not admit an analytical solution in general, and numerical methods are necessary for its solution. Using the eigenfunctions obtained from the uncontrolled Sturm-Liouville problem (relevant to the boundary feedback design), we apply the pseudospectral method to rewrite the controlled Sturm-Liouville problem as a finite dimensional matrix eigenvalue problem, which can be equivalently considered as a pole placement problem for PDE systems.

The spatial-temporal state information needed in the proposed proportional control law makes it impractical since this information is usually not available. Therefore, an observer to estimate the spatial-temporal state information is designed exploiting the availability of point measurement at one end of the physical domain. Point measurement by locating sensors at specific points of interests in the physical domain is common and feasible in engineering practice. The estimation error dynamics define a non-self-adjoint Sturm-Liouville problem (NSASLP)[16], which includes the to-be-designed observer gain. NSASLP is quite common in studying the stability of fluid mechanics and numerical mathematics research is still attracting much attention. Similarly to the controller case, Galerkin projection is used to reduce the Sturm-Liouville problem to a pole placement problem. In this case, the eigenfunctions obtained by solving the Sturm-Liouville problem associated with the uncontrolled error dynamics are used during the Galerkin projection.

Sano employed output feedback in [17] to stabilize the first order heat exchanger PDEs using Huang's result on the spectrum determined growth assumption. More work along this line can be found in [18–19] and references therein. However, the analytical study of the spectra associated with the closed-loop C_0-semigroup is complicated. The second-order nature of the parabolic PDE under consideration in this work makes the spectral analysis even more complex when designing the state observer based on the boundary measurement (by duality the feedback mechanism is similar to that in [17]).

This work proposed a novel numerical framework for the design of an explicit control law with a proportional feedback kernel function (infinite dimensional proportional control) to stabilize infinite dimensional systems. This approach avoids PDE order truncation to synthesize a finite dimensional controller. The best advantage of this method can remain the violent dynamics which need to be fed back and suppressed but carried out by high frequency components. Although order truncation is unavoidable while solving the controller synthesis equations (e.g., the infinite dimensional Lyapunov equation, Sturm-Liouville equation, and etc.), but this procedure would not neglect the information in high frequencies of the temporal-spatial state. In order to give a formal explantation of the benefit of this proposed approach, we assume a state feedback Kx, where K and x represent the feedback gain and the state, respectively. For the reduce-then-design framework, the state feedback formulation becomes $(K^* + \delta K)(x^* + \delta x)$, where $(\cdot)^*$ and $\delta(\cdot)$ represent the finite dimensional truncations and the associated truncated errors, respectively. We realize that it is a pure numerical computing problem for solving the controller synthesis equation (on K) and the numerical error can be well-controlled under a predefined error tolerance. However, the state is reflected by the system which could be driven by any possible external high frequency excitations. In another word, we can make δK as small as possible numerically but not δx. Thus, it is readily to make an extension to the design-then-reduce framework, i.e., $(K^*+\delta K)x$, where δK can be solved under any given error tolerance. The observer design can be seen as the dual formulation of the stabilization problem. Both the controller and observer designs are formulated as Sturm-Liouville problems that can be solved with the pseudospectral-Galerkin method.

The paper is organized as follows. We present the boundary control in Section 2. An infinite-dimensional interior control is presented in Section 3. A simulation study for the infinite-dimensional controller is carried out in Section 5, where both the numerical scheme and a numerical example are discussed. We close this paper by stating conclusions and future research topics in Section 6.

2 Boundary control

For the following system:

$$\begin{cases} \psi_{\rm t} = \psi_{\rm xx} + f(x)v(t) + \lambda\psi, \\ \psi_{\rm x}(0,t)=0,\ \psi_{\rm x}(1,t)=w(t),\ y(t)=\psi(1,t), \end{cases} \tag{1}$$

we consider the control Lyapunov function

$$V(\psi) := \frac{1}{2}\|\psi\|^2 = \frac{1}{2}\int_0^1 |\psi(x,t)|^2 {\rm d}x := \frac{1}{2}\langle\psi,\psi\rangle, \tag{2}$$

where $\|\cdot\|$ is the usual norm in $L^2(0,1)$, and $\langle\cdot,\cdot\rangle$ is the inner product. We compute the time derivative of the control Lyapunov function V to obtain

$$\dot V = \int_\Omega \psi\psi_{\rm t} = \int_\Omega \psi[\psi_{\rm xx} + f(x)v(t) + \lambda\psi] = \\ \psi(1)\psi_{\rm x}(1) - \int_\Omega \psi_{\rm x}^2{\rm d}x + \int_\Omega[\lambda\psi^2 + v(t)f\psi]{\rm d}x.$$

We can find that $\dot V$ can be positive to make the system (1) unstable when λ is sufficient large. To enhance the negativeness of $\dot V$, we can let $w(t) = -k_{\rm w}\psi(1,t)$, where $k_{\rm w}$ is the feedback gain. Although it may be possible to stabilize the unstable system (1) without using interior control by carefully choosing $k_{\rm w}$ high enough, in this paper we set $k_{\rm w} = 1$ to avoid high boundary control action and follow a combined boundary-interior control approach, i.e.,

$$w(t) = -\psi(1,t). \tag{3}$$

A feedback law (3) with such simplicity on the boundary is appreciated due to vulnerability on the physical boundary. Since the boundary is exposed to external environments, synthesizing complicated feedback laws is not practical if not consider environmental noises[15]. Substituting the feedback law (3) into (1), the PDE system becomes

$$\begin{cases} \psi_{\rm t} = \psi_{\rm xx} + \lambda\psi + fv, \\ \psi_{\rm x}(0,t) = \psi_{\rm x}(1,t) + \psi(1,t) = 0. \end{cases} \tag{4}$$

This is an unstable system if λ is sufficiently large, and we will use the interior control v to stabilize it in this work.

Remark 1 The boundary feedback law (3) has a quite simple form although more complicated laws can be generated by using either the weak variations approach[20] or the backstepping technique[11] to achieve better performance. In this work, we have another freedom to shape the dynamics using the interior actuator.

3 Interior actuator: control design

We propose an interior feedback control with the following proportional kernel form:

$$v(t) = -\int_\Omega k_{\rm v} f(y)\psi(y,t){\rm d}y, \tag{5}$$

where the feedback gain $k_{\rm v}$ is to be determined. Then, the closed-loop system takes the form of

$$\begin{cases} \psi_{\rm t} = \psi_{\rm xx} - \int_\Omega k_{\rm v} f(x)f(y)\psi(y,t){\rm d}y + \lambda\psi, \\ \psi_{\rm x}(0) = \psi_{\rm x}(1) + \psi(1) = 0. \end{cases} \tag{6}$$

Theorem 1 Given the unstable system (1) and the boundary feedback law (3), the interior feedback (5) can stabilize the system if the eigenvalues of the following system satisfies $\mu < 0$:

$$\begin{cases} X'' - \int_\Omega k_{\rm v} f(x)f(y)X(y){\rm d}y + \lambda X = \mu X, \\ X'(0) = X'(1) + X(1) = 0. \end{cases} \tag{7}$$

Proof Using the variable separation method ($\psi(x,t) = X(x)T(t)$), we can rewrite the system (6) as

$$\frac{\dot T(t)}{T(t)} = \\ \frac{X''(x) - \int_\Omega k_{\rm v} f(x)f(y)X(y){\rm d}y + \lambda X(x)}{X(x)} = \mu,$$

with the boundary condition given by $X'(0)T(t) = [X'(1) + X(1)]T(t) = 0$. Thus, we obtain the eigenvalue problem (6) and the temporal equation $\dot T(t) - \mu T(t) = 0$ which has exponentially stable solution if $\mu < 0$.

Therefore, the stabilization problem becomes to solve the integro-differential equation (7). Based on the feedback kernel function chosen in (5), we can prove that all the eigenvalues governed by (6) are real numbers.

Theorem 2 The eigenvalues of the Sturm-Liouville system (7) are real numbers.

Proof We introduce the operator S_1 associated with (7):

$$(S_1 g)(x) = \frac{{\rm d}^2 g}{{\rm d}x^2} - \int_\Omega k_{\rm v} f(x)f(y)g(y){\rm d}y + \lambda g(x),$$

with the domain $D(S_1) = \{g \in H^2; g'(0) = g'(1) + g(1) = 0\}$ and $H^2 = \{g; g,\ g'$ and $g'' \in L^2(0,1)\}$. We can show that S_1 is self-adjoint, i.e., given $g_1, g_2 \in D(S_1)$, it satisfies $\langle g_2, S_1 g_1\rangle = \langle g_1, S_1 g_2\rangle$:

$$\langle g_2, S_1 g_1\rangle = \\ \int_0^1 \frac{{\rm d}^2 g_1}{{\rm d}x^2} g_2{\rm d}x - \int_0^1\int_0^1 k_{\rm v} f(x)f(y)g_1(y){\rm d}y g_2(x){\rm d}x + \\ \int_0^1 \lambda g_1 g_2{\rm d}x = \\ -g_1(1)g_2(1) - \int_0^1 g_1' g_2'{\rm d}x - \\ k_{\rm v}\int_0^1 f(y)g_1(y){\rm d}y \int_0^1 f(x)g_2(x){\rm d}x + \int_0^1 \lambda g_1 g_2{\rm d}x = \\ \int_0^1 g_1 g_2''{\rm d}x - k_{\rm v}\int_0^1 f(x)g_1(x){\rm d}x \int_0^1 f(y)g_2(y){\rm d}y + \\ \int_0^1 \lambda g_2 g_1{\rm d}x = \langle g_1, S_1 g_2\rangle. \tag{8}$$

It is known that self-adjoint operators have real eigenvalues.

Therefore, the stabilization problem is to find a feedback gain k_{v} such that all the eigenvalues of the operator S_1 are negative. However, the associated Sturm-Liouville problem for feedback design can not be solved explicitly for a general control function $f(x)$ and numerical methods are necessary. The Sturm–Liouville problem of (6), when $k_{\mathrm{v}}=0$, is

$$\phi_n''=-\gamma_n^2\phi_n,\ \phi_n'(0)=\phi_n'(1)+\phi_n(1)=0,\ n\in\mathbb{N}. \tag{9}$$

We assume that the solution to (7) can be approximated as $X(x)\approx\sum_{i=1}^{I_{\mathrm{c}}}a_i\phi_i(x)$, where I_{c} is a truncation number of the infinitely many basis functions provided by (9), and $a_i(i=1,2,\cdots,I_{\mathrm{c}})$ are constants. Then, we can multiply both sides of (7) by ϕ_j and integrate over $[0,1]$ to obtain

$$\begin{aligned}&-\sum_{i=1}^{I_{\mathrm{c}}}a_i\phi_i(1)\phi_j(1)-\int_0^1\sum_{i=1}^{I_{\mathrm{c}}}a_i\phi_i'(x)\phi_j'(x)\mathrm{d}x-\\&k_{\mathrm{v}}\sum_{i=1}^{I_{\mathrm{c}}}a_i\int_0^1\int_0^1 f(x)f(y)\phi_i(y)\phi_j(x)\mathrm{d}y\mathrm{d}x+\\&\lambda\sum_{i=1}^{I_{\mathrm{c}}}a_i\int_0^1\phi_i(x)\phi_j(x)\mathrm{d}x=\\&-\sum_{i=1}^{I_{\mathrm{c}}}[\phi_i(1)\phi_j(1)]a_i-\sum_{i=1}^{I_{\mathrm{c}}}[\int_0^1\phi_i'(x)\phi_j'(x)\mathrm{d}x]a_i-\\&k_{\mathrm{v}}\sum_{i=1}^{I_{\mathrm{c}}}[f_if_j]a_i+\lambda\sum_{i=1}^{I_{\mathrm{c}}}[\int_0^1\phi_i(x)\phi_j(x)\mathrm{d}x]a_i=\\&\mu\sum_{i=1}^{I_{\mathrm{c}}}[\int_0^1\phi_i(x)\phi_j(x)\mathrm{d}x]a_i.\end{aligned} \tag{10}$$

We introduce the matrix notation

$$\boldsymbol{A}_1(i,j)=\phi_i(1)\phi_j(1), \tag{11}$$

$$\boldsymbol{A}_2(i,j)=\int_0^1\phi_i'(x)\phi_j'(x)\mathrm{d}x \tag{12}$$

$$\boldsymbol{A}_3(i,j)=\int_0^1\int_0^1 f(x)f(y)\phi_i(y)\phi_j(x)\mathrm{d}y\mathrm{d}x, \tag{13}$$

$$\boldsymbol{A}_4(i,j)=\int_0^1\phi_i(x)\phi_j(x)\mathrm{d}x, \tag{14}$$

$$\boldsymbol{a}=[a_1\ \cdots\ a_{I_{\mathrm{c}}}]^{\mathrm{T}}, \tag{15}$$

and rewrite (10) to obtain the finite dimensional representation of (7):

$$(-\boldsymbol{A}_1-\boldsymbol{A}_2-k_{\mathrm{v}}\boldsymbol{A}_3+\lambda\boldsymbol{A}_4)\boldsymbol{a}=\mu\boldsymbol{A}_4\boldsymbol{a}. \tag{16}$$

To make equation (16) have non-trivial solutions, we must satisfy the following equation with respect to μ:

$$\det[(\mu-\lambda)\boldsymbol{A}_4+\boldsymbol{A}_1+\boldsymbol{A}_2+k_{\mathrm{v}}\boldsymbol{A}_3]=0. \tag{17}$$

Therefore, the stabilization problem is to find a control gain k_{v} to place the roots of (17) on the left half plane ($\Re(\mu)<0$).

4 Interior actuator: observer design

We can assume that the observer takes the following form:

$$\begin{cases}\hat\psi_{\mathrm{t}}=\hat\psi_{\mathrm{xx}}+f(x)v+\lambda\hat\psi+k_{\mathrm{o}}(x)[\psi(1)-\hat\psi(1)],\\ \hat\psi_{\mathrm{x}}(0)=\hat\psi_{\mathrm{x}}(1)+k_{\mathrm{w}}\psi(1)=0,\end{cases}$$

where $k_{\mathrm{o}}(x)$ is the observer gain to be designed. We define $e=\psi-\hat\psi$, which is governed by

$$\begin{cases}e_{\mathrm{t}}=e_{\mathrm{xx}}+\lambda e-k_{\mathrm{o}}(x)e(1),\\ e_{\mathrm{x}}(0)=e_{\mathrm{x}}(1)=0.\end{cases} \tag{18}$$

We use the variable separation method $(e(x,t)=Y(x)T(t))$ to obtain the Sturm–Liouville problem of (18),

$$\begin{cases}Y''-k_{\mathrm{o}}(x)Y(1)+\lambda Y=\nu Y,\\ Y'(0)=Y'(1)=0,\end{cases} \tag{19}$$

where ν can be a complex number since the operator

$$S_2\varphi=\frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2}-k_{\mathrm{o}}(x)\varphi(1)+\lambda\varphi(x)$$

over the domain $D(S_2)=\{\varphi\in H^2;\varphi'(0)=\varphi'(1)=0\}$ is non self-adjoint, i.e., $\langle\varphi_2,S_2\varphi_1\rangle\neq\langle\varphi_1,S_2\varphi_2\rangle$, $\forall\varphi_1,\varphi_2\in D(S_2)$. We can explicitly solve the eigenvalue problem $S_2\varphi=\nu\varphi$, $\nu\in\mathbb{C}$, when $k_{\mathrm{o}}(x)$ is a simple function, such as a constant or a harmonic function as discussed below. However, a numerical approach is needed in the general case.

4.1 Constant gain

If the feedback gain in the observer is a constant k_{o}, the Sturm-Liouville problem (19) becomes

$$\frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2}-k_{\mathrm{o}}\varphi(1)+\lambda\varphi(x)=\nu\varphi(x), \tag{20}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}(0)=\frac{\mathrm{d}\varphi}{\mathrm{d}x}(1)=0. \tag{21}$$

We first check that $\nu=\lambda$ is not the eigenvalue. If $\nu=\lambda$, we have the solution $\varphi(x)=C_1x^2+C_2x+C_3$, where C_1,C_2,C_3 are constants determined by $\varphi'(0)=C_2=0$, $\varphi'(1)=2C_1=0$, but $\varphi''(x)=2C_1=0\neq k_{\mathrm{o}}\varphi(1)$. Then $\nu\neq\lambda$ and the general solution is given by

$$\begin{aligned}\varphi(x)=\ &C_1\cos(\sqrt{\nu-\lambda}x)+C_2\sin(\sqrt{\nu-\lambda}x)+\\&\frac{k_{\mathrm{o}}\varphi(1)}{\nu-\lambda},\end{aligned} \tag{22}$$

where C_1 and C_2 are constants. When $C_1=C_2=0$, we have a constant solution,

$$\varphi(x)=\frac{-k_{\mathrm{o}}\varphi(1)}{\nu-\lambda}\Longrightarrow\nu=-k_{\mathrm{o}}+\lambda. \tag{23}$$

Then, we can find that a constant gain k_{o} can change the eigenvalue λ. The general solution (22) satisfies the boundary conditions $\varphi'(0)=\varphi'(1)=0$ where $C_1(\neq0)$ and $C_2(=0)$ are constants to be determined by the boundary conditions (21), i.e.,

$$\begin{cases}C_1\sqrt{\nu-\lambda}\sin 0-C_2\sqrt{\nu-\lambda}\cos 0=0,\\ C_1\sqrt{\nu-\lambda}\sin(\sqrt{\nu-\lambda})-\\ C_2\sqrt{\nu-\lambda}\cos(\sqrt{\nu-\lambda})=0.\end{cases}$$

Then, the eigenvalue ν is determined by $\sin(\sqrt{\nu-\lambda})=0$ which is independent of k_{o}. This is an interesting result that shows that a constant gain k_{o} can only change the eigenvalue corresponding to the first constant eigenfunction. Therefore, to design an effective observer based on the point measurement output, a gain function including positive frequency harmonics is required.

4.2 Harmonic function gain

We choose sine functions as the feedback gain, i.e., $k_o(x) = \sin(n\pi x)$, $n \in \mathbb{N}$, then (19) becomes

$$\frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2} - k_o \sin(n\pi x)\varphi(1) + \lambda\varphi(x) = \nu\varphi(x), \quad (24)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}(0) = \frac{d\varphi}{dx}(1) = 0, \quad (25)$$

whose solution is given by

$$\frac{\varphi(x)}{k_o\varphi(1)} = \frac{n\pi\sin\sqrt{\nu-\lambda}x}{\sqrt{\nu-\lambda}(\nu-\lambda-n^2\pi^2)} - \frac{\sin(n\pi x)}{\nu-\lambda-n^2\pi^2} + \frac{n\pi\cos(\sqrt{\nu-\lambda}x)(\cos\sqrt{\nu-\lambda}-\cos(n\pi))}{\sqrt{\nu-\lambda}(\nu-\lambda-n^2\pi^2)\sin(\sqrt{\nu-\lambda})}, \quad (26)$$

when $\nu \neq \lambda$, $\nu - \lambda \neq n^2\pi^2$ and $\nu - \lambda \neq n\pi$. The other cases are

$$\nu = \lambda : \frac{\varphi(x)}{k_o\varphi(1)} = \frac{\sin(n\pi x) - 1}{n^2\pi^2}, \quad (27)$$

$$\nu = \lambda + n^2\pi^2 : \quad (28)$$

$$\frac{\varphi(x)}{k_o\varphi(1)} = \frac{\mathrm{e}^{n\pi(1-x)}[\mathrm{e}^{n\pi} - (-1)^n]}{2n^2\pi^2(-1+\mathrm{e}^{2n\pi})} - \frac{\mathrm{e}^{n\pi x}[(-1)^n\mathrm{e}^{n\pi} - 1]}{2n^2\pi^2(-1+\mathrm{e}^{2n\pi})} + \frac{\sin(n\pi x)}{2n^2\pi^2}, \quad (29)$$

$$\nu = \lambda + n\pi : \quad (30)$$

$$\frac{\varphi(x)}{k_o\varphi(1)} = \frac{\mathrm{e}^{\sqrt{n\pi}x}\left[-1+(-1)^n\mathrm{e}^{\sqrt{n\pi}}\right]}{(1+n\pi)\sqrt{n\pi}\left(-1+\mathrm{e}^{2\sqrt{n\pi}}\right)} - \frac{\mathrm{e}^{\sqrt{n\pi}(1-x)}[(-1)^n - \mathrm{e}^{\sqrt{n\pi}}]}{(1+n\pi)\sqrt{n\pi}\left(-1+\mathrm{e}^{2\sqrt{n\pi}}\right)} + \frac{\sin(n\pi x)}{n\pi(1+n\pi)}. \quad (31)$$

By making $x = 1$ in (26), we can obtain the characteristic equation for ν ($n \in \mathbb{N}$, $\nu \neq \lambda$, $\nu - \lambda \neq n^2\pi^2$ and $\nu - \lambda \neq n\pi$):

$$\sqrt{\nu-\lambda}(\nu-\lambda-n^2\pi^2)\sin\sqrt{\nu-\lambda} + k_o n\pi[(-1)^n\cos\sqrt{\nu-\lambda} - 1] = 0. \quad (32)$$

We neglect the other three cases in (27)–(31), since the eigenvalue ν is positive and not of our interest. Therefore, the eigenvalues of the operator S_2 is given by $\sigma(S_2) := \{\nu : \text{ equation (32)}\} \cap \{\nu : \nu \neq \lambda\} \cap \{\nu : \nu - \lambda \neq n\pi\} \cap \{\nu : \nu - \lambda \neq n^2\pi^2\}$. The stabilization problem is to find a feedback gain k_o such that the roots of (32) in $\sigma(S_2)$ reside on the left half plane. However, this is a transcendental complex equation and not always able to be solved explicitly. Therefore, numerical computation is necessary to solve the Sturm-Liouville problem (19) associated with the observer design problem.

4.3 General function gain-numerical approach

The Sturm-Liouville problem of (18), when $k_o(x) = 0$, is

$$\varphi_n'' = -\omega_n^2\varphi_n,\ \varphi_n'(0) = \varphi_n'(1) = 0,\ n \in \mathbb{N}. \quad (33)$$

We assume that the solution to (19) can be expressed as $Y(x) \approx \sum_{i=1}^{I_o} b_i\varphi_i(x)$, where I_o is a truncation number of the infinitely many basis functions provided by (33) and $b_i (i = 1, 2, \cdots, I_o)$ are constants. Then, we can multiply both sides of (19) by φ_j and integrate over $[0, 1]$ to obtain

$$-\int_0^1 \sum_{i=1}^{I_o} b_i\varphi_i'(x)\varphi_j'(x)\mathrm{d}x - \sum_{i=1}^{I_o} b_i\varphi_i(1)\int_0^1 k_o(x)\varphi_j(x)\mathrm{d}x = (\nu-\lambda)\int_0^1 \sum_{i=1}^{I_o} b_i\varphi_i(x)\varphi_j(x)\mathrm{d}x. \quad (34)$$

We introduce the matrix notation

$$\boldsymbol{B}_1(i,j) = \langle\varphi_i, \varphi_j\rangle, \quad (35)$$

$$\boldsymbol{B}_2(i,j) = \langle\varphi_i', \varphi_j'\rangle, \quad (36)$$

$$\boldsymbol{B}_3(i,j) = \varphi_i(1)k_{o,j}, \quad (37)$$

where $k_{o,j} = \int_0^1 k_o(x)\varphi_j(x)dx$, and rewrite (34) to obtain the finite dimensional representation of (19):

$$(\lambda\boldsymbol{B}_1 - \boldsymbol{B}_2 - \boldsymbol{B}_3)\boldsymbol{b} = \nu\boldsymbol{B}_1\boldsymbol{b}. \quad (38)$$

To ensure that equation (38) has non-trivial solution $\boldsymbol{b} := [b_1\ b_2\ \cdots\ b_{I_o}]^{\mathrm{T}}$, we require

$$\det((\lambda-\nu)\boldsymbol{B}_1 - \boldsymbol{B}_2 - \boldsymbol{B}_3) = 0. \quad (39)$$

Therefore, the observer design problem is to find $k_{o,j}$ such that the roots of (39) reside on the left half plane, i.e., $\Re(\nu) < 0$.

5 Simulation study

5.1 Numerical approach

The closed-loop system is governed by the following coupled PDEs:

$$\begin{cases} \psi_t = \psi_{xx} - f(x)\int_0^1 k_v f(y)\psi(y,t)\mathrm{d}y + \lambda\psi + f(x)\int_0^1 k_v f(y)e(y,t)\mathrm{d}y, \\ e_t = e_{xx} + \lambda e - k_o(x)e(1), \\ \psi_x(0) = \psi_x(1) + \psi(1) = 0, \\ e_x(0) = e_x(1) = 0. \end{cases} \quad (40)$$

Defining an operator

$$S_0 e := f(x)\int_0^1 k_v f(y)e(y,t)\mathrm{d}y, \quad (41)$$

we can rewrite the closed-loop system (40) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\psi \\ e\end{pmatrix} = \begin{pmatrix} S_1 & S_0 \\ 0 & S_2\end{pmatrix}\begin{pmatrix}\psi \\ e\end{pmatrix}, \quad (42)$$

where the separation principle holds (see, e.g., [21]) to ensure closed-loop stability. For the numerical simulation, we can use the Galerkin method to solve the closed-loop system (40). We make the following expansion:

$$\psi^{(I_N)} = \sum_i^{I_N} z_i(t)\phi_i(x),\ \mathrm{e}^{(I_N)} = \sum_i^{I_N} \varepsilon_i(t)\varphi_i(x), \quad (43)$$

where the basis functions $\{\phi_i\}_{i=1}^{I_N}$ and $\{\varphi_i\}_{i=1}^{I_N}$ solve (9) and (33), respectively. We note that the index 'I_N' in deriving the numerical scheme in this section is always chosen larger than or equal to 'I_c' and 'I_o' in Section 3. Now we substitute (43) into the system (40) and use Galerkin projection to obtain

$$\begin{cases} \sum_{i=1}^{I_{\rm N}}\langle\phi_i,\phi_j\rangle\dot z_i=-\sum_{i=1}^{I_{\rm N}}\phi_i(1)\phi_j(1)z_i-\sum_{i=1}^{I_{\rm N}}\langle\phi_i',\phi_j'\rangle z_i- \\ \qquad k_{\rm v}\sum_{i=1}^{I_{\rm N}}f_if_jz_i+\lambda\sum_{i=1}^{I_{\rm N}}\langle\phi_i,\phi_j\rangle z_i+ \\ \qquad k_{\rm v}f_j\int_0^1 f(y){\rm e}^{(I_{\rm N})}(y,t){\rm d}y, \\ \sum_{i=1}^{I_{\rm N}}\langle\varphi_i,\varphi_j\rangle\dot\varepsilon_i=-\sum_{i=1}^{I_{\rm N}}\langle\varphi_i',\varphi_j'\rangle\varepsilon_i+\lambda\sum_{i=1}^{I_{\rm N}}\langle\varphi_i,\varphi_j\rangle\varepsilon_i- \\ \qquad \sum_{i=1}^{I_{\rm N}}\varphi_i(1)k_{{\rm o},j}\varepsilon_i. \end{cases} \tag{44}$$

We first solve the ε-equations in (44), then substitute ${\rm e}^{(I_{\rm N})}(x,t)=\sum_{i=1}^{I_{\rm N}}\varepsilon_i(t)\varphi_i(x)$ into the z-equations in (44) to solve the state equations. Defining

$$\boldsymbol z=(z_1,z_2,\cdots,z_{I_{\rm N}})^{\rm T},\ \varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{I_{\rm N}})^{\rm T}, \tag{45}$$

$$\boldsymbol A_1(i,j)=\phi_i(1)\phi_j(1),\ \boldsymbol A_2(i,j)=\int_0^1\phi_i'(x)\phi_j'(x){\rm d}x, \tag{46}$$

$$\boldsymbol A_3(i,j)=f_if_j=\int_0^1\int_0^1 f(x)f(y)\phi_i(y)\phi_j(x){\rm d}y{\rm d}x, \tag{47}$$

$$\boldsymbol A_4(i,j)=\int_0^1\phi_i(x)\phi_j(x){\rm d}x, \tag{48}$$

$$\boldsymbol B_1(i,j)=\langle\varphi_i,\varphi_j\rangle,\ \boldsymbol B_2(i,j)=\langle\varphi_i',\varphi_j'\rangle, \tag{49}$$

$$\boldsymbol B_3(i,j)=\varphi_i(1)k_{{\rm o},j},\ \boldsymbol F(j)=f_j\int_0^1 f(y){\rm e}^{(I_{\rm N})}(y,t){\rm d}y, \tag{50}$$

we can rewrite system (44) as

$$\begin{cases} \boldsymbol A_4\dfrac{{\rm d}\boldsymbol z}{{\rm d}t}=-(\boldsymbol A_1+\boldsymbol A_2+k_{\rm v}\boldsymbol A_3-\lambda\boldsymbol A_4)\boldsymbol z+\mathcal F(\varepsilon), \\ \boldsymbol B_1\dfrac{{\rm d}\varepsilon}{{\rm d}t}=-(\boldsymbol B_2-\lambda\boldsymbol B_1+\boldsymbol B_3)\varepsilon. \end{cases} \tag{51}$$

Remark 2 It is interesting to note that the closed-loop system (51) can be rewritten in the complex domain by using the Laplace transform (assuming the initial values are zeros):

$$\begin{cases} s\boldsymbol A_4\check{\boldsymbol z}=-(\boldsymbol A_1+\boldsymbol A_2+k_{\rm v}\boldsymbol A_3-\lambda\boldsymbol A_4)\check{\boldsymbol z}+\check{\mathcal F}, \\ s\boldsymbol B_1\check\varepsilon=-(\boldsymbol B_2-\lambda\boldsymbol B_1+\boldsymbol B_3)\check\varepsilon, \end{cases} \tag{52}$$

where $\check f$ is defined as the Laplace transform, i.e.,

$$\check f(s)=\int_0^\infty {\rm e}^{st}f(t){\rm d}t,\ s\in\mathbb C.$$

Then, we can obtain the characteristic equations for both the state and observer equations

$$|s\boldsymbol A_4+(\boldsymbol A_1+\boldsymbol A_2+k_{\rm v}\boldsymbol A_3-\lambda\boldsymbol A_4)|=0, \tag{53}$$

$$|s\boldsymbol B_1+\boldsymbol B_2-\lambda\boldsymbol B_1+\boldsymbol B_3)|=0, \tag{54}$$

which become the design conditions obtained in (17) (if $I_{\rm c}=I_{\rm N}$) and (39) (if $I_{\rm o}=I_{\rm N}$), respectively.

5.2 Numerical examples

In this subsection, we assume the input function

$$f(x)=\cos(0.86x)+f_{\rm h}\cos(9.53x),$$

and $\lambda=10$, where $f_{\rm h}$ is a constant. We solve the Sturm-Liouville problem (9) to obtain the first four eigenvalues $\gamma_1=0.86$, $\gamma_2=3.43$, $\gamma_3=6.44$, $\gamma_4=9.53$ and associated eigenfunctions

$$\phi_1(x)=\cos(0.86x),\ \phi_2(x)=\cos(3.43x),$$
$$\phi_3(x)=\cos(6.44x),\ \phi_4(x)=\cos(9.53x).$$

We solve the Sturm-Liouville problem (33) for $n=0$ and $n\in\mathbb N$: $\nu_n=n\pi$, $\varphi_n(x)=\cos(\nu_nx)$.

The stabilization and observer design problems are to find $k_{\rm v}\in\mathbb R$, and $k_{\rm o}(x)$ such that (17) and (39) have all the roots on the left half plane, i.e., $\Re(\mu)<0$, $\Re(\nu)<0$. For the feedback gain design, we choose $I_{\rm c}=3$ and compute the matrices defined in (11)–(15). The characteristic equation (17) becomes

$$c_0\mu^3+c_1\mu^2+c_2\mu+c_3=0,$$

where the coefficients $c_i,(i=0,1,2,3)$ are given by

$$c_0=0.2176, \tag{55}$$
$$c_1=0.2763k_{\rm v}+5.2039, \tag{56}$$
$$c_2=9.1672k_{\rm v}-54.9763, \tag{57}$$
$$c_3=15.0723k_{\rm v}-109.9055. \tag{58}$$

By using the Hurwitz stability criterion, we can obtain the stability condition with respect to the feedback gain $k_{\rm v}$:

$$c_i>0(i=2,3),\ c_1c_2>c_0c_3\Rightarrow k_{\rm v}>5.9256. \tag{59}$$

We choose that the observer gain function takes the form of $k_{\rm o}(x)=a+b\cos(\pi x)$. Let $I_{\rm c}=3$, then we compute the matrices defined in (35)–(37). The characteristic equation (39) becomes

$$d_0\nu^3+d_1\nu^2+d_2\nu+d_3=0, \tag{60}$$

where the coefficients $d_i,(i=0,1,2,3)$ are given by

$$d_0=0.25, \tag{61}$$
$$d_1=0.25a-0.50b+4.84, \tag{62}$$
$$d_2=7.34a-9.74b-74.33, \tag{63}$$
$$d_3=-0.96a+147.39b+9.61. \tag{64}$$

By using the Hurwitz stability criterion, we can obtain the stability condition with respect to (a,b):

$$d_i>0(i=1,2,3),\ d_1d_2>d_0d_3. \tag{65}$$

For the numerical simulation, we choose $I_{\rm N}=4$ and compute the matrices defined in (46)–(50):

$$\boldsymbol A_1=\begin{pmatrix} 0.43 & -0.63 & 0.64 & -0.65 \\ -0.63 & 0.92 & -0.95 & 0.95 \\ 0.64 & -0.95 & 0.98 & -0.98 \\ -0.65 & 0.95 & -0.98 & 0.99 \end{pmatrix}, \tag{66}$$

$$\boldsymbol A_2=\begin{pmatrix} 0.16 & 0.62 & -0.64 & 0.64 \\ 0.62 & 5.38 & 0.95 & -0.96 \\ -0.64 & 0.95 & 20.11 & 1.00 \\ 0.64 & -0.96 & 1.00 & 44.62 \end{pmatrix}, \tag{67}$$

$$\boldsymbol A_3=\begin{pmatrix} 1 & 0 & 0 & f_{\rm h} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f_{\rm h} & 0 & 0 & f_{\rm h}^2 \end{pmatrix},\ \boldsymbol B_3=\begin{pmatrix} a & b & 0 & 0 \\ -a & -b & 0 & 0 \\ a & b & 0 & 0 \\ -a & -b & 0 & 0 \end{pmatrix}, \tag{68}$$

$$\boldsymbol{A}_4 = \mathrm{diag}\{0.79, 0.55, 0.52, 0.51\}, \quad (69)$$

$$\boldsymbol{B}_1 = \mathrm{diag}\{1, 0.5, 0.5, 0.5\}, \quad (70)$$

$$\boldsymbol{B}_2 = \mathrm{diag}\{0, 4.93, 19.74, 44.41\}. \quad (71)$$

We choose

$$k_{\mathrm{v}} = 12,\ a = 12,\ b = 0.8$$

to satisfy the stability conditions (59)–(65). When $f_{\mathrm{h}} = 0$, i.e., there is no truncation error for the input function $f(x)$ involved in the control design, the closed-loop dynamics is shown in Fig.1, and the observer dynamics is shown in Fig.2. The observer-based feedback system dynamics is shown in Fig.3.

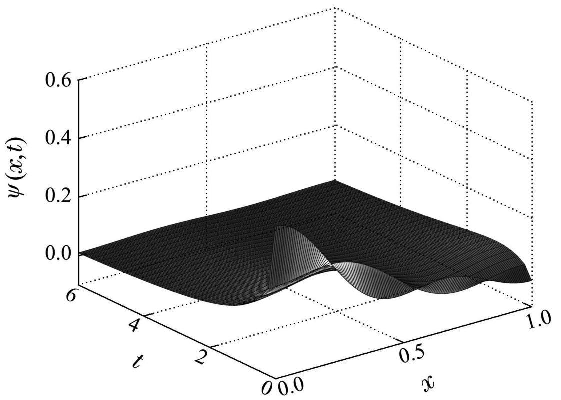

Fig. 1 Simulation of the controlled system without observer ($k_{\mathrm{v}} = 12$)

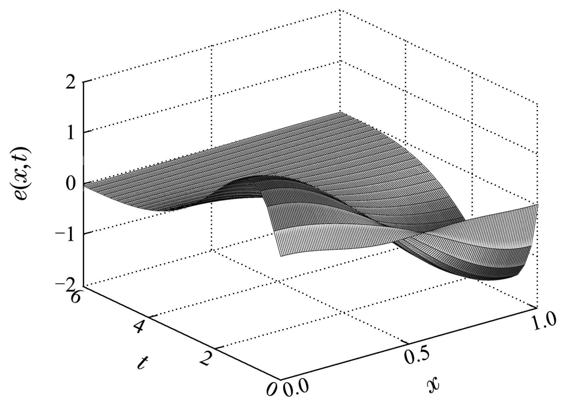

Fig. 2 Simulation of the observer equation ($a = 12, b = 0.8$)

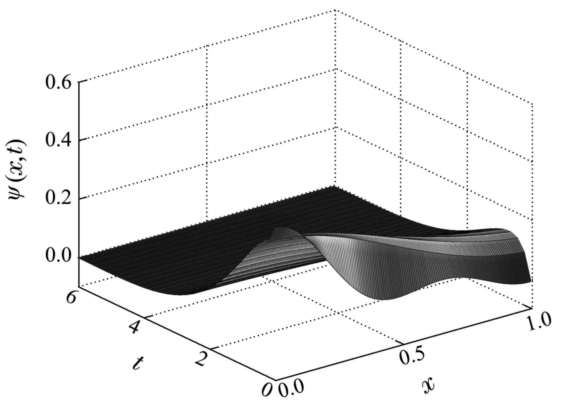

Fig. 3 Simulation of the controlled system with observer

6 Conclusions

Sturm-Liouville theory and numerical spectral analysis of differential operators are used in this work to approach the stabilization problem of an unstable parabolic PDE with constant diffusion coefficient. The stabilization mechanism includes two components: boundary (Neumann) and interior controls. In addition to the case of reaction-diffusion PDEs with constant diffusion coefficients in isotropic medias, we intend to consider the stabilization problem of dynamical systems in anisotropic medias, where the diffusion coefficients may vary with respect to the spatial coordinate. This method reduces the control synthesis for linear PDE systems to a parametric stabilization problem for a Sturm-Liouville system, which is solved using the finite dimensional truncation approach based on the pseudo-spectral method. The design of a state observer based on a boundary measurement is also approached in this work. Analytical and numerical work is carried out for the solution of the Sturm-Liouville system arising during the observer design in terms of three different scenarios for the observer gain: constant, harmonic and general gains. The analysis concludes that it is required to have harmonic components in the observer gains instead of pure constants. A numerical algorithm using the pseudo-spectral method is proposed for the observer design with general gain.

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